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Quadratic solitons in degenerate quasi-phase-matched noncollinear geometry

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We investigate optical spatial solitons in a two-dimensional quasi-phase-matched geometry involving two concurrent noncollinear quadratic processes. The model, formally equivalent to that ruling second-harmonic generation in the presence of a one-dimensional transverse nonlinear grating, supports a class of simultons with a large domain of stability. We also identify a regime where the general equations predict walking solitary waves.

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Resonant interactions through dispersive wave mixing in nonlinear media are general phenomena occurring in acoustics, plasma physics, optics, etc. [1,2]. Parametric optical interactions in second-order media support a class of self-localized waves which encompass the mutual phase locking of various frequencies [3,4]. Such quadratic solitary waves or simultons have been investigated not only for their fundamental relevance but also for potential applications to light steering and signal processing in all-optical circuits, particularly in the case of frequency-degenerate second-harmonic (SH) generation (SHG) from a fundamental frequency (FF) beam [5–7], including the occurrence of *walking solitons* when walkoff is present [8–10]. Three-wave processes require energy and momentum conservation to yield efficient frequency conversion and/or amplification; the introduction of a periodic modulation via one-dimensional (1D) quasi-phase-matching (QPM) has been proven effective in exploiting the nonlinearity of noncentrosymmetric crystals in spite of dispersion [11] even toward simulton formation [12].

When momentum conservation is satisfied in several directions of propagation for the harmonic, degenerate quasi-phase-matching (DQPM) is said to take place. DQPM can occur in a variety of photonic lattices [13–15] as well as in random-QPM structures [16–18]. Berger introduced two-dimensional purely nonlinear lattices (NLs) [13], an extension of QPM to parametric interactions with extra in-plane degrees of freedom [14].

Preliminary studies of spatial simultons via SHG in 2D-NL were recently carried out in a doubly resonant quasi-phase-matched structure, namely, hexagonally poled lithium niobate [19,20]; however, a theoretical model able to correctly predict solitary waves in a DQPM geometry is still far to be accomplished. The simplest DQPM configuration involves two noncollinear reciprocal lattice vectors $\mathbf{G}_+$ and $\mathbf{G}_-$ of equal amplitude G_o and at angle 2Ω (Fig. 1). When the FF propagates with wave vector $\mathbf{k}_u$ at a small angle θ with respect to the bisector between $\mathbf{G}_+$ and $\mathbf{G}_-$, two SH wave vectors $\mathbf{k}_w^{\pm}$ can satisfy momentum conservation with $\mathbf{G}_{\pm}=\mathbf{k}_w^{\pm}-2\mathbf{k}_u$, at the two angles $\approx \pm G_o \sin(\Omega)/k_w$ with $\mathbf{k}_u$, respectively, having assumed $\theta \ll G_o \sin(\Omega)/k_w \ll \Omega$.

In this Rapid Communication we study quadratic solitary waves in a DQPM scheme, identifying their range of existence and addressing their stability. We consider continuous-wave light beams propagating in the structure described above, with a small detuning $\Delta k_o = k_w - 2k_u - G_o \cos(\Omega)$ when $\theta=0$. In the paraxial approximation, for an FF beam of waist x_o and diffraction length $L_R = k_u x_o^2/2$, it is straightforward to derive the normalized equations ruling the evolution of FF and SH waves:

$$iu_z + \frac{u_{xx}}{2} + 2\cos[\gamma(x-\vartheta z)]u^* w = 0,$$

$$iw_z + \frac{w_{xx}}{4} - \Delta w + 2\cos[\gamma(x-\vartheta z)]u^2 = 0, \qquad (1)$$

with u and w as the FF and SH amplitudes, respectively, $\gamma = G_o x_o \sin(\Omega)$, $\vartheta = 2L_R/x_o\theta$ and $\Delta = \Delta_o - (\gamma/2)^2$, with $\Delta_o = 2L_R \Delta k_o$. Clearly, the problem reduces to SHG in a 1D-NL along the transverse coordinate (x). Even though transverse lattices have been addressed in linear and nonlinear system, including second-order materials with a modulated refractive index [21,22], a purely quadratic transverse modulation was never investigated before.

Intuitively, SHG in a DQPM geometry can be described as the interaction of the FF with two distinct SH components, each of them combining momentum with one of the two reciprocal vectors $G_{\pm}$, i.e., each undergoing a phase mismatch $\Delta_o \mp \vartheta\gamma$ and walkoff $\pm\gamma/2$. This applies to plane waves in the no-depletion regime, as readily verified by substituting $w(x,z) = w(z)_+^{i\gamma(x-\vartheta z)} + w(z)_-^{-i\gamma(x-\vartheta z)}$ in the second of Eq. (1), or when the angle θ is large enough to decouple the two phase-matching conditions versus wavelength. Close to resonance for one of the two interactions, a regime of excitation can be identified for which the quadratic term out of resonance can be neglected or cast as an equivalent Kerr-like (cubic) term due to cascading [23–25]. In this limit the system supports standard *walking* simultons [3,9] and models soliton steering, as addressed in Refs. [19,20]. Far from these conditions, the simple superposition of two SH components cannot be correctly adopted because it does not conserve the energy of the interacting waves. Equation (1) possesses two symmetries—phase invariance and translational invariance

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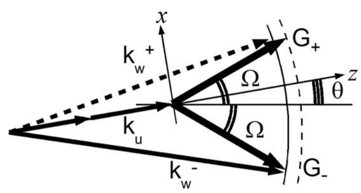

FIG. 1. Degenerate phase-matching configuration: in the drawing SHG is phase matched with the reciprocal vector G_- and slightly mismatched with G_+.

in $(x-\vartheta z)$—associated by Noether's theorem to two conserved quantities: the energy $Q=\int\{|u|^2+|w|^2\}dx$ and the combination $H'=H-\theta M$ of momentum $M=i\int\mathrm{Im}\{uu_x^* + ww_x^*/2\}dx$ and Hamiltonian $H=\int\{|u_x|^2/2+|w_x|^2/8\}dx + \Delta Q_w/2-2\int\cos[\gamma(x-\vartheta z)]\mathrm{Re}\{u^2w^*\}dx$. At variance with SHG processes with single phase matching, the Hamiltonian and momentum are not independently conserved, as the translational invariance is not decoupled in x and z. This leads to invariant solutions of the form $u=u_o(x-\vartheta z)e^{i\beta z}$ and $w=w_o(x-\vartheta z)e^{i2\beta z}$, with just one free parameter β: solitary waves are therefore expected to propagate along $x-\vartheta z$, in contrast with walking solitons.

These solutions reduce Eq. (1) to the variational problem $\delta_F(H'+\beta Q)=0$ for the eigenvalue β, with δ_F the Fréchet derivative [3,26]. For numerical integration, we can reduce the number of independent parameters by applying the transformations [3]: $\tilde{z}=\beta z$, $\tilde{x}=\sqrt{\beta}x$, $\gamma=\tilde{\gamma}\sqrt{\beta}$, $\vartheta=\sqrt{\beta}\tilde{\vartheta}$, $\alpha=\Delta_o/\beta+2-\tilde{\gamma}^2/4$, and setting $u_o=\beta\tilde{u}e^{i\vartheta x}$ and $w_o=\beta\tilde{w}e^{2i\vartheta x}$. We obtain

$$\frac{\tilde{u}_{\tilde{x}\tilde{x}}}{2}-\left(1-\frac{\tilde{\vartheta}^2}{2}\right)\tilde{u}+2\cos(\tilde{\gamma}\tilde{x})\tilde{u}\tilde{w}=0,$$

$$\frac{\tilde{w}_{\tilde{x}\tilde{x}}}{4}-(\alpha-\tilde{\vartheta}^2)\tilde{w}+2\cos(\tilde{\gamma}\tilde{x})\tilde{u}^2=0, \tag{2}$$

where $\tilde{u}$ and $\tilde{w}$ can be taken real since all the involved parameters are real. We focus on bright solitary waves, their range of existence being determined by the conditions $\tilde{\vartheta}<\sqrt{2}$ and $\alpha>\vartheta^2$ for evanescent tails in $x\to\pm\infty$; such conditions define a cutoff $\beta_{cut}=1/2\max\{\vartheta^2,(\gamma/2)^2+\vartheta^2-\Delta_o\}$ in the nonlinear phase shift β, as in the case of a single noncollinear interaction yielding walking solitons [9,27].

When $\tilde{\vartheta}=0$,

$$\frac{\tilde{u}_{\tilde{x}\tilde{x}}}{2}-\tilde{u}+2\cos(\tilde{\gamma}\tilde{x}+\phi)\tilde{u}\tilde{w}=0,$$

$$\frac{\tilde{w}_{\tilde{x}\tilde{x}}}{4}-\alpha\tilde{w}+2\cos(\tilde{\gamma}\tilde{x}+\phi)\tilde{u}^2=0, \tag{3}$$

with $\tilde{\gamma}$ as a scaling factor for the soliton transverse size as compared to the lattice period. System (2) can be cast as Eq. (3) by using the previous transformations [3] and replacing β by $(1-\tilde{\vartheta}^2/2)$; hence, Eq. (3) completely describes quadratic solitons in these DQPM structures. Owing to the connection between quadratic and nonlocal solitons [28], Eq. (3) can also model a class of solitary waves in nonlocal media with periodic boundary conditions. Let us consider the solitary solution centered in $\tilde{x}=0$, accounting for its position relatively to the NL by varying the parameter ϕ in $[0,\pi/2]$. With reference to bright one-hump FF solutions which can exist only for $\alpha>0$, some preliminary considerations can be made in the Kerr limit $\alpha\gg1$. Equation (3) reduces to the periodic nonlinear Schrödinger equation (PNLSE) [3]

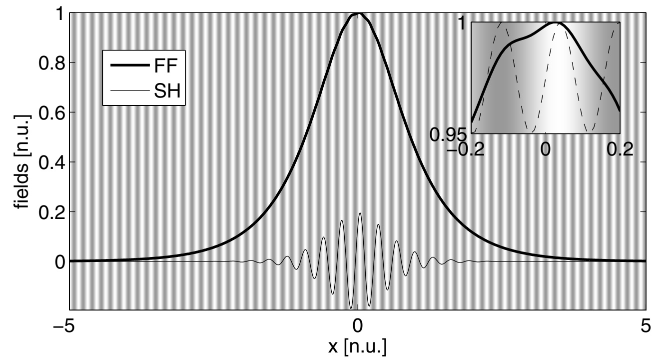

FIG. 2. Asymmetric solitary solution in the Kerr limit and for large periods in normalized units (n.u.). In this simulation $\tilde{\gamma}=20$, $\alpha=10$, and $\phi=\pi/4$. The SH (thin line) follows the grating periodicity, as visible by the gray pattern reproducing the modulation $\cos(\tilde{\gamma}\tilde{x}+\phi)$ of the quadratic nonlinearity. The FF (thick line) has a sech-type profile with a small correction due to periodicity, as emphasized in the inset enlargement with the dashed line graphing the (effective) nonlinear refractive index modulation $\cos(\tilde{\gamma}\tilde{x}+\phi)^2$.

$$\frac{\tilde{u}_{\tilde{x}\tilde{x}}}{2}-\tilde{u}+\frac{4}{\alpha}\cos(\tilde{\gamma}\tilde{x}+\phi)^2\tilde{u}^3=0,$$

$$\tilde{w}=\frac{2}{\alpha}\cos(\tilde{\gamma}\tilde{x}+\phi)\tilde{u}^2, \tag{4}$$

where the nonlinearity is periodic with $\tilde{\gamma}$ and SH changes sign accordingly. The PNLSE [Eq. (4)] has been investigated in contexts much wider than optics [29,30], but some theoretical results can be adapted to our case [31]. When $\tilde{\gamma}$ is small, the soliton is narrower than the period and stable solutions can always be found centered at the maximum of the nonlinear refractive index modulation $\cos(\tilde{\gamma}\tilde{x}+\phi)^2$. Solitary waves centered in the minimum $(\phi=\pi/2)$ can also be retrieved, but these are unstable as they tend to move toward regions with a higher nonlinearity. In the limit $\tilde{\gamma}\gg1$ the solitons are wider than the period, the first of Eqs. (4) can be recast as the homogeneous NLSE for the first-order expansion in $1/\tilde{\gamma}$ of $\tilde{u}$, with the average Kerr coefficient $2/\alpha$ [31]. Stable solitary solutions can be derived from the NLSE at any ϕ using Kerr mean nonlinear properties, the periodicity only intervening as a second-order correction in $1/\tilde{\gamma}$. Consistently with these results, in the limits $\tilde{\gamma}\gg1$ and $\alpha\gg1$ sech-type solutions are found for the FF, with the SH exhibiting the same zeroes of the periodicity. As previously pointed out, even the FF undergoes a small correction due to the grating. Figure 2 shows an example for $\tilde{\gamma}=20$ and $\alpha=10$. It is interesting to notice that the average $\chi^{(2)}$ nonlinearity is zero, nonetheless the FF experiences a Kerr equiva-

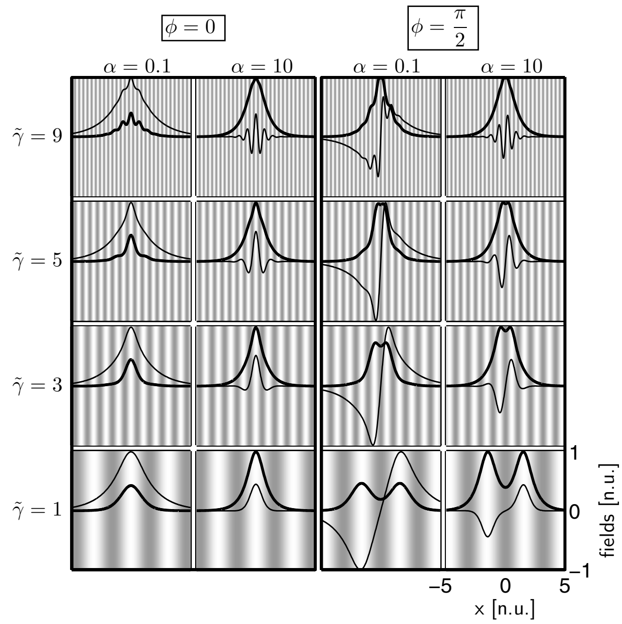

FIG. 3. Even (two leftmost panels) and odd (two rightmost panels) solitary solutions for $\phi=0$ and $\phi=\pi/2$, respectively, for $\tilde{\gamma}=1,3,5,9$, and $\alpha=0.1$ and $\alpha=10$. Thick (thin) lines refer to the FF (SH) wave and the gray pattern indicates the periodicity in quadratic response.

lent response with a non-null mean value because the SH modifies its sign with the nonlinear grating.

Away from large $\tilde{\gamma}$ and large α, solitary waves exist for both $\phi=0$ and $\phi=\pi/2$. In these cases, since the structure has a symmetry with respect to the origin, it is convenient to look for odd and even solutions. Two classes of solutions adiabatically varying with α exist for every $\alpha>0$, with either even or odd grating symmetries, as summarized in Fig. 3. The SH shares its zeroes with the modulation. For $\tilde{\gamma}\to 0$, even and odd classes collapse into the family of one-hump nonwalking solitary waves in quadratic homogeneous media and in the null solution, respectively. For large $\tilde{\gamma}$ solitary solutions exist although the average $\chi^{(2)}$ nonlinearity is zero. When the parameter $\alpha \gg 1$, both classes fall back in the limit of Fig. 2. As α diminishes, the SH energy increases compared to the FF and the number of SH zeroes decreases. For small α the SH profile has one or two-humps for even or odd parities, respectively, while the FF has no zeroes. Some general features of simultons are preserved: SH energy and relative weight increase as the mismatch α reduces [Figs. 4(a) and 4(b)].

Higher energies are required to obtain solitary waves for larger $\tilde{\gamma}$ at given α, as the transverse grating makes the SH diffract more than in homogeneous media.

The stability of these soliton solutions can be addressed by letting a small perturbation evolve in $\tilde{z}$ and solving the resulting linearized eigenvalue problem [26,27,32–35] by adding $i\tilde{u}_{\tilde{z}}$ ($i\tilde{w}_{\tilde{z}}$) to the left hand side of the first (second) equation in system (3). These can be linearized by substituting $\tilde{u}=u_S+\delta u e^{\lambda \tilde{z}}$ and $\tilde{w}=w_S+2\delta w e^{\lambda \tilde{z}}$, with δu and δw small with respect to the solitary solutions u_S and w_S, respectively. Note that the constrain $|\delta w| \ll |w_S|$ can be removed in the nodes of w_S, as the latter function shares its zeroes with the

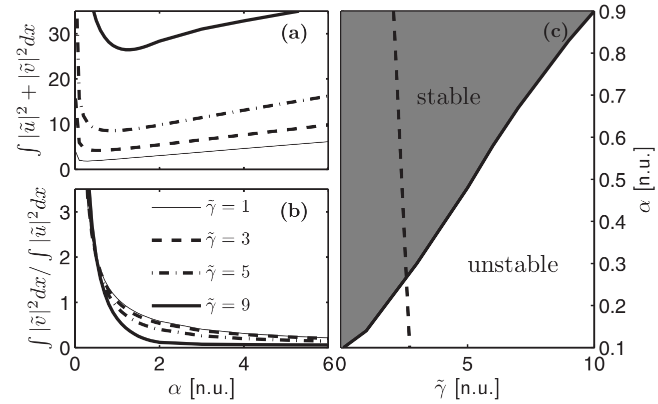

FIG. 4. Simultons for $\phi=0$ [(a) and (b)] total energy and energy ratio between FF and SH, respectively. (c) Stability range of even solitary waves with $\phi=0$. The dashed line indicates the phase-matching condition $\Delta_o=0$.

nonlinear coefficient; hence δw is a general perturbation not necessarily zero with w_S [32]. Separating real and imaginary parts, we obtain the eigenvalue system for $\mathbf{v}_R=(\delta u_R,\delta w_R)$ and $\mathbf{v}_I=(\delta u_I,\delta w_I)$:

$$\begin{matrix} \mathcal{L}_+\mathbf{v}_R=\lambda\mathcal{S}\mathbf{v}_I \\ \mathcal{L}_-\mathbf{v}_I=-\lambda\mathcal{S}\mathbf{v}_R \end{matrix}, \quad \mathcal{S}=\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \tag{5}$$

$$\mathcal{L}_\pm=\begin{pmatrix} \dfrac{\partial_{\tilde{x}\tilde{x}}}{2}-1\pm 2w_S\cos(\tilde{\gamma}\tilde{x}+\phi) & 4u_S\cos(\tilde{\gamma}\tilde{x}+\phi) \\ 4u_S\cos(\tilde{\gamma}\tilde{x}+\phi) & \dfrac{\partial_{\tilde{x}\tilde{x}}}{2}-2\alpha \end{pmatrix}, \tag{6}$$

with null boundary conditions at $x\to\pm\infty$ for the functions and their derivatives, being $\mathcal{L}_\pm$ self-adjoint operators.

The kernel of the problem consists of independent solutions equal in number to the symmetries of the original nonlinear problem (3); it possesses the only solution $\mathbf{v}_R^K=0$ and $\mathbf{v}_I^K=(u_S,w_S)$, the latter being the kernel of the operator $\mathcal{L}_-$. The operator $\mathcal{L}_+$ has a void kernel as confirmed by numerical analysis.

In relevant cases $\mathcal{L}_-$ is defined positive in the space orthogonal to its kernel. The numerical analysis shows that this happens for the class of even solutions previously illustrated for $\phi=0$: when both u_S and w_S have no zeroes (small α), this occurrence is stated by the oscillation theorem [26]. When $\mathcal{L}_-$ has this property, the quantity λ^2 is always real as it is the eigenvalue of the generalized self-adjoined problem $\mathcal{L}_+\mathbf{v}_R=-\lambda^2\mathcal{M}\mathbf{v}_R$, with $\mathcal{M}=\mathcal{S}\mathcal{L}_-^{-1}\mathcal{S}$ defined positive. As a consequence, system (5) has either purely imaginary or purely real eigenvalues λ, the latter governing unstable solutions. The stability threshold is defined by $\lambda=0$ for eigenvector $\mathbf{v}_R=0$ and $\mathbf{v}_I=(u_S,w_S)$ at some critical values $\alpha=\alpha_c$ and $\tilde{\gamma}=\tilde{\gamma}_c$. The solvability condition for a slightly unstable solution (i.e., with small λ) provides a threshold condition involving the conserved quantities. Following Pelinovsky and co-workers [33] we expand the eigenvector in λ, its zeroth order being $\mathbf{v}_R^{(0)}=0$ and $\mathbf{v}_I^{(0)}=(u_S,w_S)$. For the first-order correction of $\mathbf{v}_R^{(1)}$ we find

$$u_R^{(1)} = u_S + (2-\alpha)\partial_\alpha u_S + \frac{x}{2}\partial_x u_S - \tilde{\gamma}\partial_{\tilde{\gamma}} u_o,$$

$$w_R^{(1)} = \frac{1}{2}\left(w_S + (2-\alpha)\partial_\alpha w_S + \frac{x}{2}\partial_x w_S - \tilde{\gamma}\partial_{\tilde{\gamma}} w_S\right). \tag{7}$$

After substituting in the solvability condition

$$\int \{u_S u_R^{(1)} + 2 w_S w_R^{(1)}\} dx = 0,$$

we get

$$\frac{3+2(2-\alpha)\partial_\alpha - \tilde{\gamma}\partial_{\tilde{\gamma}}}{4}\int \{u_S^2 + w_S^2\} dx = \frac{\partial_\beta Q}{2\sqrt{\beta}} = 0. \tag{8}$$

The latter considers the energy Q of Eq. (1) and can be easily checked by direct expansion. Noticeably, relation (8) requires $\partial_\beta Q=0$, i.e., the Vakhitov-Kolokolov (VK) criterion extensively applied to various classes of solitary waves [26,32]. The stability domain obtained from Eq. (8) and numerically verified by direct integration of Eq. (5) is shown in Fig. 4(c). Although shrinking with $\tilde{\gamma}$, the stability region remains remarkably large for any $\tilde{\gamma}$. For odd solutions and $\phi=\pi/2$, the operator $\mathcal{L}_-$ is no longer defined positive and only direct numerical integration of system (5) is available. A positive eigenvalue λ exists for every $\tilde{\gamma}$ in a vast range of α and the solutions are mostly unstable. The rate of instability λ decreases asymptotically, converging to a stable condition for growing $\tilde{\gamma}$ and α. In conclusion, we presented a general model and theoretical results on quadratic spatial solitons in a degenerate phase-matched geometry, governed by two reciprocal vectors $G_\pm$ of equal modulus. By simple considerations based on Lie symmetries we conclude that two-color (FF+SH) solitary waves can rigorously occur only with propagation along the bisector between G_+ and G_-. The numerical study of solitary solutions in these quadratic lattices indicates the existence of simultons with a large stability domain as analytically defined by the VK criterion.

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