Fisher information and spin squeezing in the Lipkin-Meshkov-Glick model

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Fisher information, which lies at the heart of parameter estimation theory, was recently found to have a close relation with multipartite entanglement [L. Pezzé and A. Smerzi, Phys. Rev. Lett. **102**, 100401 (2009)]. We use Fisher information to distinguish and characterize behaviors of ground state of the Lipkin-Meskhov-Glick model, which displays a second-order quantum phase transition between the broken and symmetric phases. We find that the parameter sensitivity of the system attains the Heisenberg limit in the broken phase, while it is just around the shot-noise limit in the symmetric phase. Based on parameter estimation, Fisher information provides us a useful approach to the quantum phase transition.

I. INTRODUCTION

Parameter estimation of probability distributions is one of the most basic tasks in information theory and has been generalized to quantum regime [1,2] since the description of quantum mechanics is essentially probabilistic. How to improve the precision of parameter estimation has been focused for many years and is of important applications in quantum technology such as quantum frequency standards [3,4], measurement of gravity accelerations [5], clock synchronization [6], etc.

Consider a quantum state $\rho_\theta = U_\theta \rho_{in} U_\theta^\dagger$, where $U_\theta = \exp(i\theta \hat{K})$, $\hat{K}$ is a generator. We estimate parameter θ through proper measurements. However, the precision of our estimation is limited by the quantum Cramer-Rao (QCR) bound [1,2],

$$\Delta \hat{\theta} \geq (\Delta \theta)_{QCR} = \frac{1}{\sqrt{\nu F(\rho_{in}, \hat{K})}}, \qquad (1)$$

where ν is the number of trails, $\hat{\theta}$ is the so-called unbiased estimator, and $F(\rho_{in}, \hat{K})$ is the quantum Fisher information (QFI) [1,2,7,8]. In a sense, parameter estimation is equivalent to distinguishing neighboring states along the path in parameter space. We know QFI has close relation with Bures distance [9], the most studied distance in quantum-state space, and Bures distance is directly related to the Uhlmann fidelity [10]. The QFI is proportional to the Bures distance [11,12]. For pure states, the QFI, as well as the Bures distance ds_B^2, is just proportional to the variance of $\hat{K}$ [8], that is $F(\rho_{in}, \hat{K}) = 4ds_B^2 = 4(\Delta \hat{K})^2$. Therefore, besides increasing experimental times ν, we can improve the estimation precision $\Delta \hat{\theta}$ by choosing a proper state ρ_{in} for a given $\hat{K}$. In general, entangled states are more sensitive than separable states, i.e., the variance of $\hat{K}$ is large. In the past, many works have been devoted to improvement of parameter sensitivity by using entangled states [13–24].

Quite recently, Pezzé and Smerzi [25] found an interesting application of QFI in multipartite entanglement and the sub-shot-noise phase sensitivity in the estimation of a collective rotation angle. Consider an ensemble of spin-half particles in the state ρ_{in}. They introduced a quantity

$$\chi^2 = \frac{N}{F(\rho_{in}, S_{\vec{n}})} \qquad (2)$$

and prove that $\chi^2 < 1$ implies multipartite entanglement. Here, the generator of θ is $S_{\vec{n}} = \vec{S} \cdot \vec{n}$, which denotes the collective spin operator along direction $\vec{n}$. Namely, a sufficient condition is given for quantum entanglement. We may define a mean Fisher information as $F_m = F(\rho_{in}, S_{\vec{n}})/N$. Then, χ^2 and F_m are reciprocal to each other. The relation between χ and QCR bound is

$$\Delta \hat{\theta} \geq \frac{1}{\sqrt{F(\rho_{in}, S_{\vec{n}})}} = \frac{\chi}{\sqrt{N}} = \chi(\Delta \theta)_{SN}, \qquad (3)$$

where $(\Delta \theta)_{SN} = 1/\sqrt{N}$ is the shot-noise limit and we set $\nu = 1$. Thus, it is evident that $\chi^2 < 1$ becomes a necessary and sufficient condition for sub-shot-noise phase estimation.

In this work, we study the Fisher information of the ground state of the Lipkin-Meshkov-Glick (LMG) model [26], which has a second-order quantum phase transition (QPT) [27], between a symmetric (polarized, $h \geq 1$) phase and a broken (collective, $h < 1$) phase. The QPT in this model is interesting and was studied in field of entanglement by using concurrence [28], spin squeezing [29], entanglement entropy [30], single-copy entanglement, and geometric entanglement [31]. Furthermore, in Refs. [32,33], the results indicate that QPT can be viewed as a resource for high-precision quantum estimation. Around the critical point, the estimation of the driving parameter, which induces the QPT, is enhanced. In our work, we find that besides indicating the critical point and entanglement, χ^2 reflects the performances of ground states of these two phases in the sense of parameter sensitivity. In the symmetric phase, χ^2 approaches to 1 with the increasing of h and is independent of N, which means $(\Delta \theta)_{QCR} \sim (\Delta \theta)_{SN}$. In the broken phase, we find $\chi^2 \simeq 1/N$, thus $(\Delta \theta)_{QCR} \simeq 1/N$ attaining the Heisenberg limit.

This paper is organized as follows. In Sec. II, we give brief discussions about the relations between spin squeezing and χ^2. Then in Sec III, we study χ^2 and spin squeezing for the ground state of the LMG model in both isotropic ($\gamma = 1$) and anisotropic ($\gamma \neq 1$) cases. In isotropic case, the LMG model is diagonal in Dicke states. For Dicke states, we

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find that χ^2 and the spin squeezing parameter defined by Kitagawa and Ueda [34] are reciprocal to each other. In anisotropic case, we use Holstein-Primakoff transformation and derive χ^2 in the thermodynamic limit. The finite-size behaviors of χ^2 and the spin-squeezing parameter in the critical point are also obtained. The numerical results coincide well with the analytical ones.

II. FISHER INFORMATION AND SPIN SQUEEZING PARAMETERS

Fisher information is related to spin squeezing and there are two spin squeezing parameters, respectively, given by Kitagawa and Ueda [34] and Wineland [35],

$$\xi_1^2 = \frac{4(\Delta S_{\vec{n}_\perp})^2}{N}, \quad \xi_2^2 = \frac{N(\Delta S_{\vec{n}_\perp})^2}{|\langle S_{\vec{n}} \rangle|^2}, \quad (4)$$

where subscript $\vec{n}_\perp$ refers to an arbitrary axis perpendicular to the mean spin $\langle \vec{S} \rangle$, where the minimum value of $(\Delta S)^2$ is obtained. The inequality $\xi_i^2 < 1$ $(i=1,2)$ indicates that the state is spin squeezed. Spin-squeezed states can be used to reduce the measurement uncertainty [34,36] and improve the measurement precision of the atomic clock transition [37,38]. The spin-squeezing inequality is a criterion for multipartite entanglement [39,40]. For an arbitrary multiqubit separable state, it was found that $\xi_2^2 \geq 1$ and thus $\xi_2^2 < 1$ implies quantum entanglement.

As proved in [25],

$$F(\rho, S_{\vec{n}'_\perp})(\Delta S_{\vec{n}_\perp})^2 \geq |\langle S_{\vec{n}} \rangle|^2, \quad (5)$$

where directions $\vec{n}'_\perp$, $\vec{n}_\perp$, $\vec{n}$ are orthogonal to each other. $F(\rho, S_{\vec{n}'_\perp}) = 4(\Delta \hat{R})^2$, where $\hat{R}$ is determined by $\hat{R}\rho + \rho\hat{R} = i(\rho S_{\vec{n}'_\perp} - S_{\vec{n}'_\perp}\rho)$. In general, $(\Delta \hat{R})^2 \leq (\Delta S_{\vec{n}'_\perp})^2$ and the equality is obtained only for pure states. Then Eq. (5) reduces to the usual uncertainty relation,

$$(\Delta S_{\vec{n}'_\perp})^2(\Delta S_{\vec{n}_\perp})^2 \geq \frac{|\langle S_{\vec{n}} \rangle|^2}{4}, \quad (6)$$

for pure states. The above inequality can be written in terms of the inverse of the mean QFI and the squeezing parameter ξ_2^2 as

$$\xi_2^2 = \frac{N(\Delta S_{\vec{n}_\perp})^2}{|\langle S_{\vec{n}} \rangle|^2} \geq \frac{N}{4(\Delta S_{\vec{n}'_\perp})^2} = \frac{1}{F_m} = \chi^2. \quad (7)$$

Both the inequalities $\xi_2^2 < 1$ and the mean QFI $F_m > 1$ $(\chi^2 < 1)$ indicate the presence of entanglement.

Furthermore, $\xi_1^2 \leq \xi_2^2$ and there is no similar relation between ξ_1^2 and χ^2 like Eq. (7). However, we find that

$$\xi_1^2 \chi^2 = \frac{(\Delta S_{\vec{n}_\perp})^2}{(\Delta S_{\vec{n}'_\perp})^2} \leq 1, \quad (8)$$

since $(\Delta S_{\vec{n}'_\perp})^2$ $[(\Delta S_{\vec{n}_\perp})^2]$ is the maximum (minimum) variance. As proved in [40], if the pure state is of exchange symmetry, $\xi_1^2 < 1$ implies entanglement. Then, from the above equation, $\chi^2 > 1$ implies entanglement. We know that

$\chi^2 < 1$ indicates entanglement too. Therefore, a pure symmetric state is entangled when $\chi^2 \neq 1$ $(\xi_1^2 \neq 1)$.

When the mean spin direction is along z direction, the squeezing parameters ξ_1^2 and χ^2 become

$$\xi_1^2 = \frac{4 \min \langle S_\perp^2 \rangle}{N}, \quad \chi^2 = \frac{N}{4 \max \langle S_\perp^2 \rangle}, \quad (9)$$

where

$$S_\perp = \cos \theta S_x + \sin \theta S_y. \quad (10)$$

Furthermore, if $\langle \{S_x, S_y\} \rangle = 0$, for instance, in the LMG model [28], we have

$$\xi_1^2 = \frac{4 \min(\langle S_x^2 \rangle, \langle S_y^2 \rangle)}{N}, \quad \chi^2 = \frac{N}{4 \max(\langle S_x^2 \rangle, \langle S_y^2 \rangle)}, \quad (11)$$

thus we only need to derive $\langle S_x^2 \rangle$ and $\langle S_y^2 \rangle$ to determine the squeezing parameter and quantity χ^2 in the following discussions of QPTs in LMG model.

III. FISHER INFORMATION AND SQUEEZING IN THE LMG MODEL

The LMG model, originally introduced in nuclear physics, has been widely studied in other fields: statistical mechanics of quantum spin system [41], Bose-Einstein condensates [42], and magnetic molecules such as Mn_{12} acetate [43]. Recently, some quantum-information concepts, such as entanglement entropy [30], single-copy entanglement and geometric entanglement [31], and quantum fidelity [44,45], have been studied in this model, aiming at characterizing its QPT. It is an exactly solvable [46,47] many-body interacting quantum system as well as one of the simplest to show a quantum transition in the regime of strong coupling.

A. LMG Hamiltonian

The Hamiltonian of the LMG model reads

$$H = -\frac{\lambda}{N}(S_x^2 + \gamma S_y^2) - h S_z, \quad (12)$$

where $S_\alpha = \sum_{i=1}^{N} \sigma_\alpha^i / 2$ $(\alpha = x, y, z)$ are the collective spin operators, σ_α^i are the Pauli matrices, N is the total spin number, and γ is the anisotropic parameter. λ and h are the spin-spin interaction strength and the effective external field, respectively. Here, we focus on the ferromagnetic case ($\lambda > 0$) and without loss of generality, we set $\lambda = 1$ and $0 \leq \gamma \leq 1$. As the spectrum is invariant under the transformation $h \leftrightarrow -h$, we only consider $h \geq 0$.

The QPT of this model roots in the competition between the spin-spin interaction and the external field. To understand this in an easy picture, by using a mean-field approach [28], we can see that when $h > 1$, all spins tend to be polarized in the external field direction, when $h < 1$, the interaction energy is dominant, the system is twofold degenerate. Therefore, a spontaneous symmetry breaking occurs at $h=1$, which is a second-order QPT point between the so-called symmetric ($h \geq 1$) phase and broken ($h < 1$) phase. However, by considering the quantum effects, the exact ground state is not degenerate ($\gamma \neq 1$) in the broken phase, since the Hamil-

tonian is of spin-flip symmetry, i.e., $[H, \Pi_{i=1}^{N} \sigma_z^i] = 0$. We have

$$\langle S_x \rangle = \langle S_y \rangle = 0, \quad \langle S_x S_z \rangle = \langle S_y S_z \rangle = 0, \quad (13)$$

the mean spin direction is along the z axis. In addition, $[H, \mathbf{S}^2] = 0$ and the ground state lies in the $S = N/2$ symmetric section.

B. Isotropic case and Dicke state

We begin with the simple isotropic case, $\gamma = 1$. The Hamiltonian reduces to

$$H = -\frac{1}{N}(\mathbf{S}^2 - S_z^2) - hS_z, \quad (14)$$

which is diagonal in the standard eigenbasis $\{|S, M\rangle\}$ of $\mathbf{S}^2$ and S_z. For $S = N/2$, the energy eigenvalue is

$$E(M, h) = \frac{2}{N}\left(M - \frac{hN}{2}\right)^2 - \frac{N}{2}(1 + h^2), \quad (15)$$

and the ground state $|S, M_0\rangle$ is readily obtained when [45,48]

$$M_0 = \begin{cases} N/2 & \text{for } h \geq 1 \\ N/2 - R[N(1-h)/2] & \text{for } 0 \leq h < 1, \end{cases} \quad (16)$$

where $R(x) \equiv \text{round}(x)$ gives the nearest integer of x. Then one can see level crossings exist at $h = h_j$, where $h_j = 1 - (2j+1)/N$, between the two states $|S, S-j\rangle$ and $|S, S-j-1\rangle$.

As the ground state is actually a Dicke state $|S, M\rangle$, $\langle S_x^2 \rangle = \langle S_y^2 \rangle = (S^2 + S - M^2)/2$, then

$$\chi^2 = \frac{N}{2(S^2 + S - M^2)} = \frac{1}{N/2 + 1 - 2M^2/N} \leq 1, \quad (17)$$

the equality is obtained for $M = \pm S$. Immediately, we have

$$\xi_1^2 = 1/\chi^2 \geq 1. \quad (18)$$

As we know that, when $M \neq \pm S$, the Dicke states are entangled but not spin squeezed, since $\xi_2^2 > \xi_1^2 > 1$. Numerical results of ξ_1^2 and χ^2 for the isotropic LMG model are shown in Fig. 1(d). We can see that, in the broken phase, $M < S$, $\xi_1^2 = 1/\chi^2$, while in the symmetric phase, the ground state is $|S, S\rangle$, thus $\chi^2 = \xi_1^2 = 1$.

By considering χ^2 in Eq. (17), when M is close to $\pm S$, χ^2 is just a bit lower than 1, thus $\Delta\theta$ is not improved much than $(\Delta\theta)_{\text{SN}}$. When $|M| \leq \sqrt{S}$, we have

$$2/(N+2) \leq \chi^2 \leq 2/N, \quad (19)$$

and thus

$$(\Delta\theta)_{\text{QCR}} = \chi/\sqrt{N} \sim 1/N, \quad (20)$$

which attains the Heisenberg limit. Although $|S, \pm S\rangle$ is not entangled, the Schrödinger "cat state" (or GHZ state)

$$|\psi\rangle = (|S, S\rangle + |S, -S\rangle)/\sqrt{2} \quad (21)$$

is entangled and is useful in phase estimation [16]. Under $|\psi\rangle$, $\langle S_\alpha \rangle = 0$, for $\alpha = x, y, z$, thus there is no spin squeezing. We find the maximum variance $(\Delta S_z)^2 = S^2$, then
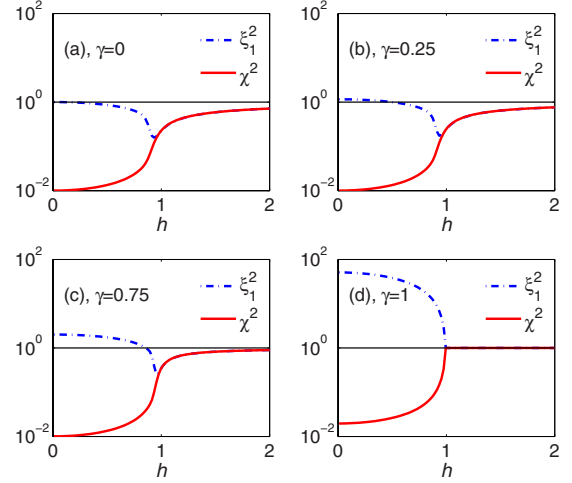

FIG. 1. (Color online) ξ_1^2 and χ^2 as functions of h for various γ, with system size $N = 100$. The crossing points of ξ_1^2 and the horizontal line in the broken phase are $h = \sqrt{\gamma}$.

$$\chi^2_{|\psi\rangle} = 1/N, \quad (\Delta\theta)_{\text{QCR}} = 1/N, \quad (22)$$

beating the Heisenberg limit. From the above analysis we know that, for typical symmetry multipartite states, Dicke states, there are no spin squeezings, while $\chi^2 < 1$ indicates that they are entangled and are useful resources for phase estimation.

C. Anisotropic case

Now we consider the anisotropic case, $0 \leq \gamma < 1$. The spin expectation values $\langle S_\alpha^2 \rangle$ cannot be obtained analytically. By treating the quantum effect as small fluctuations, approximate results can be obtained by using the Holstein-Primakoff (H-P) transformation [49] in the thermodynamic limit and by using the continued unitary transformation method [50–52] for finite-size case.

In the thermodynamic limit, the quantum fluctuations are small, we can use the H-P approximation. This method requires one to determine the semiclassical magnetization $\langle \vec{S} \rangle$, which is not along z axis in the broken phase in the thermodynamic limit. Following conventional steps, we first employ a mean-field approach and define a spin coherent state

$$|\theta, \phi\rangle = \bigotimes_{l=1}^{N}\left[e^{-i\phi/2}\cos\frac{\theta}{2}|0\rangle_l + e^{i\phi/2}\sin\frac{\theta}{2}|1\rangle_l\right], \quad (23)$$

under which

$$\langle\theta, \phi|\vec{S}|\theta, \phi\rangle = \frac{N}{2}(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \quad (24)$$

The Hamiltonian is rewritten as

$$H = -\frac{N}{4}[\sin^2\theta(\cos^2\phi + \gamma\sin^2\phi) + 2h\cos\theta]. \quad (25)$$

As $\max[\cos^2\phi + \gamma\sin^2\phi] = \max(1, \gamma) = 1, (\gamma \leq 1)$, we have

$$\min H = -\frac{N}{4}\max[\sin^2\theta + 2h\cos\theta], \quad (26)$$

then we conclude: (i) symmetric phase, $h \geq 1$, $\theta_0 = 0$, for all

γ; (ii) broken phase, $0 \le h < 1$, $\theta_0 = \arccos h$, $\phi = 0, \pi$, for $\gamma \ne 1$. We emphasize that the mean spin direction is along the z axis when the system is finite.

We rotate the z axis to the semiclassical magnetization

$$\begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} = \begin{pmatrix} \cos\theta_0 & 0 & \sin\theta_0 \\ 0 & 1 & 0 \\ -\sin\theta_0 & 0 & \cos\theta_0 \end{pmatrix} \begin{pmatrix} \widetilde{S}_x \\ \widetilde{S}_y \\ \widetilde{S}_z \end{pmatrix}. \quad (27)$$

As presented in [28], $\theta_0 = 0$ for $h > 1$ so that $\mathbf{S} = \widetilde{\mathbf{S}}$ and $\theta_0 = \arccos h$ for $h \le 1$. The transformed Hamiltonian reads

$$\widetilde{H} = -hm\widetilde{S}_z - \frac{1}{N}\left[\frac{m^2 + \gamma}{2}\widetilde{\mathbf{S}}^2 - \frac{3m^2 + \gamma - 2}{2}\widetilde{S}_z^2 \right]$$

$$+ \frac{h\sqrt{1-m^2}}{2}(\widetilde{S}_+ + \widetilde{S}_-) - \frac{m^2 - \gamma}{4N}(\widetilde{S}_+^2 + \widetilde{S}_-^2)$$

$$- \frac{m\sqrt{1-m^2}}{2N}(\widetilde{S}_+\widetilde{S}_z + \widetilde{S}_z\widetilde{S}_+ + \widetilde{S}_-\widetilde{S}_z + \widetilde{S}_z\widetilde{S}_-), \quad (28)$$

where $m = \cos\theta_0$. Then we introduce the H-P transformation

$$\widetilde{S}_z = N/2 - a^\dagger a, \quad \widetilde{S}_- = \sqrt{N}a^\dagger\sqrt{1 - a^\dagger a/N} = (\widetilde{S}_+)^\dagger. \quad (29)$$

The Hamiltonian can be written as

$$\widetilde{H}^{(0)} = \frac{2hm - 3m^2 - \gamma + 2}{2}a^\dagger a - \frac{m^2 - \gamma}{4}(a^{\dagger 2} + a^2) + \frac{1 - m^2}{4} \quad (30)$$

up to the 0th order of N. We neglect the terms of the 1th order of N as they are constant. Now we use the Bogoliubov transformation

$$a^\dagger = \cosh\left(\frac{\theta}{2}\right)b^\dagger + \sinh\left(\frac{\theta}{2}\right)b. \quad (31)$$

To diagonalize $\widetilde{H}^{(0)}$, we find

$$\tanh\theta = \varepsilon = \frac{m^2 - \gamma}{2hm - 3m^2 - \gamma + 2}. \quad (32)$$

The rotated spins are written under the H-P representation,

$$\widetilde{S}_x = \frac{\sqrt{N}}{2}\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{1/4}(b^\dagger + b) + O(1/N),$$

$$\widetilde{S}_y = \frac{i\sqrt{N}}{2}\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{1/4}(b^\dagger + b) + O(1/N),$$

$$\widetilde{S}_z = \frac{N}{2} + \frac{1}{2}\left(1 - \frac{1}{\sqrt{1-\varepsilon^2}}\right) - \frac{1}{\sqrt{1-\varepsilon^2}}\left[b^\dagger b + \frac{\varepsilon}{2}(b^{\dagger 2} + b^2)\right]. \quad (33)$$

For symmetric phase, $m = 1$, $S_\alpha = \widetilde{S}_\alpha$, we have

$$\langle S_x^2 \rangle = \langle \widetilde{S}_x^2 \rangle = \frac{N}{4}\sqrt{\frac{h - \gamma}{h - 1}},$$

$$\langle S_y^2 \rangle = \langle \widetilde{S}_y^2 \rangle = \frac{N}{4}\sqrt{\frac{h - 1}{h - \gamma}}, \quad (34)$$

while for broken phase, $m = h$, we need to rotate $\widetilde{S}_x$ back to S_x as

$$S_x = \widetilde{S}_x \cos\theta_0 + \widetilde{S}_z \sin\theta_0 = h\widetilde{S}_x + \sqrt{1-h^2}\widetilde{S}_z, \quad (35)$$

then we have

$$\langle S_x^2 \rangle = (1 - h^2)\langle \widetilde{S}_z^2 \rangle + h^2\langle \widetilde{S}_x^2 \rangle = \left(\frac{N^2}{4} + \frac{N}{2}\right)(1 - h^2)$$

$$+ \frac{N}{4}\frac{(1-\gamma)h^2 - (2 - h^2 - \gamma)(1-h^2)}{\sqrt{(1-h^2)(1-\gamma)}}. \quad (36)$$

We insert the above results into Eq. (11). For polarized phase, $h > 1$, we have

$$\xi_1^2 = \chi^2 = \sqrt{\frac{h - 1}{h - \gamma}} < 1. \quad (37)$$

When h is far from the critical point, ξ_1^2 and χ^2 approach to 1, then $\Delta\theta \sim (\Delta\theta)_{\text{SN}}$. For broken phase, $h < 1$, we get the spin squeezing parameter,

$$\xi_1^2 = \sqrt{\frac{1 - h^2}{1 - \gamma}}, \quad (38)$$

while

$$\chi^2 = \frac{N}{4\langle S_x^2 \rangle} \simeq \frac{1}{(N+2)(1-h^2)} \simeq \frac{1}{N}. \quad (39)$$

Thus $(\Delta\theta)_{\text{QCR}} \simeq 1/N$. When h approaches to the critical point $h_c = 1$, there are two limits in Eq. (36), that is, $(1-h)$ tends to be zero and N tends to be infinite. To overcome this problem, we need to expand the Hamiltonian in higher order of $1/N$. Fortunately, the finite-size behaviors of the spin squeezing and χ^2 at the critical point can be derived by using the results obtained in [28], where the authors employ the continued unitary transformations and get

$$\left.\frac{4\langle S_x^2 \rangle}{N^2}\right|_{h=1} \sim \frac{a_{xx}^{(0)}}{N^{2/3}}, \quad \left.\frac{4\langle S_y^2 \rangle}{N^2}\right|_{h=1} \sim \frac{a_{yy}^{(0)}}{N^{4/3}}, \quad (40)$$

where $a_{xx}^{(0)}$ and $a_{yy}^{(0)}$ are constant independent of N. Now we have

$$\xi^2|_{h=1} \sim \frac{a_{yy}^{(0)}}{N^{2/3}}, \quad \chi^2|_{h=1} \sim \frac{a_{xx}^{(0)}}{N^{2/3}}, \quad (41)$$

then for large N, ξ^2 and χ^2 converge to zero as $1/N^{2/3}$ and $(\Delta\theta)_{\text{QCR}} \sim 1/N^{5/6}$.

To verify these analytical predictions, in Figs. 1 and 2, we show numerical results for ξ_1^2 and χ^2 as functions of h with different γ for finite-size system. As shown in Fig. 1, in the symmetric phase, $\xi_1^2 = \chi^2 < 1$, while in the broken phase, χ^2 and ξ_1^2 behave very differently. In most of parameter ranges, $\chi^2 < 1$, which indicates entanglement, while for $h \le \sqrt{\gamma}$, $\xi_1^2 \ge 1$, and thus the system is not spin squeezed [Fig. 1(d)]. For the isotropic case, there is no spin squeezing. In Fig. 2, we plot χ^2 and ξ_1^2 for $N = 500$ and the thermodynamic limit and find that the numerical results coincide well with the analyti-
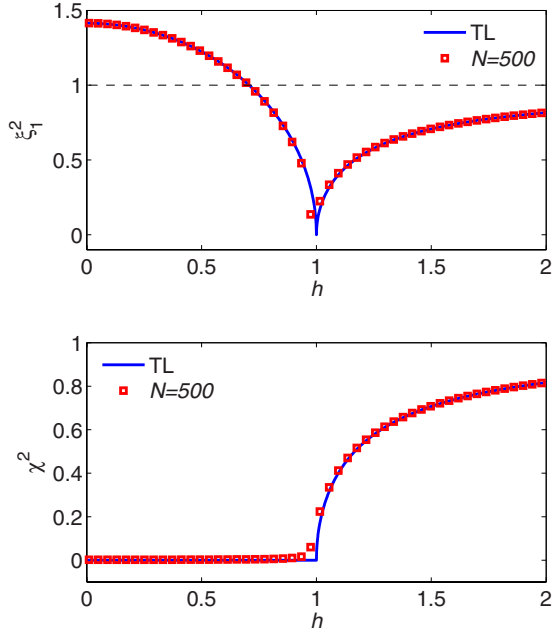
FIG. 2. (Color online) Comparison between analytical and numerical results for ξ_1^2 and χ^2 in the case of $\gamma = 1/2$. Solid lines are analytical results for the thermodynamic limit (TL).

cal ones obtained by H-P transformation method.

As shown in Figs. 3 and 4, χ^2 is nearly independent of larger N in the symmetric phase and approaches to $1/N$ as h being away from the critical point in the broken phase. Therefore, entanglement characterized by χ^2 is very different in the two phases, especially when we treat them as resources for quantum estimation. In the symmetric phase, as shown in Fig. 3, χ^2 is nearly independent of N and the parameter sensitivity is at the level of $(\Delta\theta)_{SN}$, while in the broken phase, the ground states are more sensitive to parameter. In Fig. 4, we show numerical results for χ^2 in the broken phase at $h=1/2$, $\gamma=1/2$, and we see clearly that $\chi^2 \propto 1/N$. Therefore, the parameter estimation in the broken phase is enhanced to the Heisenberg limit.

One can use concurrence [28], entanglement entropy [30], single-copy entanglement, and geometric entanglement [31]
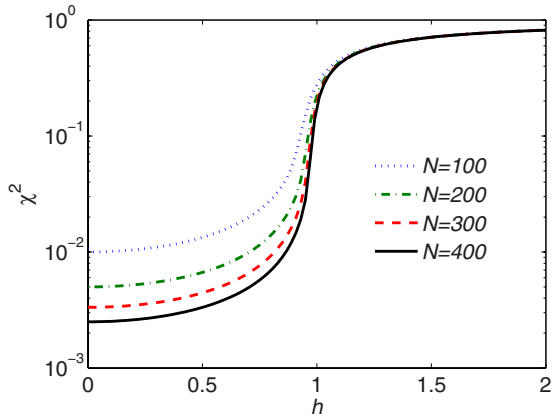

FIG. 3. (Color online) Plot of χ^2 as a function of h with $\gamma = 1/2$ for different system sizes $N = 100, 200, 300, 400$ (from top to bottom). χ^2 is nearly independent of N for $h > 1$, while it drops with the increasing of N in the broken phase, $h < 1$.
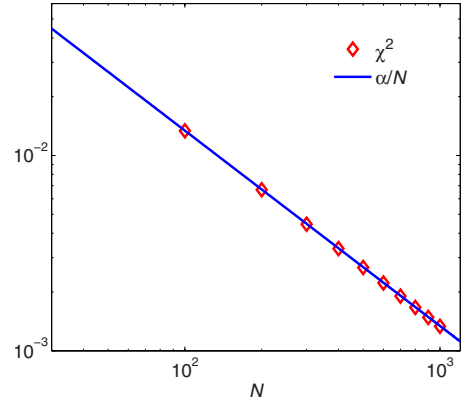

FIG. 4. (Color online) Scaling of $1/\chi^2$ as a function of N with $h=\gamma=1/2$. Diamonds are numerical results, while the solid line is a linear fit with α a constant. It is obvious that $\chi^2 \propto 1/N$.

to quantity the entanglement in the LMG model, while these quantities cannot tell us whether the entanglement of the ground state is useful in parameter estimation. From the results of χ^2, we can see that the entanglements in these two phases are different according to their performances in estimation. On one hand, we can use the collective behavior of the LMG model to improve the phase estimation precision and on the other hand, the differences of the parameter sensitivities can be used to characterize and distinguish the two quantum phases.

IV. CONCLUSION

We have analyzed χ^2 and spin squeezing parameters ξ_i^2 in the ground state of the LMG model. For isotropic case, the Hamiltonian is diagonal in Dicke states, for which we have $\xi_2^2 \geq \xi_1^2 = 1/\chi^2$. For anisotropic case, our results indicate that χ^2 classifies states in different phases in the sense of quantum phase estimation. Hence we can use χ^2 to distinguish and characterize the behaviors of the two phases of the LMG model. In the symmetric phase, χ^2 is independent of N and approaches to 1 with the increase of h, thus $(\Delta\theta)_{QCR} \sim (\Delta\theta)_{SN}$, that is just a bit lower than the shot-noise limit. In the broken phase, we find $\chi^2 \simeq 1/N$ and $(\Delta\theta)_{QCR} \simeq 1/N$, which attains the Heisenberg limit.

Fisher information, being related to the Cramer-Rao inequality, is used to measure how much information that we know about some certain parameters in a probability distribution. From present results, we see that Fisher information can also characterize the QPT by distinguishing the entangled ground states in the sense of parameter sensitivity. This approach is promising and expected to be applicable to other spin systems undergoing a QPT.

ACKNOWLEDGMENTS

[1] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976).

[2] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982).

[3] J. J. Bollinger, Wayne M. Itano, D. J. Wineland, and D. J. Heinzen, Phys. Rev. A **54**, R4649 (1996).

[4] S. F. Huelga, C. Macchiavello, T. Pellizzari, A. K. Ekert, M. B. Plenio, and J. I. Cirac, Phys. Rev. Lett. **79**, 3865 (1997).

[5] A. Peters, K. Y. Chung, and S. Chu, Nature (London) **400**, 849 (1999).

[6] R. Jozsa, D. S. Abrams, J. P. Dowling, and C. P. Williams, Phys. Rev. Lett. **85**, 2010 (2000).

[7] M. Hübner, Phys. Lett. A **163**, 239 (1992); **179**, 226 (1993).

[8] M. Hayashi, *Quantum Information: An Introduction* (Springer-Verlag, Berlin, 2006).

[9] D. Bures, Trans. Am. Math. Soc. **135**, 199 (1969).

[10] A. Uhlmann, Rep. Math. Phys. **9**, 273 (1976); **24**, 229 (1986).

[11] W. K. Wootters, Phys. Rev. D **23**, 357 (1981).

[12] S. L. Braunstein and C. M. Caves, Phys. Rev. Lett. **72**, 3439 (1994).

[13] B. Yurke, S. L. McCall, and J. R. Klauder, Phys. Rev. A **33**, 4033 (1986).

[14] J. P. Dowling, Phys. Rev. A **57**, 4736 (1998).

[15] P. Kok, S. L. Braunstein, and J. P. Dowling, J. Opt. B: Quantum Semiclassical Opt. **6**, S811 (2004).

[16] V. Giovannetti, S. Lloyd, and L. Maccone, Science **306**, 1330 (2004).

[17] V. Giovannetti, S. Lloyd, and L. Maccone, Phys. Rev. Lett. **96**, 010401 (2006).

[18] S. Boixo, S. T. Flammia, C. M. Caves, and J. M. Geremia, Phys. Rev. Lett. **98**, 090401 (2007).

[19] S. M. Roy and S. L. Braunstein, Phys. Rev. Lett. **100**, 220501 (2008).

[20] S. Boixo, A. Datta, M. J. Davis, S. T. Flammia, A. Shaji, and C. M. Caves, Phys. Rev. Lett. **101**, 040403 (2008).

[21] H. F. Hofmann, Phys. Rev. A **79**, 033822 (2009).

[22] M. Rosenkranz and D. Jaksch, Phys. Rev. A **79**, 022103 (2009).

[23] J. Estève, C. Gross, A. Weller, S. Giovanazzi, and M. K. Oberthaler, Nature (London) **455**, 1216 (2008).

[24] G. R. Jin and S. W. Kim, Phys. Rev. Lett. **99**, 170405 (2007).

[25] L. Pezzé and A. Smerzi, Phys. Rev. Lett. **102**, 100401 (2009).

[26] H. J. Lipkin, N. Meshkov, and A. J. Glick, Nucl. Phys. **62**, 188 (1965).

[27] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, England, 1999); M. Vojta, Rep. Prog. Phys. **66**, 2069 (2003).

[28] S. Dusuel and J. Vidal, Phys. Rev. B **71**, 224420 (2005).

[29] J. Vidal, G. Palacios, and R. Mosseri, Phys. Rev. A **69**, 022107 (2004).

[30] T. Barthel, S. Dusuel, and J. Vidal, Phys. Rev. Lett. **97**, 220402 (2006).

[31] R. Orús, S. Dusuel, and J. Vidal, Phys. Rev. Lett. **101**, 025701 (2008).

[32] C. Invernizzi, M. Korbman, L. C. Venuti, and M. G. A. Paris, Phys. Rev. A **78**, 042106 (2008).

[33] P. Zanardi, M. G. A. Paris, and L. Campos Venuti, Phys. Rev. A **78**, 042105 (2008).

[34] M. Kitagawa and M. Ueda, Phys. Rev. A **47**, 5138 (1993).

[35] D. J. Wineland, J. J. Bollinger, W. M. Itano, and D. J. Heinzen, Phys. Rev. A **50**, 67 (1994).

[36] D. J. Wineland, J. J. Bollinger, W. M. Itano, F. L. Moore, and D. J. Heinzen, Phys. Rev. A **46**, R6797 (1992).

[37] P. G. Petrov, D. Oblak, Carlos L. Garrido Alzar, N. Kjærgaard, and E. S. Polzik, Phys. Rev. A **75**, 033803 (2007).

[38] S. Chaudhury, G. A. Smith, K. Schulz, and P. S. Jessen, Phys. Rev. Lett. **96**, 043001 (2006).

[39] A. Sørensen, L. M. Duan, J. I. Cirac, and P. Zoller, Nature (London) **409**, 63 (2001).

[40] X. Wang and B. C. Sanders, Phys. Rev. A **68**, 012101 (2003).

[41] R. Botet, R. Jullien, and P. Pfeuty, Phys. Rev. Lett. **49**, 478 (1982).

[42] J. I. Cirac, M. Lewenstein, K. Mølmer, and P. Zoller, Phys. Rev. A **57**, 1208 (1998).

[43] D. A. Garanin, X. Martinez Hídalgo, and E. M. Chudnovsky, Phys. Rev. B **57**, 13639 (1998).

[44] H.-M. Kwok, W.-Q. Ning, S.-J. Gu, and H.-Q. Lin, Phys. Rev. E **78**, 032103 (2008).

[45] J. Ma, L. Xu, H.-N. Xiong, and X. Wang, Phys. Rev. E **78**, 051126 (2008).

[46] F. Pan and J. P. Draayer, Phys. Lett. B **451**, 1 (1999).

[47] J. Links, H.-Q. Zhou, R. H. McKenzie, and M. D. Gould, J. Phys. A **36**, R63 (2003).

[48] J. I. Latorre, R. Orús, E. Rico, and J. Vidal, Phys. Rev. A **71**, 064101 (2005).

[49] T. Holstein and H. Primakoff, Phys. Rev. **58**, 1098 (1940).

[50] F. Wegner, Ann. Phys. **3**, 77 (1994).

[51] S. D. Głazek and K. G. Wilson, Phys. Rev. D **48**, 5863 (1993).

[52] S. D. Głazek and K. G. Wilson, Phys. Rev. D **49**, 4214 (1994).