Decoherence without classicality in the resonant quantum kicked rotor

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(Received 4 May 2009; published 3 August 2009)

We study the quantum kicked rotor in resonance subjected to a unitary noise defined through Kraus operators. We show that this type of decoherence does not, in general, lead to the classical diffusive behavior. We find exact analytical expressions for the density matrix and the variance in the primary resonances. The variance does not loose its ballistic behavior; however, the coherence decays as a power law. The secondary resonances are treated numerically, obtaining a power-law decay for the variance and an exponential-law decay for the coherence.

DOI: 10.1103/PhysRevA.80.022102 PACS number(s): 03.65.Yz, 03.67.−a, 05.45.Mt

I. INTRODUCTION

The development of experimental techniques has made possible the trapping of samples of atoms using resonant exchanges in momentum and energy between atoms and laser light [1]. This progress has been accompanied by the development of the interdisciplinary fields of quantum computation and quantum information [2].

The study of open quantum systems is an outstanding topic of quantum mechanics. In particular the transition from the quantum to the classical world has intrinsic importance. On the other hand the advent of quantum computation makes of decoherence a central problem in the interaction of the quantum computer with its surroundings.

Simple theoretical and experimental models such as the quantum kicked rotor (QKR) and the quantum walk (QW) may play an important role in this frame. Although the existent experiments have high accuracy in both coherent storage and manipulation of the atoms, the interaction with the surroundings introduces different degrees of decoherence influencing the unitary evolution of the system.

The QKR is a milestone in the study of chaos at the quantum level [3]. The behavior of the QKR depends on whether the period of the kick is a rational or irrational multiple of 2π (in convenient units) [4]. For rational multiples the behavior of the system is resonant with ballistic spreading and has no classical analog; its standard deviation σ has the time dependence $\sigma(t) \sim t$. For irrational multiples the average energy of the system grows in a diffusive manner for a short time and then dynamical localization takes place. The quantum resonances and the dynamical localization of the QKR have been experimentally observed in samples of cold atoms interacting with a far-detuned standing wave of laser light [5], and in particular the secondary resonances have been recently observed by Kanem *et al.* [6].

The QKR as a simple toy model allows studying of the complexity of decoherence both analytically and numerically; these studies have a 25-yr-old history [7–9]. On the other hand the first experimental observation of environment induced decoherence in the QKR was reported by Ammann *et al.* [10].

In this line we recently investigated the QKR in resonance subjected to (a) decoherence with a Lévy waiting-time distribution [11,12] and (b) an excitation that follows an aperiodic Fibonacci prescription [13]. In both cases we find that the secondary resonances have a subballistic behavior [$\sigma(t) \sim t^c$, $1/2 < c < 1$] while the principal resonances maintain the well-known ballistic behavior. These results are very surprising since one expects diffusive behavior when decoherence occurs. Other authors also investigated the QKR subjected to noises with a Lévy distribution [14,15] and almost periodic Fibonacci sequence [16], showing that this decoherence never fully destroys the dynamical localization of the system but leads to a subdiffusion regime for a short time before localization appears.

In this work we want to study the decoherence effect of a unitary operation described by Kraus operators [17] acting on the density matrix. Our route is similar to that followed by Brun *et al.* [18] with the QW but our results in the QKR are very different.

II. KICKED ROTOR

In this section we briefly review the dynamical equations for the QKR [4]. Its Hamiltonian is

$$H = \frac{P^2}{2I} + K \cos \theta \sum_{n=1}^{\infty} \delta(t - nT), \quad (1)$$

where the external kicks occur at times $t=nT$, with n as the integer and T as the kick period, I is the moment of inertia of the rotor, P is the angular-momentum operator, K is the strength parameter, and θ is the angular position. In the angular-momentum representation, $P|\ell\rangle = \ell\hbar|\ell\rangle$, the matrix element of the time-step evolution operator U is

$$U_{\ell j} \equiv \langle \ell | U(\kappa) | j \rangle = i^{-(j-\ell)} e^{-ij^2 \varepsilon T/\hbar} J_{j-\ell}(\kappa), \quad (2)$$

where $\varepsilon = \hbar^2/2I$, J_m is the mth order cylindrical Bessel function, and its argument is the dimensionless kick strength $\kappa \equiv K/\hbar$. The resonance condition does not depend on κ and takes place when the frequency of the driving force is commensurable with the frequencies of the free rotor. Inspection of Eq. (2) shows that the resonant values of the scale parameter $\tau \equiv \varepsilon T/\hbar$ are the set of the rational multiples of 2π, $\tau = 2\pi\ p/q$. In what follows we assume that the resonance

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condition is satisfied; therefore, the evolution operator depends on κ, p, and q. We call a resonance primary when p/q is an integer and secondary when it is not.

III. KICKED ROTOR DYNAMICS WITH DECOHERENCE

In order to generate the dynamics of the system we consider that the decoherence is introduced through a completely positive map, which is defined by a set of Kraus operators $\{A_n\}$ [18]. To preserve the trace of the quantum operation these operators satisfy

$$\sum_{n=1}^{N} A_n A_n^{\dagger} = I. \quad (3)$$

Let us take two values of the strength parameter κ: κ_1 and κ_2. The corresponding time-step operators $U_1 \equiv U(\kappa_1)$ and $U_2 \equiv U(\kappa_2)$ are used to define

$$A_1 \equiv \sqrt{\alpha} U_1, \quad (4)$$

$$A_2 \equiv \sqrt{\beta} U_2, \quad (5)$$

as a particular set of Kraus operators, where $\alpha \in [0,1]$ and $\beta = 1-\alpha$ in order to satisfy Eq. (3). Then the following map for the time evolution of the density matrix is proposed

$$\rho(n+1) = \alpha U_1 \rho(n) U_1^{\dagger} + \beta U_2 \rho(n) U_2^{\dagger}, \quad (6)$$

where n indicates the time $t=nT$. When α (or β) vanishes, Eq. (6) reduces to the well-known evolution of the usual kicked rotor in quantum resonance. In other cases α (or β) may be thought of as the probability per time step to apply the operator U_1 (or U_2) to the density matrix.

We shall study the previous map in the case when the operators U_1 and U_2 commute ($[U_1, U_2]=0$), as is the case in the primary resonances; in what follows we use the principal resonance for simplicity. In this case it is easy to prove that, using mathematical induction, the solution of the map [Eq. (6)] is

$$\rho(n) = \sum_{j=0}^{n} \binom{n}{j} \alpha^{n-j} \beta^{j} U_1^{n-j} U_2^{j} \rho(0) U_2^{j\dagger} U_1^{(n-j)\dagger}, \quad (7)$$

where $\binom{n}{j} = \frac{n!}{j!(n-j)!}$. It is important to point out that Eq. (7) is a generic solution of Eq. (6) for any couple of unitary operators that commute. This means that the solution of the map is independent of the details of the model.

The probability for the angular-momentum value ℓ at time n is $P(\ell,n) \equiv \langle l|\rho(n)|l\rangle$. We shall calculate this probability for the first principal resonance. The matrix elements of U_1 and U_2 are expressed as $\langle l|U_1|j\rangle = i^{-(j-\ell)} J_{j-\ell}(\kappa_1)$ and $\langle l|U_2|j\rangle = i^{-(j-\ell)} J_{j-\ell}(\kappa_2)$. Then using the above equation [Eq. (7)] with the initial condition $\rho(0)=|0\rangle\langle 0|$, the probability is

$$P(\ell,n) = \sum_{j=0}^{n} \binom{n}{j} \alpha^{n-j} \beta^{j} \langle l|U(r_{nj})|0\rangle\langle 0|U^{\dagger}(r_{nj})|l\rangle, \quad (8)$$

where $r_{nj}=(n-j)\kappa_1 + j\kappa_2$ and $\langle l|U(r_{nj})|0\rangle = i^{-l} J_l(r_{nj})$. The moments of the angular momentum are

$$\langle \ell^m(n)\rangle = \sum_{\ell=-\infty}^{\ell=\infty} \ell^m P(\ell,n). \quad (9)$$

We want to study the time behavior of the variance $\sigma^2 = \langle \ell^2\rangle - \langle \ell\rangle^2$. The first moment vanishes due to the symmetry of the initial condition $\rho(0)$. Using the properties of the Bessel functions, the following value for the variance is obtained

$$\sigma^2(n) = \frac{1}{4}[(\alpha\kappa_1 + \beta\kappa_2)^2 n^2 + (\kappa_1 - \kappa_2)^2 \alpha\beta n]. \quad (10)$$

In the case when $\kappa_1=\kappa_2$ the system reduces to the usual kicked rotor in resonance and its variance has the well-known ballistic behavior characteristic of this case [4]. When $\kappa_1 \neq \kappa_2$ the coherence of the system is lost, as it is shown below, because the probabilistic map is effectively working. Equation (10) leads us to some interesting results. It shows that the ballistic behavior is maintained with this decoherence; but additionally there appears the diffusive term $(\kappa_1 - \kappa_2)^2 \alpha\beta n$. In particular if the parameters verify $\alpha\kappa_1 + \beta\kappa_2 = 0$, the behavior of the variance is totally diffusive as in the classical random walk. Then we can conclude that this decoherence always affects the behavior of the variance but, in general, does not break its ballistic growth.

The degree of coherence of the system can be measured by several means. We choose the following:

$$C(n) \equiv \mathrm{Tr}\{\rho^2(n)\} = \sum_{l=0} \langle l|\rho^2(n)|l\rangle. \quad (11)$$

Substituting Eq. (7) in the above equation, and using the properties of the Bessel function, the equation for the coherence is obtained

$$C(n) = \sum_{j=0}^{n} \sum_{i=0}^{n} \binom{n}{j} \binom{n}{i} \alpha^{n-j} \beta^{j} \alpha^{n-i} \beta^{i} J_0^2(\Delta\kappa_{ij}), \quad (12)$$

where $\Delta\kappa_{ij} = (i-j)\Delta\kappa$, with $\Delta\kappa = \kappa_1 - \kappa_2$. From Eq. (12) it is easy to prove that $C(0)=1$ and $C(n)<1$ for $n>0$ but in general this equation will be difficult to reduce to a more simple expression. However, we can get some additional information when $\Delta\kappa$ is very large. In this case $J_0^2(\Delta\kappa_{ij})$ goes to zero, except when $i=j$ because $J_0(0)=1$. Then in this limit Eq. (12) reduces to

$$C(n) \simeq \sum_{j=0}^{n} \binom{n}{j}^2 \alpha^{2(n-j)} \beta^{2j}. \quad (13)$$

Here we observe the interesting result that the coherence is independent of the strength parameters of the system if $\Delta\kappa$ is sufficiently large.

We made numerical studies of the long-time behavior for Eq. (12) as a function of the parameter α. In Fig. 1 the function $C(n)$ is plotted for large $\Delta\kappa$ and different values of α. This figure shows a power-law decay for the coherence $C(n) \sim \frac{1}{\sqrt{n}}$ independently of the value of α. The same results were obtained using Eq. (13). Therefore, in Eq. (13) we may choose a particular value of α to calculate its long-time decay. Taking $\alpha=1/2$ and using the sums of the binomial coefficients, we obtain

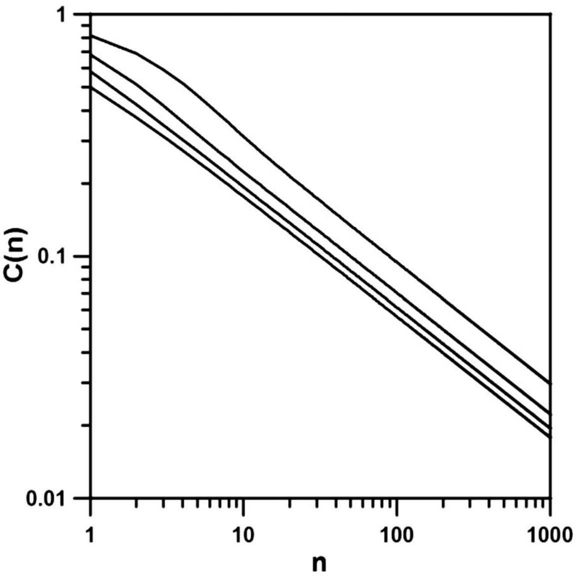

FIG. 1. The coherence $C(n)$ as a function of the dimensionless time n in log-log scales for $\Delta\kappa=1000$. The coherence was calculated, from top to bottom, for $\alpha=0.1$, $\alpha=0.2$, $\alpha=0.3$, and $\alpha=0.5$. The straight stretches with slopes of −0.5 show a power-law behavior $C(n)\sim\frac{1}{\sqrt{n}}$.

$$C(n) \simeq \binom{2n}{n}\left(\frac{1}{2}\right)^{2n}. \tag{14}$$

For large n it is possible to use the Stirling formula to obtain analytically the following expression for the coherence

$$C(n) \simeq \frac{1}{\sqrt{\pi n}}, \tag{15}$$

confirming our numerical result.

We also studied the coherence for several smaller values of $\Delta\kappa$. We obtained that, for each fixed value of $\Delta\kappa$, the coherence always decays as the power law $C(n)\sim n^{-\gamma}$ with $\gamma>0$ (see Fig. 2). Additionally we observed that the exponent γ is always independent of α like in Fig. 1; therefore, γ only depends on $\Delta\kappa$. We observe that the exponent γ grows with $\Delta\kappa$ with its values in the interval [0.3,0.5]. We can conclude that the qualitative behavior of Eqs. (12) and (13) are the same for all values of $\Delta\kappa$.

Now we inquire the incidence of decoherence on the secondary resonances. In this case the commutativity between the evolution operators U_1 and U_2 is lost, and the expressions for the variance and the density matrix become very cumbersome. Then we study the decoherence numerically using Eq. (6) for several values of the parameters κ and α.

In Fig. 3 the standard deviation σ is presented, for fixed values of κ_1 and κ_2 and for different values of α. It is seen that σ has power-law decay with an exponent c that depends on α. This parametric dependence is very different from that given by Eq. (10) in the primary resonances where $c=1$. The values of c were adjusted for the last thousand values of n and we found that they are near $c=1$. For other values of κ_1, κ_2, and α the exponent c varies between 0, 4, and 1.2. Considering all the cases studied we conclude that the exponent c does not show a clear rule of dependence with the parameters.

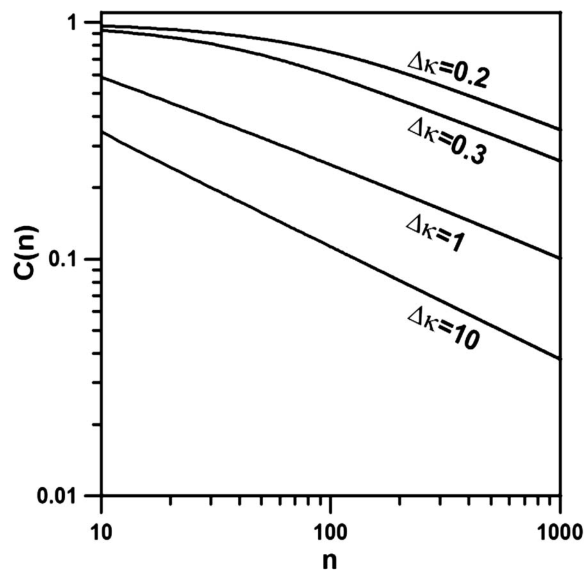

FIG. 2. The coherence $C(n)$ as a function of the dimensionless time n for $\alpha=0.1$ in log-log scales. For large n the curve satisfy a power law $C(n)\sim n^{-\gamma}$. The parameters of the curves, from top to bottom, are (1) $\Delta\kappa=0.2$ and $\gamma=0.36$, (2) $\Delta\kappa=0.3$ and $\gamma=0.37$, (3) $\Delta\kappa=1$ and $\gamma=0.4$, and (4) $\Delta\kappa=10$ and $\gamma=0.47$.

The numerical study of the coherence $C(n)$, for the same range of values of the parameters as for σ, showed that its time decay is better approximated by an exponential than by a power law, i.e., $C(n)\sim\exp(-\delta n)$ with $\delta>0$. Therefore, the coherence of the system in the secondary resonances is lost faster than in the primary ones.

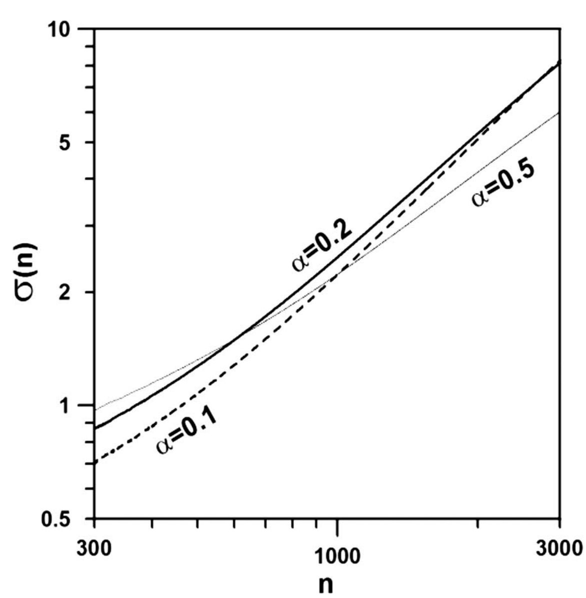

FIG. 3. The standard deviation $\sigma(n)$ as a function of the dimensionless time n. The parameters are $\kappa_1=0.1$, $\kappa_2=0.2$, and $p/q=1/3$ in all cases. Dashed line $\alpha=0.1$ ($c=1.2$), thick line $\alpha=0.2$ ($c=1.0$), and thin line $\alpha=0.5$ ($c=0.9$).

IV. DISCUSSION AND CONCLUSION

Decoherence in quantum systems as QKR or QW has been extensively studied. Analytical and numerical results [18–21] on the effect of different kinds of noise have shown that quantum properties are highly sensitive to random events. In particular the linear increase in the standard deviation $\sigma(t) \sim t$ can be eventually substituted by a diffusive behavior $\sigma(t) \sim t^{1/2}$ as in the classical random walk.

The linear increase in the standard deviation of the QKR in resonance is usually accepted as a direct consequence of the quantum coherence, in other words, a consequence of the unitary evolution. This work shows explicitly that unitary decoherence does not break the temporal linear increase in σ.

The absence of diffusive behavior in presence of decoherence has already been shown in our previous works [11–13]. There we have studied the QKR subject to different types of noise with a Lévy waiting-time distribution and we found that the system has a subballistic wave-function spreading and its standard deviation has a power-law tail. However, in that opportunity the coherence had not been studied.

Here we have considered a special type of decoherence in the QKR as a unitary map acting on the density matrix. We obtain an analytical expression for the density matrix when the Kraus operators commute. We prove that the decoherence affects the variance but its ballistic growth persists in spite of an additional linear term. Therefore, asymptotically the linear behavior of the standard deviation is not suppressed by the noise. On the other hand the coherence $C(n)$ has a power-law decay for all values of the parameters. We want to underline that the density matrix [Eq. (7)], solution of Eq. (6), only depends on the commutativity of the unitary operators U_1, U_2, and it is independent of their detail. This allows extending of the use of this expression for other quantum models such as the QW. In previous works [11–13,22], we have established a parallelism between the QKR in resonance with the discrete QW. Then the type of treatment presented in this paper could be applied to the QW.

When the Kraus operators do not commute we have no usable analytical expressions, it is necessary to make numerical studies. We establish that (a) the standard deviation has no simple dependence with the parameters of the system, (b) the standard deviation has (in the long-time limit) a continuous range of behaviors from diffusive to ballistic, and (c) the coherence $C(n)$ shows a exponential-law decay.

We can conclude that the effect of decoherence of the type studied in this work does not necessarily transform our quantum system into a dissipative system such as a Markov process. In more general terms, the mere presence of noise does not assure the passage from the quantum to the classical world.

ACKNOWLEDGMENTS

We acknowledge the support from PEDECIBA, PDT Contract No. S/C/IF/54/5, ANII, and thank V. Micenmacher for his comments and stimulating discussions.

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