Asymptotic approach to the analysis of mode-hopping in semiconductor ring lasers

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We investigate the directional mode-hopping in semiconductor ring lasers in the limit of small noise intensity. We show how only long time scales are involved in the hopping process. In such limit, the initial stochastic rate equations can be reduced to an auxiliary Hamiltonian system, and the dependence of the hopping on the laser parameters can be investigated. The predictions made from the reduced model agree well with the results obtained by simulating the full rate-equation system.

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I. INTRODUCTION

Semiconductor ring lasers (SRLs) are currently the subject of many experimental and theoretical investigations, ranging from fundamental studies of its nonlinear dynamical behavior to multiple practical applications [1–9]. In particular, the bistable character of SRLs makes them attractive for the purpose of encoding information in the direction of emission [1,3,6].

Recently, highly performing all-optical flip-flops based on two coupled microrings have been fabricated [7]. The information is stored in the direction of emission and a counterpropagating signal could be used to switch the operation [7]. On the other hand, a device based on a single ring would benefit of a smaller footprint and could take advantage of the recently demonstrated switching schemes based on one-side injection [8]. However, single-ring devices present a reduced stability when compared to coupled rings designs. In particular, undesired dynamical regimes such as alternate oscillations [3] have been reported, as well as noise-induced mode-hopping [9]. The presence of noise-induced hopping is especially detrimental for applications such as the realization of optical memories.

In particular, the results of a full bifurcation analysis performed in Ref. [10] suggest that the shape of the manifolds separating the basins of attraction of the two counterpropagating modes is controlled by the phase of the backscattering parameters. However, an investigation of the stability versus stochastic fluctuation has not been addressed yet. Especially, the important role of the linear backscatter phase between the two counterpropagating modes remains an open question. An asymptotic approach to the activation problem consists in the use of an auxiliary Hamiltonian system and in the solution of an associate boundary value problem [11]. However, the large dimensionality of the system makes this analysis extremely intricate in the case of SRLs. On the other hand, the use of a phenomenological model such as a double-well potential requires the fitting of a large number of parameters and loses a direct connection with the physically meaningful parameters of the system. A similar problem has been investigated in solid-state ring lasers [12] to reveal stochastic resonance, but it relied on a phenomenological double-well potential which kept hidden the real parameters of the system.

In this paper we propose an asymptotic approach to the problem of stochastic mode-hopping by focusing on the limit of vanishing noise intensity. Two independent limits are considered: first the full rate-equation set are asymptotically reduced to a two-dimensional reduced model as described in [13,14]; second, the asymptotic limit of vanishing noise intensity is considered [15–17]. As a result, we obtain an auxiliary Hamiltonian system which describes the mode-hopping process and contains the dependence on the original parameters of the system without any fitting or any further phenomenology. The role of the bias current and the important backscattering phase are investigated. The limit of small backscattering amplitude is addressed numerically and analytically.

The paper is structured as follows. In Sec. II we introduce the rate-equation model to describe the operation of a SRL [3], and we perform the two asymptotic limits of slow-dynamics and vanishing-noise intensities. In Sec. III we discuss the optimal transition paths that are the results of the Hamiltonian approach. Changes in the qualitative features of the optimal paths are discussed when the model parameters are changed.

II. THEORY

A semiconductor ring laser is a semiconductor laser whose active cavity have a circular geometry [1,3,6] as schematically depicted in Fig. 1. Two directional mode are supported by the cavity geometry: one which propagates clockwise (CW) and a counterclockwise (CCW) one. A bus waveguide is integrated on the same chip in order to couple out power from the ring cavity.

We consider a SRL operating in a single-longitudinal single-transverse mode. The rate-equation model is formu-

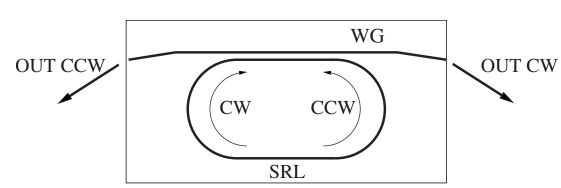

FIG. 1. A schematic view of the semiconductor ring laser. SRL: semiconductor ring laser cavity; WG: bus waveguide; CW and CCW: clockwise and counterclockwise modes propagating in the cavity; OUT CW and CCW: output beams.

lated mathematically in terms of two complex equations for the slowly varying amplitudes $E_{1,2}$ and one real equation for the carrier number N. In what follows, in order to fix the notation we will assume that mode 1 is rotating clockwise, whereas mode 2 is rotating counterclockwise. The equations for $E_{1,2}$ and N read [1,3,13] as

$$\dot{E}_{1,2} = \kappa(1+i\alpha)[N(1-s|E_{1,2}|^2 - c|E_{2,1}|^2) - 1]E_{1,2} - ke^{i\phi_k}E_{2,1}, + \xi_{1,2}(t), \tag{1}$$

$$\dot{N} = \gamma[\mu - N - N(1-s|E_1|^2 - c|E_2|^2)|E_1|^2 - N(1-s|E_2|^2 - c|E_1|^2)|E_2|^2], \tag{2}$$

where the dot represents differentiation with respect to t, κ is the field decay rate, γ is the carrier decay rate, α is the linewidth enhancement factor, and μ is the renormalized injection current with $\mu \approx 0$ at transparency and $\mu \approx 1$ at lasing threshold. The two counterpropagating modes are considered to saturate both their own and each other gain due to, e.g., spectral hole burning effects. Self- and cross-saturation effects are added phenomenologically and are modeled by s and c. For a realistic device the cross saturation term is larger that the self-saturation.

In Eq. (1), the spontaneous emission noise has been introduced phenomenologically as complex uncorrelated zero-mean stochastic terms described by the correlation terms: $\langle \xi_i(t+\tau)\xi_j^*(t)\rangle = 2D'\delta_{ij}\delta(\tau)$, where $i,j=1,2$ and D' is the noise intensity. Carrier noise has been disregarded as its relevance in directional mode-hopping was proved negligible [18].

For a small size ring laser, reflection of the counterpropagating modes occurs where the ring cavity and coupling waveguide meet and can also occur at the end facets of the coupling waveguide. These localized reflections result in a linear coupling between the two fields characterized by an amplitude k and a phase shift ϕ_k [19].

The set of stochastic rate Eqs. (1) and (2) can be solved numerically using a predictor-corrector scheme [31] for different values of the model parameters and the noise intensity D'. Examples of stochastic time series of the modal intensities $|E_1|^2$ and $|E_2|^2$, as well as the total emitted intensity $|E_1|^2 + |E_2|^2$ are shown in Fig. 2

It is clear from Fig. 2 that the SRL emits almost steadily in one of the two modes until an abrupt transition leads to lasing in the counterpropagating mode. However, during such mode-hoppings, the total intensity $|E_1|^2 + |E_2|^2$ is conserved. Such conservation of power has also been observed experimentally [1]. A further analysis of the time series shown in Fig. 2 reveals that (for the parameters chosen) two qualitatively different transitions are possible. The first kind of transition is characterized by a long permanence in one of the states until the fluctuations induce a hopping to the opposite mode. The same time series reveal short excursions to the opposite mode; however, as shown in the inset of Fig. 2 the system does not settle in the opposite mode, and it always returns to lase in the initial state.

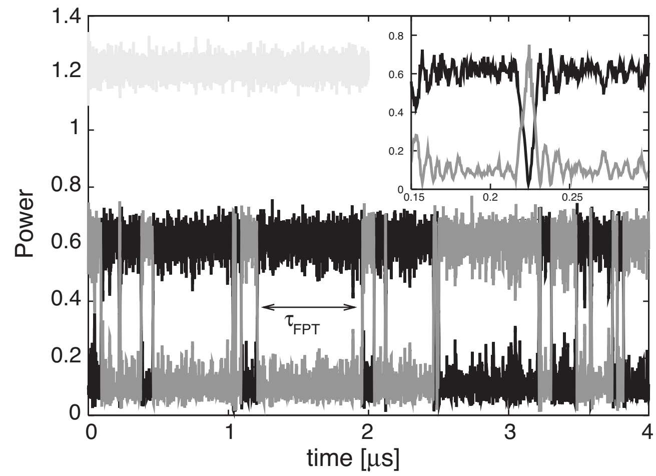

FIG. 2. Example of a stochastic trajectory solution of Eqs. (1) and (2) for the following parameter's choice: $K=0.44$ ns^{-1}, $\phi_k = 1.5$, $\mu=1.72$, $\alpha=3.5$. and $D'=7.7\times 10^{-4}$ ns^{-1}. The intensities of the CW mode (black) and the CCW mode (gray) are shown together with the total intensity P_1+P_2 (light gray). The total power is shifted upward for clarity. The arrow indicates an example of first passage time. A short excursion to the opposite mode is shown in the inset.

The different features of these two transitions have been investigated in a previous publication [9] using topological arguments. In particular, it has been shown that the transitions of the first kind are the result of a noise-induced activation process, whereas the short excursions are a consequence of the finiteness of the noise.

In this paper, we aim to address the first kind of mode-hopping by formulating it in the form of an activation in a nonequilibrium bistable system. It is known from the theory of stochastic systems [16] that activation problems can be characterized by the mean first passage time $\langle \tau_{FPT}\rangle$ across the boundary of the basin of attraction of the initial state. An example of first passage time is given in Fig. 2. The mean first passage time can be expressed to logarithmic accuracy by a generalized nonequilibrium potential [17,20–23]

$$\langle \tau_{FPT}\rangle \propto e^{S/D}, \tag{3}$$

where S is the nonequilibrium potential and D is the intensity of the stochastic force that drives the fluctuations. It is important to understand the dependence of the quasipotential S on the parameters of systems (1) and (2). However, Eqs. (1) and (2) are too complicated for an analytical or quasianalytical approach. Nevertheless, one cannot introduce any *ad hoc* model such as a double-well potential as this would introduce phenomenological model parameters which are not related to the physically relevant quantities such as k, ϕ_k, or α.

However, from the observation that the escape events are rare when compared to the other time scales of the system and the quasiconservation of the total power during the hopping, we propose that the hopping problem can be successfully described in the framework of an asymptotically reduced model [13].

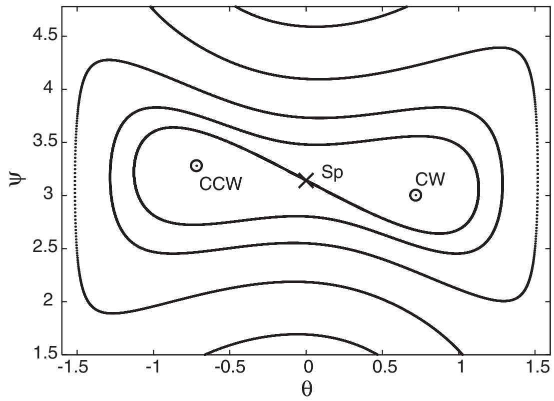

FIG. 3. Phase space portrait of Eqs. (5) and (6). The two stationary points are marked by circles and correspond to CW and CCW unidirectional operations. The SP is marked by a cross and the stable manifold of SP is marked as a solid line. The unstable manifold of SP is not shown. The parameters are as follow $k=0.44$, $\phi_k=1.5$, $\mu=1.65$, and $\alpha=3.5$.

On time scales slower than the relaxation oscillations, it has been shown [13] that the total intensity are indeed asymptotically conserved:

$$|E_1|^2+|E_2|^2=\mu-1>0. \tag{4}$$

The long time scale dynamics is then described by the time evolution of two auxiliary angular variables:

$$\theta' = K_\theta(\theta,\psi) = -2\sin\phi_k\sin\psi + 2\cos\phi_k\cos\psi\sin\theta + J\sin\theta\cos\theta + \xi_\theta, \tag{5}$$

$$\cos\theta\psi' = \cos\theta K_\psi(\theta,\psi) = \alpha J\sin\theta\cos\theta + 2\cos\phi_k\sin\psi + 2\sin\phi_k\cos\psi\sin\theta + \xi_\psi, \tag{6}$$

where $\theta=2\arctan\sqrt{|E_2|^2/|E_1|^2}-\pi/2\in[-\pi/2,\pi/2]$ represents the relative modal intensity and $\psi\in[0,2\pi]$ is the phase difference between the counterpropagating modes. Prime now denotes derivation to the slow time scale $\tau=kt$. Finally, in this reduced model the pump current has been rescaled as $J=\kappa(c-s)(\mu-1)/k$. As the phase space of Eqs. (5) and (6) is restricted to two dimensions, it allows for a clear physical picture of the influence of all parameters on the dynamical evolution of the variables in a plane. For a comprehensive analysis of Eqs. (5) and (6), including a complete bifurcation analysis, we refer to [10].

We make here the assumption of uncorrelated white Gaussian noise terms $\xi_{\theta,\psi}$ for the angular variables too. This assumption is indeed unphysical, as multiplicative terms and cross correlations are expected to appear in the noise. However, the agreement with the results of the full model (see below) and the benefits in simplicity will justify this assumption.

The phase space portrait of Eqs. (5) and (6) when the laser operates in unidirectional regime is characterized by four stationary solutions as shown in Fig. 3 [8,9]: an unstable in-phase bidirectional state in (0,0) (not shown); two symmetric stable states CW and CCW at $\psi\approx\pi$, both corresponding to unidirectional operation; and a saddle point (SP) in $(0,\pi)$ which corresponds to unstable lasing in the out-of-phase bidirectional mode.

The basins of attraction of the CW and CCW modes are separated by the stable manifold of SP. It is clear from Fig. 3 that the shape of the basin boundary can be complicated, and its topological features have been used to explain unexpected behaviors of SRL [8,9]. The unstable manifold of the saddle (not shown in Fig. 3) is made up of two branches, each of them connecting the saddle with one of the stable states.

Asymptotic analysis of the Fokker-Planck equation

In order to investigate the stochastic properties of Eqs. (5) and (6), we study the Fokker-Planck equation (FPE) for the probability density $\rho(\theta,\psi)$. The FPE is written as follows:

$$\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial\theta}K_\theta\rho - \frac{\partial}{\partial\psi}K_\psi\rho + D\frac{\partial^2}{\partial\theta^2}\rho + D\frac{\partial^2}{\partial\psi^2}\rho, \tag{7}$$

where D is the noise intensity for the reduced system which is the small parameter in our theory. The value of D is the only quantity that require a fitting in our analysis.

In the limit of vanishing noise intensity $D\to0$, an asymptotic WKB expansion can be performed for the probability density ρ [24]. We stress here that we consider two separate asymptotic limits in our approach. The first limit (discussed above) is related to time scale separation and depends on the laser parameters. Such first asymptotic does not affect the noise properties of the system. In what follows, we introduce a second asymptotic limit that depends on the noise intensity and not on the laser parameters.

We consider the following ansatz for ρ:

$$\rho = Ze^{-S/D}D\to0, \tag{8}$$

where S is an auxiliary function. S is the analogous of a potential in a gradient system and it is referred to as *quasi-potential* in nonequilibrium systems such as Eqs. (5) and (6). It is known from the theory of stochastic systems that, at the leading order $\frac{1}{D}$, the quasipotential S satisfies a Hamilton-Jacobi equation for a classical action:

$$\frac{\partial S}{\partial t} = -H\left(\theta,\psi,\frac{\partial S}{\partial\theta},\frac{\partial S}{\partial\psi}\right), \tag{9}$$

with the Hamiltonian $H(\theta,\psi,\frac{\partial S}{\partial\theta},\frac{\partial S}{\partial\psi})$ defined as

$$H = \frac{1}{2}\left(\frac{\partial S}{\partial\psi}\right)^2 + \frac{1}{2}\left(\frac{\partial S}{\partial\theta}\right)^2 + \frac{\partial S}{\partial\theta}K_\theta + \frac{\partial S}{\partial\psi}K_\psi. \tag{10}$$

In what follows, we will refer to the function S indifferently as action or nonequilibrium potential. The analogy with the case of a classical Hamiltonian problem is completed by defining the auxiliary momenta:

$$p_\theta = \frac{\partial S}{\partial\theta}, \quad p_\psi = \frac{\partial S}{\partial\psi}, \tag{11}$$

leading to the Hamiltonian

$$H(p_\theta,p_\psi,\theta,\psi) = \frac{p_\theta^2}{2} + \frac{p_\psi^2}{2} + K_\theta p_\theta + K_\psi p_\psi. \tag{12}$$

With this approach, initial stochastic systems (5) and (6) can be mapped into a deterministic Hamiltonian system

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = K_\theta + p_\theta, \tag{13}$$

$$\dot{\psi} = \frac{\partial H}{\partial p_\psi} = K_\psi + p_\psi, \tag{14}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -\frac{\partial K_\theta}{\partial \theta}p_\theta - \frac{\partial K_\psi}{\partial \theta}p_\psi, \tag{15}$$

$$\dot{p}_\psi = -\frac{\partial H}{\partial \psi} = -\frac{\partial K_\theta}{\partial \psi}p_\theta - \frac{\partial K_\psi}{\partial \psi}p_\psi. \tag{16}$$

The action S evolves along the solutions of Eqs. (13)–(16) following the equation

$$\frac{dS}{dt} = \frac{1}{2}p_\psi^2 + \frac{1}{2}p_\theta^2. \tag{17}$$

The action S calculated along a certain trajectory can be interpreted as a "cost" for such transition to take place and ultimately its probability. Consider a transition between an initial point x_i and a final point x_f. A large action indicates trajectory with low probability whereas a smaller action indicates a more likely trajectory. In the limit of vanishing noise intensity $D\to 0$, the transitions corresponding to the minimum actions become exponentially more likely than any other transition path, and only trajectories with minimum actions become relevant in the calculation of the probability distribution. In other words, only those trajectories corresponding to a global minimum can be observed in a physical experiment in the zero noise-intensity limit [11,25,26].

III. OPTIMAL ESCAPE PATHS IN SEMICONDUCTOR RING LASERS

In the previous sections, we reviewed the general theory for stochastic fluctuation in the limit of vanishing noise intensity. In this section we formulate the problem for the mode-hopping in SRLs, and we discuss the general topological features of the escape trajectories that realize the hopping.

During the regular operation of SRL, the system spends the majority of its time in the close vicinity of one of the stationary states. In order for a mode-hop to take place, the spontaneous emission noise must drag the system outside the basin of attraction of the original state (for instance CW) to the basin of attraction of the counterpropagating mode (for instance CCW). As the basins of attraction of the CW and CCW modes are separated by the stable manifold of the saddle SP (see Fig. 3), the mode-hop is realized by a trajectory solution of Eqs. (13)–(16) that connects the stationary state CW with the stable manifold of the saddle.

In general there are infinitely many solutions of Eqs. (13)–(16) that emanates from the initial state and reach the basin boundary. According to the discussion in Sec. IIA, in the limit of small noise intensity $D\to 0$, the escape takes place with overwhelming probability along the trajectory that minimizes the action S. Such path is know in literature as the *most probable escape path* (MPEP) [11,15,27–30].

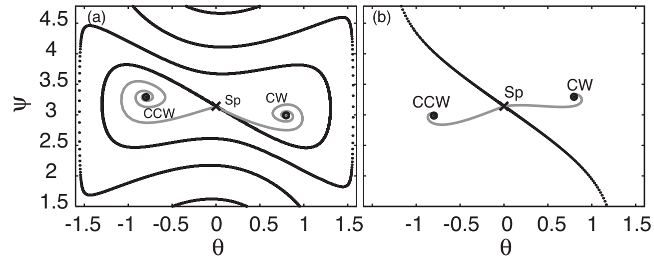

FIG. 4. MPEP paths for different values of the backscattering phase: (a) ϕ_k=1.5; (b) ϕ_k=1.08. The other parameters are as in Fig. 3.

As the motion along the stable manifold of SP is deterministic and deterministic motion does not increase the action, the minimal action along the stable manifold of the saddle coincide with the saddle itself [11,15,27–29]. Therefore, the MPEP satisfies the following boundary conditions: it emanates from the initial state CW (CCW) at time $t\to -\infty$ and converges to the saddle SP at time $t\to\infty$. The transition to the opposite mode is completed deterministically by following the unstable manifold of SP.

We calculated the MPEP for a SRL by minimization of action functional (17) along the solutions of Eqs. (13)–(16) for different values of the system parameters.

Some examples of MPEPs are shown in Fig. 4 for transitions from the CW state to the CCW state. As expected according to the previous discussion, the MPEP connects the CW state to the saddle SP and a deterministic relaxation completes the mode-hop to the CCW mode.

It is clear from Fig. 4, that the backscattering phase ϕ_k plays a major role in determining the actual shape of the MPEP. In the next sections, we will discuss the role of ϕ_k and the other parameters of the system in the mode-hopping.

IV. DEPENDENCE OF THE ACTIVATION ENERGY ON THE PARAMETERS OF THE SYSTEM

In order to validate our analysis and to provide insight in the dependence of the mode-hopping versus the parameters of the system, we calculate the activation energy for different values of the current and the backscattering phase using both numerical simulation and the Hamiltonian theory.

The full set of rate Eqs. (1) and (2) is solved using a Monte Carlo method [31] for different values of the noise intensity D'. The activation energy is calculated by fitting the mean-first-passage-time versus $1/D'$. The dependence of the activation energy versus current μ and phase ϕ_k is shown in Fig. 5 by markers.

The theoretical activation energy is calculated by solution of the boundary value problem associated with Eqs. (13)–(16) and by minimization of the function S [11]. The parameters are the same as in Eqs. (1) and (2), excluding the noise intensity D which is fitted. The results of this calcula-

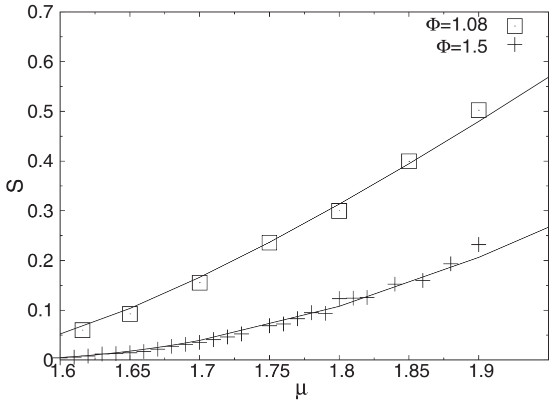

FIG. 5. Activation energy as a function of the current μ for different values of the backscattering phase. Markers are used to indicate the results of the integration of the full rate-equation model [Eqs. (1) and (2)], whereas the solid lines represent the solution of auxiliary Hamiltonian systems (13)–(16).

tion are shown in Fig. 5 with lines. The agreement between theoretical and numerical values of the activation energy is good.

In Fig. 6, we show the dependence of the activation energy calculated by solution of the boundary value problem as a function of the backscattering phase ϕ_k for different values of the bias current μ. We observe that the activation energy increases with μ in accordance with the results of Fig. 5 and the intuitive argument that increasing the bias current stabilizes the laser operation. On the other hand, the dependence of S on the backscattering phase is nonmonotonous, indicating that the stability of the directional modes versus random fluctuations decreases exponentially from a maximum at $\phi_k^{\max}\approx 0.85$. The loss of stability of the directional modes can be understood by considering the bifurcation scenario of a SRL [10]. According to [10], for the physically relevant parameters considered in Fig. 6, stable unidirectional lasing is possible between a pitchfork bifurcation taking place at $\phi_k \approx 0.2$ and a subcritical Hopf bifurcation taking place at $\phi_k \approx 1.6$. Therefore, the decrease in S for $\phi_k < \phi_k^{\max}$ corresponds to a loss of stability of the system when approaching the pitchfork bifurcation line, whereas the decrease in S for $\phi_k > \phi_k^{\max}$ is consistent with a loss of stability of the direc-

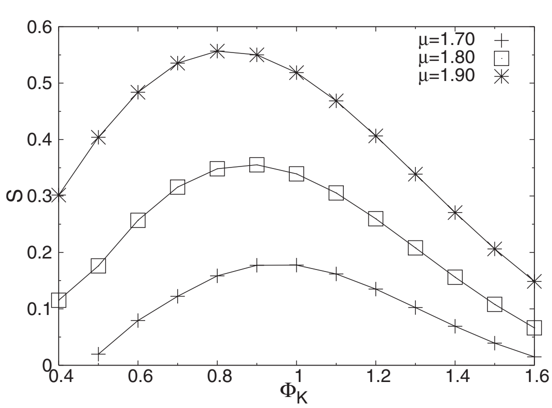

FIG. 6. Activation energy as a function of the phase of the backscattering ϕ_K for different values of the bias current μ.

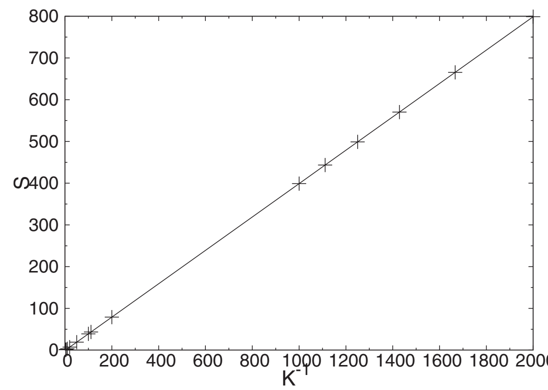

FIG. 7. Activation energy S as a function of the inverse backscattering $1/|K|$ obtained as a solution of Eqs. (13)–(16). The current is set at $\mu=1.8$ and the phase at $\phi_k=1.3$.

tional modes when approaching the subcritical Hopf bifurcation.

The asymptotic limit $J\to\infty$ can be treated analytically, which corresponds to either $\mu\to\infty$ or $K\to 0$. In this limit, we can neglect the terms depending on the backscattering phase ϕ_K in Eqs. (5) and (6) and obtain

$$\theta' = +J \sin\theta \cos\theta, \tag{18}$$

$$\cos\theta\psi' = \alpha J \sin\theta \cos\theta. \tag{19}$$

Therefore, the dynamics of the variable θ is decoupled by the variable ψ which becomes a follower of θ. The activation energy S can be calculated by noticing that $\theta' = -\nabla U$, with $U = -\frac{1}{2}J\cos(2\theta)$. Therefore $S=J/2$. In this limit, the action is linearly dependent on J and inversely proportional to $|K|$. In Fig. 7 the dependence of the activation energy versus the inverse of the backscattering magnitude for $\mu=1.8$ and $\phi_k = 1.3$ is shown as the solution of Eqs. (13)–(16). The linearity is evident.

Finally, we investigate the effect of a change in the linewidth enhancement factor in the laser medium. In Fig. 8(a), the activation energy is calculated numerically by Monte Carlo simulations of the full rate-equation model [Eqs. (1) and (2)]. A monotonic increase in the activation energy with α is evident. An unexpected feature of the dependence of S versus α is the abrupt change in the gradient of S. Such a change in scaling can be interpreted by investigating the topology of the MPEP in the reduced model. A typical dependence of S on α calculated using the Hamilton's Eqs. (13)–(16) is exemplified in Fig. 8(b). The same change in scaling as in Fig. 8(a) is observed. Moreover, this can be explained by studying the different topology of the MPEP for parameters corresponding to the different branches of the S-α curve.

For values of α lower than a critical value α_c, the MPEP connects the initial stationary state with the saddle directly [Fig. 8(c)], whereas, for $\alpha > \alpha_c$ the MPEP surrounds the unstable in-phase bidirectional mode [Fig. 8(d)]. Therefore, for low values of α a small excursion in the phase different ψ is expected, whereas, for values of α larger than α_c an excursion in ψ larger that 2π is expected. We conjecture the exis-

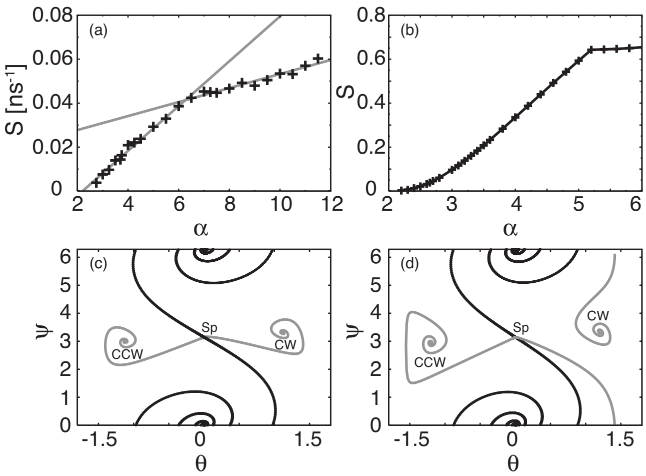

FIG. 8. Activation energy S as a function of the linewidth enhancement factor α when calculated (a) by fitting the MFPT obtained from Monte Carlo simulations of Eqs. (1) and (2) and (b) from the Hamiltonian formalism. Both approaches reveal the "knee." The current is set at $\mu=1.8$, the backscattering amplitude to $k=0.44$ and the phase at $\phi_k=1.2$. (Bottom) Qualitatively different MPEP for (c) $\alpha=5$ and (d) $\alpha=6$ as calculated with Hamiltonian systems (13)–(16).

tence of other critical values of α corresponding to excursions in $\psi>4\pi,6\pi,\ldots$ and so forth, leading to corresponding changes in the scaling of the activation energy. At the critical value α_c, the MPEP is degenerate and two qualitatively different trajectories reach the saddle with the same probability.

We remark here that simpler double-well models cannot predict such change in the gradient of the activation energy.

V. CONCLUSION

In conclusion, we have investigated the stochastic mode-hopping in semiconductor ring lasers. We started with a five-dimensional rate-equation system, and we proved that the main features of the mode-hopping can be understood in the framework of a reduce two-dimensional model.

Stochastic planar systems (5) and (6) was subsequently mapped into a four-dimensional Hamiltonian system [Eqs. (13)–(16)] and the optimal escape paths could be calculated. As expected from the general theory of escape in nonlinear systems [11,25,30], the most probable escape paths connects the initial stationary lasing operation with a saddle in the phase space. The results of the reduced model were confirmed by the numerical simulations of the full rate-equation model [Eqs. (1) and (2)].

Opposite to the adoption of a phenomenological bistable system such as a double-well potential, the use of the asymptotic reduction Eqs. (5) and (6) does not require the introduction of phenomenological parameters to be matched with Eqs. (1) and (2). In this way we investigated the dependence of the activation energy on the principal parameters as shown in Figs. 5–8. In particular, our asymptotic method allowed us to understand the change in scaling of the dependence of S on α which cannot be explained with other phenomenological models such as a double-well potential.

Our approach can be straightforwardly extended to analyze the problem of fluctuations in other bistable optical systems [22,32–35] or in the recently discovered multistable regimes of operation of SRLs [36].

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[1] M. Sorel, G. Giuliani, A. Scire, R. Miglierina, S. Donati, and P. J. R. Laybourn, IEEE J. Quantum Electron. **39**, 1187 (2003).
[2] V. R. Almeida and M. Lipson, Opt. Lett. **29**, 2387 (2004).
[3] M. Sorel, J. P. R. Laybourn, A. Sciré, S. Balle, G. Giuliani, R. Miglierina, and S. Donati, Opt. Lett. **27**, 1992 (2002).
[4] Z. Wang, G. Yuan, G. Verschaffelt, J. Danckaert, and S. Yu, IEEE Photonics Technol. Lett. **20**, 1228 (2008).
[5] S. Furst and M. Sorel, IEEE Photonics Technol. Lett. **20**, 366 (2008).
[6] J. J. Liang, S. T. Lau, M. H. Leary, and J. M. Ballantyne, Appl. Phys. Lett. **70**, 1192 (1997).
[7] M. T. Hill, H. J. S. Dorren, T. de Vries, X. J. M. Leijtens, J. H. den Besten, B. Smalbrugge, Y. S. Oei, H. Binsma, G. D. Khoe, and M. K. Smit, Nature (London) **432**, 206 (2004).
[8] L. Gelens, S. Beri, G. Van der Sande, J. Danckaert, N. Calabretta, H. J. S. Dorren, R. Nötzel, E. A. J. M. Bente, and M. K. Smit, Opt. Express **16**, 10968 (2008).
[9] S. Beri, L. Gelens, M. Mestre, G. Van der Sande, G. Verschaffelt, A. Scirè, G. Mezosi, M. Sorel, and J. Danckaert, Phys. Rev. Lett. **101**, 093903 (2008).
[10] L. Gelens, G. Van der Sande, S. Beri, and J. Danckaert, Phys. Rev. E **79**, 016213 (2009).
[11] S. Beri, R. Mannella, D. G. Luchinsky, A. N. Silchenko, and P. V. E. McClintock, Phys. Rev. E **72**, 036131 (2005).
[12] B. McNamara, K. Wiesenfeld, and R. Roy, Phys. Rev. Lett. **60**, 2626 (1988).
[13] G. Van der Sande, L. Gelens, P. Tassin, A. Scirè, and J. Danckaert, J. Phys. B **41**, 095402 (2008).
[14] T. Erneux, J. Danckaert, K. Panajotov, and I. Veretennicoff, Phys. Rev. A **59**, 4660 (1999).
[15] D. Ludwig, SIAM Rev. **17**, 605 (1975).
[16] A. D. Wentzel and M. I. Freidlin, *Fluctuations in Dynamic Systems: Effects of Small Random Perturbations* (Nauka, Moscow, 1979).
[17] P. Hänggi, P. Talkner, and M. Borkovec, Rev. Mod. Phys. **62**, 251 (1990).

[18] P. Lett and L. Mandel, J. Opt. Soc. Am. B **2**, 1615 (1985).
[19] R. J. C. Spreeuw, R. C. Neelen, N. J. van Druten, E. R. Eliel, and J. P. Woerdman, Phys. Rev. A **42**, 4315 (1990).
[20] R. S. Maier and D. L. Stein, Phys. Rev. E **48**, 931 (1993).
[21] V. I. Mel'nikov, Phys. Rep. **209**, 1 (1991).
[22] B. Nagler, M. Peeters, J. Albert, G. Verschaffelt, K. Panajotov, H. Thienpont, I. Veretennicoff, J. Danckaert, S. Barbay, G. Giacomelli, and F. Marin, Phys. Rev. A **68**, 013813 (2003).
[23] B. Nagler, M. Peeters, I. Veretennicoff, and J. Danckaert, Phys. Rev. E **67**, 056112 (2003).
[24] M. I. Dykman, P. V. E. McClintock, V. N. Smelyanski, N. D. Stein, and N. G. Stocks, Phys. Rev. Lett. **68**, 2718 (1992).
[25] V. N. Smelyanskiy, M. I. Dykman, and R. S. Maier, Phys. Rev. E **55**, 2369 (1997).
[26] D. G. Luchinsky, P. V. E. McClintock, and M. I. Dykman, Rep. Prog. Phys. **61**, 889 (1998).
[27] A. N. Silchenko, S. Beri, D. G. Luchinsky, and P. V. E. McClintock, Phys. Rev. Lett. **91**, 174104 (2003).
[28] M. I. Dykman and M. A. Krivoglaz, Sov. Phys. JETP **50**, 30 (1979).
[29] A. N. Silchenko, S. Beri, D. G. Luchinsky, and P. V. E. McClintock, Phys. Rev. E **71**, 046203 (2005).
[30] M. I. Dykman, M. M. Millonas, and V. N. Smelyanskiy, Phys. Lett. A **195**, 53 (1994).
[31] R. Mannella, in *Stochastic Processes in Physics, Chemistry, and Biology* (Springer, Berlin, 2000).
[32] M. Ohtsu, Y. Teramachi, Y. Otsuka, and A. Osaki, IEEE J. Quantum Electron. **22**, 535 (1986).
[33] G. Giacomelli and F. Marin, Quantum Semiclassic. Opt. **10**, 469 (1998).
[34] M. B. Willemsen, M. P. van Exter, and J. P. Woerdman, Phys. Rev. Lett. **84**, 4337 (2000).
[35] G. Giacomelli, M. Giudici, S. Balle, and J. R. Tredicce, Phys. Rev. Lett. **84**, 3298 (2000).
[36] L. Gelens, S. Beri, G. Van der Sande, G. Mezosi, M. Sorel, J. Danckaert, and G. Verschaffelt, Phys. Rev. Lett. **102**, 193904 (2009).