Generalized Jaynes-Cummings model as a quantum search algorithm

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We propose a continuous time quantum search algorithm using a generalization of the Jaynes-Cummings model. In this model the states of the atom are the elements among which the algorithm realizes the search, exciting resonances between the initial and the searched states. This algorithm behaves like Grover's algorithm; the optimal search time is proportional to the square root of the size of the search set and the probability to find the searched state oscillates periodically in time. In this frame, it is possible to reinterpret the usual Jaynes-Cummings model as a trivial case of the quantum search algorithm.

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I. INTRODUCTION

One of the most simple and interesting quantum models that studies the interaction between radiation and matter is the Jaynes-Cummings model (JCM) [1]. The model considers the interaction between a single two-level atom with a single mode of the electromagnetic field. The coupling between the atom and the field is characterized by a Rabi frequency, and a loss of excitation in the atom appears as a gain in excitation of the field oscillator. The collapse and the eventual revival of the Rabi oscillation, described by the analytical solution of the JCM, shows a direct evidence of the quantum nature of radiation. The use of JCM has permitted to elucidate basic properties of quantum entanglement as well as some aspects of the relationship between classical and quantum physics. Since it was proposed, the pattern has been of permanent interest in the quantum theory of interactions. In the decade of the eighties it was discovered that the model exhibited highly nonclassical behavior, and the possibility of experimental realization appeared. The relative simplicity of the JCM and its extensions has drawn much attention in the physics community and recently in the field of the quantum computing [2,3]. In this work we use a generalization of the JCM to an N state atom interacting with a single field mode [4].

In 1994, Shor [5] described a quantum algorithm to decompose a number in its prime factors more efficiently than any classical algorithm. It was exponentially faster than the best known classical counterpart. In 2001 the experimental development of this algorithm has had a very interesting advance: Vandersypen *et al.* [6] using a seven-qubit molecule manipulated with nuclear magnetic resonance techniques has reported the factorization of the number 15 into its prime factors 3 and 5. This algorithm illustrates a part of the theoretical challenge of quantum computation, i.e., to learn how to work with quantum properties to obtain more efficient algorithms. Tools such as quantum parallelism, unitary transformations, amplification techniques, interference phenomena, quantum measurements, resonances, etc., must be used by the new computation science. Grover, in 1997, devised an algorithm [7] which can locate a marked item from an unsorted list of N items, in a number of steps proportional to $\sqrt{N}$, that is quadratically faster than any classical algorithm [8]. Continuous time search algorithms have been investigated by a number of researchers [9–11]. The essential content of these proposals is to built a Hamiltonian (or alternatively a unitary operator) with the aim to change the initial average state $|\phi\rangle=\Sigma_m|\varphi_m\rangle/\sqrt{N}$ into another state $|\varphi_s\rangle$ that also belongs to the same set $\{|\varphi_m\rangle\}$ of N vector in the base of the Hilbert space [2]. This last state is recognized by an unitary operator called oracle that is part of the global unitary evolution $\exp(-iHt)$, where the Hamiltonian is expressed as

$$H=|\phi\rangle\langle\phi|+|\varphi_s\rangle\langle\varphi_s|. \tag{1}$$

The probability to obtain the searched state is $|\langle\varphi_s|\exp(-iHt)|\phi\rangle|^2$, and it equals 1 after a time $\pi\sqrt{N}/2$. In this frame, the search algorithm is seen as a rotation in the Bloch sphere from the initial average state to the searched state. Recently an alternative search algorithm was developed [12–15] that uses a Hamiltonian to produce a resonance between the initial and the searched states, having the same efficiency than the Grover algorithm. It can be implemented by using any Hamiltonian with a discrete energy spectrum, and it was shown to be robust [12] when the energy of the searched state has some uncertainty. The responses of this algorithm to an external monochromatic field and to the decoherences introduced through measurement processes was also analyzed in [13]. However we do not know of any experimental implementation, not even for a small search set. In this paper we present a resonant quantum search algorithm implemented with a generalization of the two-level JCM.

The paper is organized as follows. In Sec. II we consider a generalization of the JCM, in Sec. III we develop the search model. In Sec. IV we present numerical results for our model. Finally in Sec. V we draw some conclusions.

II. GENERALIZED JAYNES-CUMMINGS MODEL

We shall consider the generalization of the JCM to an N state atom interacting with a single field mode with frequency ω synthesized by the following Hamiltonian [4]:

$$H=\hbar\omega a^{\dagger}a+\sum_{k=1}^{N}\varepsilon_k S_{kk}+\frac{\hbar}{2}\Omega_0\sum_{k=1}^{N}(a^{\dagger}S_{jk}+S_{kj}a). \tag{2}$$

The photon creation and annihilation operators $a^{\dagger}$ and a act on the photon number state $|n\rangle$ verifying

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$$[a,a^\dagger]=1, \tag{3}$$

$$a^\dagger a|n\rangle = n|n\rangle, \tag{4}$$

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \tag{5}$$

$$a|n\rangle = \sqrt{n}|n-1\rangle. \tag{6}$$

ε_k is the energy of atomic state $|\varphi_k\rangle$, Ω_0 is the atom-field coupling constant and it is fixed by physical considerations such as the cavity volume and the atomic dipole moment, S_{jk} is a transition operator acting on atomic states defined by

$$S_{lk}|\varphi_m\rangle = \delta_{km}|\varphi_l\rangle, \tag{7}$$

where δ_{km} is the Kronecker delta. In what follows the subindex j shall indicate the initial state of the atom $|\varphi_j\rangle$. This state will be the starting state for the search algorithm and it can be chosen as the ground state for experimental purposes.

Let us call $|\varphi_s\rangle$ the unknown searched state whose energy ε_s is known. This knowledge is equivalent to "mark" the searched state in the Grover algorithm [9–11]. Our task is to find the eigenvector $|\varphi_s\rangle$ with transition energy $\omega_{sj}=(\varepsilon_j-\varepsilon_s)/\hbar$ from the given initial state $|\varphi_j\rangle$. Then it is necessary to tune the frequency of the photon field with the frequency of the transition $|\varphi_j\rangle \to |\varphi_s\rangle$. This means that the frequency of the cavity mode is selected as $\omega \equiv \omega_{sj}$. The transition between the atomic states is governed by the interaction term of the Hamiltonian (2),

$$W = \frac{\Omega_0}{2}\sum_{k=1}^{N}(a^\dagger S_{jk} + S_{kj}a). \tag{8}$$

The transition probability to pass from the initial atomic state $|\varphi_j\rangle$ with m photons to any final state $|\varphi_i\rangle$ with n photons is proportional to $|\langle n\varphi_i|W|m\varphi_j\rangle|^2$. After some steps we get

$$|\langle n\varphi_i|W|m\varphi_j\rangle|^2 = (\Omega_0^2/4)\{(m+1)\delta_{nm+1} + m\delta_{nm-1}\delta_{ij}\}. \tag{9}$$

To calculate this transition probability independently of the initial and final numbers of photons the statistical weight of the photons must be incorporated,

$$P_{ji} = \frac{1}{\lambda^2}\sum_n\sum_m p(n)p(m)|\langle n\varphi_i|W|m\varphi_j\rangle|^2, \tag{10}$$

where $p(n)$ is the normalized photon number distribution and λ is an unknown constant. Using Eq. (9) in Eq. (10) we obtain the dependence of P_{ji} with the average number of photons and the parameter Ω_0,

$$P_{ji} = (1/\lambda^2)(\Omega_0^2/4)\{\langle n\rangle + 1 + \delta_{ij}\langle n\rangle\}, \tag{11}$$

where $\langle n\rangle = \Sigma_n np(n)$. Taking into account the normalization condition $\Sigma_{i=1}^N P_{ji}=1$, the dependence Ω_0 with the number of atom levels is obtained,

$$\Omega_0 = 2\lambda/\sqrt{\langle n\rangle(N+1)+N} \sim 2\lambda/\sqrt{\langle n\rangle N}, \tag{12}$$

The last step is valid for large N and $\langle n\rangle$. Note that Ω_0 depends on the photon distribution function only through the mean value of n.

III. RESONANCES

In the previous section we have determined the dependence of the atom-field coupling constant Ω_0 with the number of atomic states N and the mean number of photons $\langle n\rangle$. Now we want to study how this coupling constant determines the characteristic period of the dynamics and subsequently the waiting time for the search algorithm.

The dynamics of the system is given by the Schrödinger equation for the wave function $|\Psi(t)\rangle$,

$$i\hbar\partial|\Psi(t)\rangle/\partial t = H|\Psi(t)\rangle, \tag{13}$$

where H is given by Eq. (2). The global Hilbert space of the system is built as the tensor product of the spaces for the photons and the atom. Therefore atom-field wave function $|\Psi(t)\rangle$ is expressed as a linear combination of the basis $\{|\varphi_m\rangle|n\rangle\}$,

$$|\Psi(t)\rangle = \sum_m\sum_n b_{mn}\exp[-i(\varepsilon_m+\omega n)t/\hbar]|\varphi_m\rangle|n\rangle, \tag{14}$$

where $\{|n\rangle\}$ is the basis for the photons and $\{|\varphi_m\rangle\}$ is the eigenvector basis for the atomic Hamiltonian without electromagnetic field. The phase factor is introduced to simplify the final differential equations. Substituting Eq. (14) into Eq. (13) and projecting on the state $\langle\varphi_l|\langle k|$ the following set of differential equations for the time depended amplitudes $b_{lk}(t)$ are obtained:

$$\frac{2i}{\Omega_0}\frac{db_{lk}}{dt} = \sqrt{k}\exp[-i(\omega_{lj}-\omega_{sj})t/\hbar]b_{jk-1} + \delta_{lj}\sqrt{k+1}\sum_{m=1}^{N} b_{mk+1}\exp[-i(\omega_{lm}+\omega_{sj})t/\hbar]. \tag{15}$$

This set of equations shall be solved numerically in the next section. Here we proceed to study their qualitative behavior. These equations have two time scales involved, a fast scale associated to the Bohr frequencies ω_{lk}, and a slow scale associated to the amplitudes $b_{lk}(t)$. If we are interested in the slow scale all terms that have fast phase in the previous equations can be ignored; the most important terms are the ones with zero phase. In this approximation the previous set of equations Eq. (15) becomes

$$(2i/\Omega_0)(db_{sk}/dt) = \sqrt{k}b_{jk-1}, \tag{16}$$

$$(2i/\Omega_0)(db_{jk}/dt) \simeq \sqrt{k+1}b_{sk+1}. \tag{17}$$

These equations represent two oscillators that are coupled so that their population probabilities alternate in time. As we notice the coupling is established between the initial and the searched for states. To uncouple the previous equations we combine them to obtain

$$d^2b_{jk}/dt^2 \simeq -(k+1)(\Omega_0^2/4)b_{jk}, \tag{18}$$

$$d^2b_{sk}/dt^2 \simeq -k(\Omega_0^2/4)b_{sk}. \tag{19}$$

Solving these equations, for a given number k of photons and with the initial conditions $b_{jk}(0)=1$ and $b_{sk}(0)=0$, the following results for the amplitudes are obtained:

$$b_{jk}(t) \simeq \cos[(\Omega_0/2)\sqrt{k+1}t], \tag{20}$$

$$b_{sk+1}(t) \simeq -i \sin[(\Omega_0/2)\sqrt{k+1}t]. \quad (21)$$

Therefore the probabilities to obtain the initial and the searched states, independently of the initial number of photons, should be calculated as $P_j(t)=\Sigma_k p(k)|b_{jk}(t)|^2$, $P_s(t)=\Sigma_k p(k)|b_{sk}(t)|^2$, which satisfy the conditions $P_j(0)=1$, $P_s(0)=0$. Averaging over the number of photons and using Eq. (12) these probabilities are

$$P_j(t) \simeq \cos^2(\Omega t), \quad (22)$$

$$P_s(t) \simeq \sin^2(\Omega t), \quad (23)$$

where the new angular frequency is

$$\Omega = \lambda/\sqrt{N}. \quad (24)$$

Then we see that the probabilities of the initial state $P_j(t)$ and the searched state $P_s(t)$ oscillate harmonically with a frequency Ω and period $T=\frac{2\pi}{\lambda}\sqrt{N}$, while the probability of the other elements of the search set are neglected. If we let the system evolve during a time

$$\tau \equiv T/4 = (\pi/2\lambda)\sqrt{N}, \quad (25)$$

and at this precise moment we make a measurement, we have probability 1 to obtain the searched state. It is important to indicate that this approach is valid only in the adiabatic approximation [16], this means that all the frequencies ω_{nm} are much larger than Ω_0. Therefore the efficiency of our search algorithm is the same as that of the Grover algorithm and additionally it is independent of the number of photons.

In the next section, we implement numerically Eq. (15) and we show that this agrees with the above theoretical developments of the system in resonance.

IV. NUMERICAL RESULTS

The JCM has been used to understand the behavior of circular Rydberg atoms, where the valence electron is confined near the classical Bohr orbit [17]. This suggests to choose for our purpose an atomic model with an attractive potential whose quantum energy eigenvalues are $\varepsilon_n=-\varepsilon_0/n^2$, where n is the principal quantum number and ε_0 is a parameter. In this frame the Bohr transition frequency can be expressed as function of the parameter ε_0 as $\omega_{nm}=\varepsilon_0(1/n^2-1/m^2)$.

We shall choose the numerical values of the parameters taking into account some previous experimental data. In Refs. [17,18], the Rabi oscillation of circular Rydberg atoms was observed. The frequency of the single electromagnetic mode was tuned with the transition between adjacent circular Rydberg states with principal quantum numbers 51 and 50, where the fast scale associated to this Bohr frequency was $\omega_{50,51}\sim 100\pi$ GHz, that is very large compared with the fundamental Rabi frequency $\Omega_0\sim 50\pi$ kHz. In the micromaser configuration of Ref. [19] the field frequency of 21 GHz produced the Rydberg transitions used in this experiment, where the principal quantum number was about 63 and the Rabi frequency 10 kHz. From the above we conclude that the ratio between the Bohr transition energy and the vacuum Rabi frequency $\delta \equiv \varepsilon_0/\Omega_0$ to be used in our search algorithm should be taken as $\sim 10^6$.

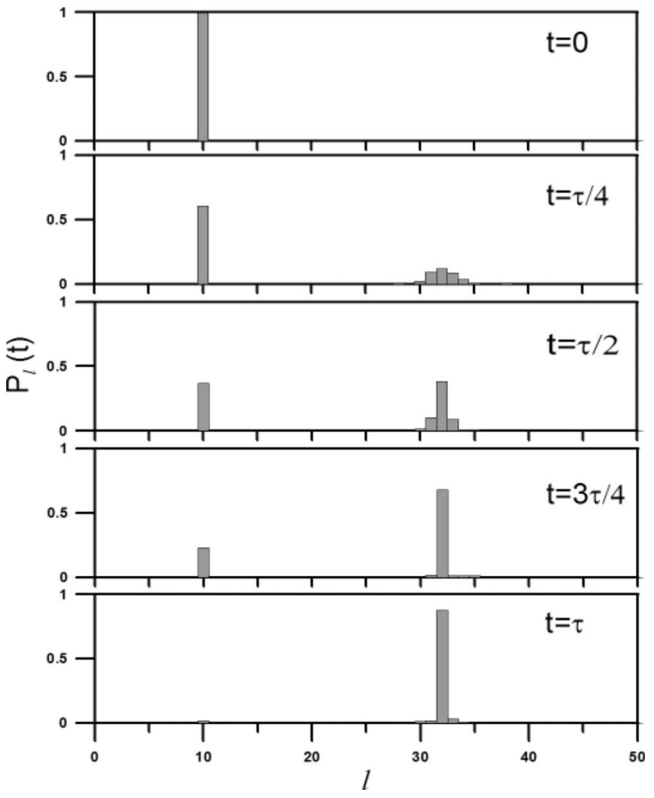

FIG. 1. Probability distribution for all atomic levels at five different times. The search set $\{l\}$, with $l=1,2,\ldots,N$, has $N=50$ elements (levels), $\delta=40000$ and τ is the optimal search time proportional to $\sqrt{N}$. The initial state was taken to be $j=10$ and the searched state $s=32$. Note that the distribution probability is essentially shared between the initial and the searched states.

We have integrated numerically Eq. (15) varying the parameter δ in a range between 5×10^3 and 10^6. The initial conditions are (i) a uniform distribution for the photons and (ii) $b_{jk}(0)$ is chosen in such a way that $P_j(0)=1$. The calculations were performed using a standard fourth order Runge-Kutta algorithm. The procedure consisted in choosing at random the energy of the searched state and then to follow the dynamics of the set and of the initial state. We verified for several values of δ that the most important coupling is between the initial and the searched states; the other couplings may be totally neglected.

In Fig. 1 we show the probabilities for all levels at five different fractions of the time τ. We see that the dispersion among the states neighboring the searched state is relatively small for $\delta=40\,000$. For higher values of δ the dispersion is even smaller, this confirms that our theoretical approximation of two coupled modes is correct. Furthermore we conclude that the flux transfer process is essentially an interchange between the initial and the searched states and that the optimal time to measure the searched state is τ. At other times we have less chance of perform a measurement of the searched state.

Figure 2 shows the oscillation of the probability flux between the initial and the searched states as a function of time for three values of δ. The time is normalized for the theoretical characteristic time τ. The evolution shows for the lower

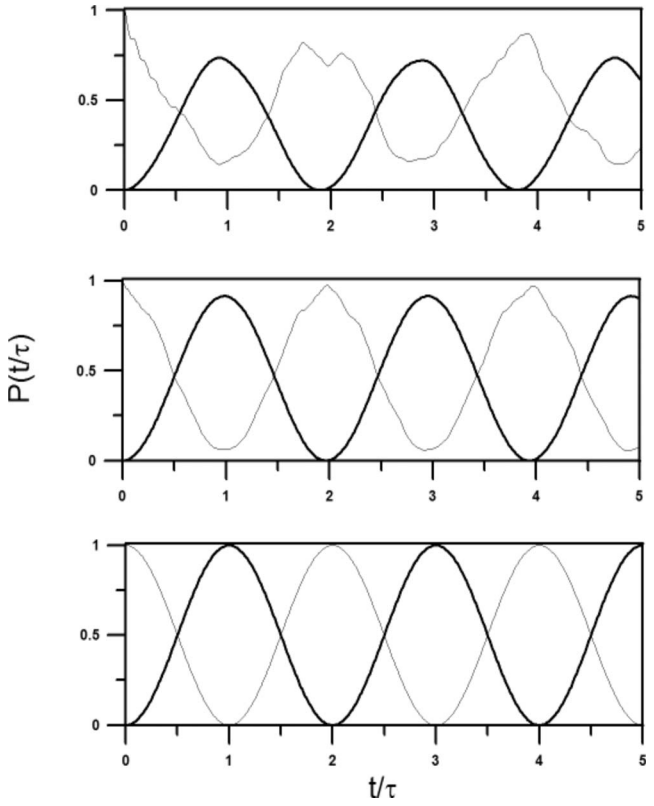

FIG. 2. Probability distributions as functions of time for the initial (thin line) and the searched (thick line) states. The parameter δ takes the values, from top to bottom, 5×10^3, 10^4 and 10^6. The size of the search set is $N=50$.

values of δ an almost periodic behavior, however for the highest value $\delta=10^6$, the behavior is completely harmonic. In this last case there is clearly a characteristic time when the probability of the searched state is maximum and very near 1 and the initial state probability is near 0. The optimal time agrees with our theoretical prediction τ. This periodic behavior and the proportionality between τ and $\sqrt{N}$ are also found in the Grover algorithm [2].

V. CONCLUSIONS

In this work we show how a generalized JCM to an N state atom interacting with a single field mode can be thought of as a quantum search algorithm that behaves like the Grover algorithm; in particular the optimal search time is proportional to the square root of the size of the search set and the probability to find the searched state oscillates periodically in time.

In the past, the biggest difficulty to build a JCM has been to obtain a single electromagnetic mode that interacts with the atomic transition. This difficulty has been overcome in the last decades thanks to the experimental advances in the handling of Rydberg atoms and to the building of microcavity for microwaves [17–22]. In the frame of this work we can interpret these devices as experimental realizations of the "analog" [9] Grover algorithm in the trivial case of the search of a marked item in an unsorted list of 2 elements. However this new way of looking at the problem is different from the usual point of view and opens new possibilities for the JCM. In summary, in this paper we reinterpret the JCM as the first step to build a more generic search algorithm with a large number of elements

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