Conditions on systems of interacting qubits for classical behavior of their macroscopic variables

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(Received 2 May 2009; published 29 July 2009)

It is shown that the conditions for classical behavior, in the sense of zero dispersion of all macroscopic variables, for systems of qubits with different dynamics, depend on whether the interaction is local or global. In the case of local interaction, macroscopic coarse-graining is enough. On the other hand, coarse-grained macroscopic variables of globally interacting large systems of qubits have negligible dispersion only if the system is exposed to some form of the environmental decoherence.

DOI: 10.1103/PhysRevA.80.014102 PACS number(s): 03.65.Yz, 05.45.Mt

The relation between classical behavior of macroscopic systems and the quantum nature of their microscopic constituents is one of the most important questions of our scientific world view. The problem is as old as the quantum mechanics itself and has been studied with different tools and aims (see for example the reviews in [1,2]). Nevertheless the problem is far from solved. In fact, it is particularly alive today [3] because the modern technology enables its experimental study and due to its relevance in realistic quantum information processing.

Coarse graining of the state space [1,4] of large systems and the environmental decoherence [5], including continuous observations [6], are the main mechanisms that induce classical behavior. The effects of these two mechanisms on the quantum to classical transition have been studied using for example phase space densities [7] and methods of the theory of open quantum systems, such as quantum trajectories [6,8,9], or stochastic master equations [10–12]. However, different systems display different aspects of the classical behavior under different, system dependent, conditions.

In this Brief Report we shall report the results of numerical investigations of the conditions that are sufficient for dispersionless evolution of all macroscopic variables using examples of systems of interacting qubits. This is enough to guarantee macrorealism in the sense of [13]. Our main result is that for systems with qualitatively different but local interaction the macroscopic coarse graining is enough to induce the dispersionless evolution of the macroscopic variables, while for systems with long-range global interaction some form of environmental decoherence is also required.

Consider the system of N qubits with the Hamiltonian of the following form:

$$H(\omega_z,\omega_x,J) = \sum_i^N \omega_z\sigma_z^i + \sum_i^N \omega_x\sigma_x^i + J\sum_{i\neq j}\gamma_{i,j}\sigma_x^i\sigma_x^j. \quad (1)$$

In Eq. (1) $\sigma_{x,y,z}^i$ are Pauli sigma matrices of the ith qubit and $\omega_z,\omega_x,\gamma_{i,j},J$ are parameters. We shall see that as far as the classical behavior of macro variables is considered the main difference is between the following two cases: the case of local nearest-neighbor interaction $\gamma_{i,j}=\delta_{i,i+1}$ and the case of global all to all interaction $\gamma_{i,j}=1$. On the other hand, the qualitative difference between the local symmetric and integrable $H(\omega_z\neq 0,\omega_x=0,J\neq 0)$ vs local nonsymmetric and nonintegrable $H(\omega_z\neq 0,\omega_x\neq 0,J\neq 0)$ [14], which is clearly manifested in qualitatively different dynamics of entanglement [14–17], is not relevant for the aspect of macroscopic behavior treated in this Brief Report.

Macroscopic variables for the system (1) are defined as

$$m_{x,y,z} = \frac{1}{N}\sum_{i=1}^N \sigma_{x,y,z}^i. \quad (2)$$

Classical macroscopic systems are always in such a state that all macroscopic variables have well defined and sharp values. In other words, the dispersions $\Delta m_{x,y,z}=\sqrt{\mathrm{Var}\, m_{x,y,z}} =[\langle m_{x,y,z}^2\rangle-\langle m_{x,y,z}\rangle^2]^{1/2}$ should be negligible in all pure states during the evolution for the system (1) to appear as classical.

With the standard definition of the time average

$$\overline{\Delta m_{x,y,z}} = \lim_{T\to\infty}\frac{1}{T}\int_0^T \Delta m_{x,y,z}dt \quad (3)$$

we can write the following necessary condition that is expected of the classical macro system of the form (1):

$$\lim_{N\to\infty}\overline{\Delta m_{x,y,z}} = 0. \quad (4)$$

The macroscopic variables satisfy relations $\lim_{N\to\infty}[m_k,m_l]=0$, $k,l=x,y,z$, so that it is expected that the dispersions $\Delta m_{x,y,z}$ are simultaneously arbitrarily small in the large-N limit. However, the behavior of $\Delta m_{x,y,z}$ during the evolution generated by different Hamiltonians can be radically different with respect to the large-N limit. Two extreme situations have been discussed recently in [18], and are represented by the following Hamiltonians:

$$H_c = \frac{\omega}{2}\sum_i \sigma_x^i \quad \text{class } c \quad (5)$$

and

$$H_q = \frac{i}{2}(\sigma_-^{\otimes N} - \sigma_+^{\otimes N}) \quad \text{class } q. \quad (6)$$

The system of qubits with the Hamiltonian (6) is always in a superposition of macroscopically distinct states $|\uparrow,\uparrow,\ldots,\uparrow\rangle$ and $|\downarrow,\downarrow,\ldots,\downarrow\rangle$. In the class c the variances of all three

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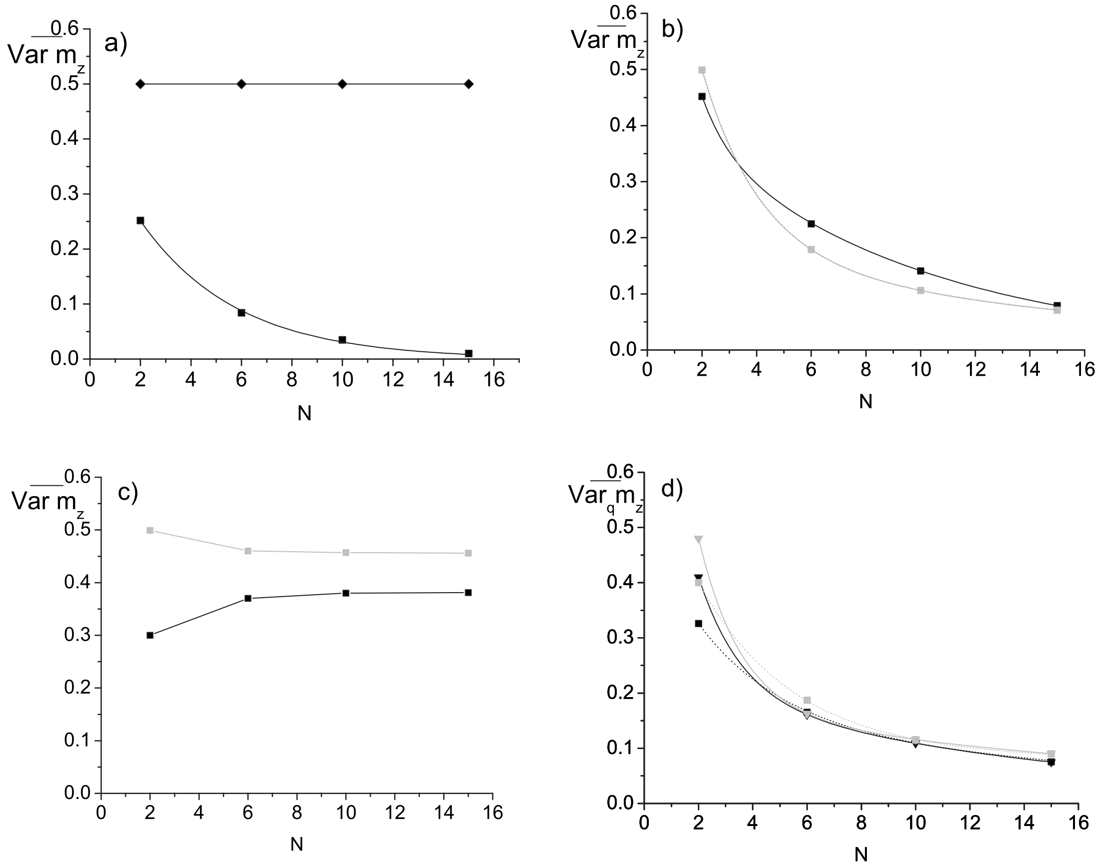

FIG. 1. Figures illustrate dependence of the time averaged variance $\overline{\mathrm{Var}\, m_z}$ on the number of qubits N. Lines in 1b and 1d represent double exponential functions fitting the data calculated at $N=2,6,10,15$. In 1a the data are fitted by an exponential (case c, boxes) and a constant (case q, diamonds). Lines in 1c only connect the data of the same type. In 1b, 1c, and 1d black boxes and black line are for the quantum nonintegrable and gray boxes and gray line for the integrable local [Fig. 1(b)] and global [Fig. 1(c)] Hamiltonians. Figure 1(d) is for the open nonsymmetric (black) and symmetric (gray) globally interacting systems with thermal (diamonds with full line) and dephasing (boxes with dotted line).

variables $\Delta^2 m_{x,y,z}$ decrease exponentially with N, and in the class q the amplitude and the time average of $\Delta^2 m_{x,y,z}$ remain constant as N is increased, as is illustrated in Fig. 1(a). The initial state in Fig. 1(a) and in all other figures is $|\uparrow\rangle^{\otimes N}$. The two examples [Eqs. (5) and (6)] represent two different typical situations and we would like to see which of the two types of behavior appears in the systems with local or global interactions [Eq. (1)].

Numerical calculations clearly show that the systems with local or global interactions display quite different dependence on N of the amplitudes and the time averages of the variances $\overline{\Delta^2 m_{x,y,z}}$. Figure 1(b) illustrates the time averaged variances $\overline{\Delta^2 m_z}$ as functions of N for the integrable and nonintegrable local interaction. Figure 1(c) illustrates the results for the Hamiltonians with global interaction. Similar type of dependence on N is observed for the other two dispersions. As is clearly seen, the variances in the case of the Hamiltonians with local interaction approach zero as N is increasing. The dependence is well fitted with the double exponential: $\overline{\Delta^2 m_z}(N)=A_1 \exp(-B_1 N)+A_2 \exp(-B_2 N)$. On the other hand, the evolution with the global interaction implies that the variances $\Delta^2 m_{x,y,z}$ remain large independently of N [Fig. 1(c)].

We can conclude that macroscopic coarse graining is enough to obtain classical behavior of dispersions of the macroscopic variables for systems of qubits with, integrable or not, but local interaction. Of course, sufficiently precise resolution would show perfectly quantum evolution of the system's microstate or its microvariables. It is important to notice that the qualitative difference of the evolution, although displayed in the properties of the microstate entanglement, has no relevance for the classical properties of the dispersions of the macrovariables. Thus, as far as the behavior of dispersions is considered, the systems of qubits with local interaction belong to the class c and the systems with global interaction to the class q.

In the sequel we shall consider open system with global interaction and report results that show how the environmental decoherence produces the classical behavior of the dispersions of macrovariables. The most general evolution equation of an open quantum system that satisfies the Markov property is of the Linblad form [19],

$$\frac{d\rho(t)}{dt} = -i[H,\rho] - \frac{1}{2}\sum_k [L_k\rho, L_k^\dagger] + [L_k, \rho L_k^\dagger], \qquad (7)$$

where $\rho(t)$ is the mixed state of the open system, H is the Hamiltonian and L_k are Linblad operators that describe different possible influences of the environment. Typical examples, that we shall consider, are the influence of the local dephasing or thermal environments represented by [19]

$$L_{dp} = \sum_{i=1}^{N} \gamma_{dp}\sigma_i^+\sigma_i^- \quad L_{th} = \sum_{i=1}^{N} \frac{\gamma_{th}}{2}[(\bar{n}+1)\sigma_i^- - \bar{n}\sigma_i^+], \quad (8)$$

where γ_{dp}, γ_{th}, and $\bar{n}$ are treated here as phenomenological parameters.

Dispersion of the macroscopic variable in the mixed state $\rho(t)$ is given by

$$\Delta^2 m_{x,y,z} = \mathrm{Tr}[\rho(t)m_{x,y,z}^2] - (\mathrm{Tr}[\rho(t)m_{x,y,z}])^2, \quad (9)$$

and can be represented as a sum of quantum and classical parts [19], $\Delta^2 m_{x,y,z} = \Delta_q^2 m_{x,y,z} + \Delta_c^2 m_{x,y,z}$. Namely, for any $t = t_0$ the state $\rho(t_0)$ can be represented, in an infinite number of ways, as convex mixture of pure states $\rho(t_0) = \Sigma_i p_i |i\rangle\langle i|$. This suggests the definitions $\Delta_q^2 A = \Sigma_i p_i(\langle i|A^2|i\rangle - \langle i|A|i\rangle^2)$ and $\Delta_c^2 A = \Sigma_i p_i \langle i|A|i\rangle^2 - (\Sigma_i p_i \langle i|A|i\rangle)^2$. $\Delta_q^2 A$ represent the average variance in pure states that appear in the resolution of ρ. Thus, it is a measure of average intrinsic quantum variance. On the other hand, $\Delta_c^2 A$ is the variance of the c-number $\langle i|A|i\rangle$ and represent statistical fluctuation of this classical quantity. In order to illustrate the influence of decoherence we shall use the quantum dispersion $\Delta_q m_{x,y,z}$ as well as the total dispersion $\Delta m_{x,y,z}$.

In order to calculate $\Delta_q m_{x,y,z}(t)$ at different times we need an evolution equation for the open system in terms of pure states that is equivalent to the Limblad master equation (7). There are many such pure state evolution equations that appear in different contexts [12,19–21]. We shall use the stochastic Schrödinger equation equivalent to Eq. (7) that appears in the theory of quantum state diffusion (QSD) [22]. This equation is singled out among other types of pure state unravelings of the density evolution by its unique relation to the Limblad equation (7). The QSD stochastic Schrödinger equation for the Limblad equation with Limblad operators L_k is unique and reads

$$|d\psi\rangle = -i\hat{H}|\psi\rangle dt + \left[\sum_k 2\langle\hat{L}_k^\dagger\rangle\hat{L}_k - \hat{L}_k^\dagger\hat{L}_k - \langle\hat{L}_k^\dagger\rangle\langle\hat{L}_k\rangle\right]|\psi(t)\rangle dt + \sum_k (\hat{L}_k - \langle L_k\rangle)|\psi(t)\rangle dW_k \quad (10)$$

where $\langle\ \rangle$ denotes the quantum expectation in the state $|\psi(t)\rangle$ and dW_k are independent increments of complex Wiener c-number processes $W_k(t)$ satisfying

$$E[dW_k] = E[dW_k dW_{k'}] = 0, \quad E[dW_k d\bar{W}_{k'}] = \delta_{k,k'}dt. \quad (11)$$

Here $E[\cdot]$ denotes the expectation with respect to the probability distribution given by the multidimensional process W, and $\bar{W}_k$ is the complex conjugate of W_k. By definition of an unraveling the stochastic state $\psi(t)$ satisfies $\mathrm{Tr}[\rho(t)A] = E[\langle\psi(t)|A|\psi(t)\rangle]$ for arbitrary observable A. The quantum and the classical dispersions of A are given in terms of the stochastic state $\psi(t)$ as

$$\Delta_q^2 A = E[\langle\psi(t)|A^2|\psi(t)\rangle - \langle\psi(t)|A|\psi(t)\rangle^2] \quad (12)$$

and

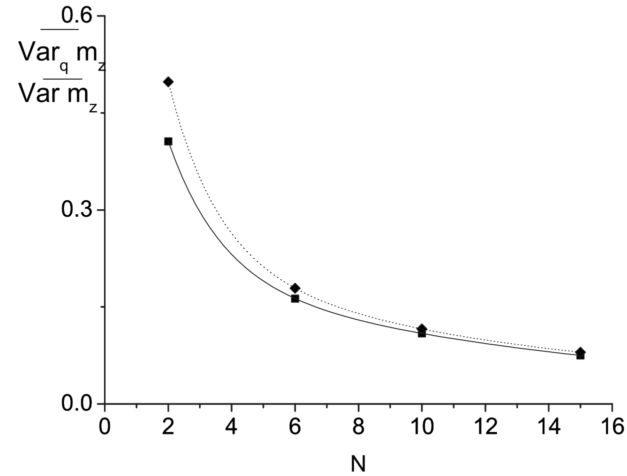

FIG. 2. Comparison of the time averaged quantum $\overline{\mathrm{Var}_q\, m_z}$ (boxes and full line) and total $\overline{\mathrm{Var}\, m_z}$ (diamonds and dotted line) variances for the example of thermal environment and nonsymmetric globally interacting system. Lines are as in Fig. 1 double exponential functions numerically fitted to the data calculated at $N = 2, 6, 10, 15$.

$$\Delta_c^2 A = E[\langle\psi(t)|A|\psi(t)\rangle^2] - (E[\langle\psi(t)|A|\psi(t)\rangle])^2. \quad (13)$$

Description of the evolution in terms of the stochastic Eq. (10) for an ensemble of pure states is numerically much less consuming than the equivalent Linblad master equation.

Our computations using the QSD equation are illustrated in Figs. 1(d) and 2. Relation between the quantum and the total variances is illustrated in Fig. 2 using the example Hamiltonian $H(\omega_z = 1.4,\ \omega_x = 1.4,\ J = 1)$ with global interaction and the thermal environment. Quantum variance (12) is illustrated in Fig. 1(d) for both types of Hamiltonians $H(\omega_z = 1.4,\ \omega_x = 1.4,\ J = 1)$ and $H(\omega_z = 0,\ \omega_x = 1.4,\ J = 1)$ with the global interaction and both types of the environments. Similar results are obtained for other two macrovariable $m_{x,y}$. The coupling parameter for the case of dephasing is set to $\gamma_{dp} = 1$ so that the Linblad and the Hamiltonian part in the master equation are of the same order. The average number of excitations $\bar{n}$ in the thermal environment is proportional to the temperature. We considered the case when $\bar{n} \gg 1$ and the coupling is such that $\gamma_{th}\bar{n} = 1$. Results presented in Fig. 1(d) clearly show that the quantum dispersions decrease to zero as N is increased. The curves can be fitted with double exponentials like in the case of isolated locally interacting systems (with different parameters). We can conclude that the decoherence by either the thermal or the dephasing environment plus macroscopic coarse graining are enough to guarantee dispersionless evolution of all macrovariables $m_{x,y,z}$ even for the systems of qubits with the global interaction.

In summary, we have shown that isolated system of qubits with local interaction of an integrable or nonintegrable type display classical behavior of macroscopic coarse-grained variables. On the other hand, if the qubits are interacting globally the macroscopic variables have nonvanishing dispersions. In this case the environmental decoherence by either thermal or dephasing environments is needed for the classical behavior of the quantum dispersions of macrovariables. Simultaneous annihilation of quantum dispersions for all macrovariables is only a necessary condition for the classical behavior of a quantum system. There are other independent properties of classical systems which need not appear under the same conditions as those related to the dispersionless evolution of the macrovariables.

ACKNOWLEDGMENTS

This work is partially supported by the Serbian Ministry of Science Contract No. 141003. I would also like to acknowledge useful discussions with Č. Brukner and J. Koefler and the support of WUS Austria and ÖAD.

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