

Entanglement cost of nonlocal measurements

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For certain joint measurements on a pair of spatially separated particles, we ask how much entanglement is needed to carry out the measurement exactly. For a class of orthogonal measurements on two qubits with partially entangled eigenstates, we present upper and lower bounds on the entanglement cost. The upper bound is based on a recent result by Berry [Phys. Rev. A **75**, 032349 (2007)]. The lower bound, based on the entanglement production capacity of the measurement, implies that for almost all measurements in the class we consider, the entanglement required to perform the measurement is strictly greater than the average entanglement of its eigenstates. On the other hand, we show that for any complete measurement in $d \times d$ dimensions that is invariant under all local generalized Pauli operations, the cost of the measurement is exactly equal to the average entanglement of the states associated with the outcomes.

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I. INTRODUCTION

In this paper we ask how much entanglement is required to perform a measurement on a pair of spatially separated systems if the participants are allowed only local operations and classical communication (LOCC). That is, we want to find the “entanglement cost” of a given measurement. (We give a precise definition of this term in Sec. I A.) Our motivation can be traced back to a 1999 paper entitled “Quantum nonlocality without entanglement,” which presents a complete orthogonal measurement that cannot be performed using only LOCC, even though the eigenstates of the measurement are all unentangled [1]. That result shows that there can be a kind of nonlocality in a quantum measurement that is not captured by the entanglement of the associated states. Here we wish to quantify this nonlocality for specific measurements. Though the measurements we consider here have outcomes associated with *entangled* states, we find that the entanglement cost of the measurement often exceeds the entanglement of the states themselves.

The 1999 paper just cited obtained an upper bound on the cost of the specific nonlocal measurement presented there, a bound that has recently been improved and generalized by Cohen [2]. In addition, there are in the literature at least three other avenues of research that bear on the problem of finding the entanglement cost of nonlocal measurements. First, there are several papers that simplify or extend the results in Ref. [1], for example, by finding other examples of measurements with product-state outcomes that cannot be carried out locally [2–12]. A related line of research asks whether or not a given set of orthogonal bipartite or multipartite states (not necessarily a complete basis and not necessarily unentangled) can be distinguished by LOCC [7,13–31], and if not, how well one *can* distinguish the states by such means [32–36]. Finally, a number of authors have investigated the cost in entanglement or the entanglement production capacity of various bipartite and multipartite operations [37–54].

In this paper we consider three specific cases: (i) a class of orthogonal measurements on two qubits, in which the four eigenstates are equally entangled, (ii) a somewhat broader

class of orthogonal measurements with unequal entanglements, and (iii) a general, nonorthogonal, bipartite measurement in $d \times d$ dimensions that is invariant under all local generalized Pauli operations. For the first of our three cases we present upper and lower bounds on the entanglement cost. For the second case we obtain a lower bound, and for the last case we compute the cost exactly: it is equal to the average entanglement of the states associated with the outcomes. Throughout the paper, we mark our main results as propositions.

The upper bound in case (i) can be obtained directly from a protocol devised by Berry [40]—a refinement of earlier protocols [39,44]—for performing a closely related nonlocal unitary transformation. Our bound is therefore the same as Berry’s bound. However, because we are interested in performing a measurement rather than a unitary transformation, we give an alternative protocol consisting of a sequence of local measurements.

To get our lower bounds, we use a method developed in papers on the local distinguishability of bipartite states [15,16,55]. The average entanglement between two parties cannot be increased by LOCC, so in performing the measurement, the participants must consume at least as much entanglement as the measurement can produce. This fact is the basis of all except one of our lower bounds. The one exception is in Sec. III, where we use a more stringent condition, a bound on the success probability of local entanglement manipulation, to put a tighter bound on the cost for a limited class of procedures.

A. Statement of the problem

To define the entanglement cost, we imagine two participants, Alice and Bob, each holding one of the two objects to be measured. We allow them to do any sequence of local operations and classical communication, but we do not allow them to transmit quantum particles from one location to the other. Rather, we give them, as a resource, arbitrary shared entangled states, and we keep track of the amount of entanglement they consume in performing the measurement.

At this point, though, we have a few options in defining the problem. Do we try to find the cost of performing the measurement only once or do we imagine that the same measurement will be performed many times (on many different pairs of qubits) and look for the asymptotic cost per trial? And how do we quantify the amount of entanglement that is used up? In this paper we imagine that Alice and Bob will perform the given measurement only once. (In making this choice we are following Cohen [2].) However, we suppose that this measurement is one of many measurements they will eventually perform (not necessarily repeating any one of the measurements and not necessarily knowing in advance what the future measurements will be), and we assume that they have a large supply of entanglement from which they will continue to draw as they carry out these measurements. In this setting it makes sense to use the standard measure of entanglement for pure states, namely, the entropy of either of the two parts [56]. Thus, for a pure state $|\psi\rangle$ of a bipartite system AB , the entanglement is

$$\mathcal{E}(|\psi\rangle) = -\text{tr } \rho_A \log \rho_A, \tag{1}$$

where ρ_A is the reduced density matrix of particle A : $\rho_A = \text{tr}_B |\psi\rangle\langle\psi|$. In this paper, the logarithm will always be base two, so the entanglement is measured in ebits. By means of local operations and classical communication, Alice and Bob can create from their large supply of entanglement any specific state that they need. For example, if they create and completely use up a copy of the state $|\phi_c^+\rangle = c|00\rangle + d|11\rangle$, this counts as a cost of $\mathcal{E}(|\phi_c^+\rangle) = -(c^2 \log c^2 + d^2 \log d^2)$. On the other hand, if their procedure converts an entangled state into a less entangled state, the cost is the difference, that is, the amount of entanglement lost.

A general measurement is specified by a positive-operator-valued measure (POVM), that is, a collection of positive semidefinite operators Π_i that sum to the identity, each operator being associated with one of the outcomes of the measurement. In this paper we restrict our attention to *complete* measurements, that is, measurements in which each operator is of rank one, so each Π_i is of the form $\alpha_i |\phi_i\rangle\langle\phi_i|$ for some α_i in the range $0 < \alpha_i \leq 1$. In a complete *orthogonal* measurement, each operator is a projection operator ($\alpha_i = 1$) that projects onto a single vector (an eigenvector $|\phi_i\rangle$ of the measurement). Now, actually performing a measurement will always entail performing some operation on the measured system. All that we require of this operation is that Alice and Bob both end up with an accurate classical record of the outcome of the measurement. In particular, we do not insist that the measured system be collapsed into some particular state or even that it survive the measurement.

We allow the possibility of probabilistic measurement procedures, in which the probabilities might depend on the initial state of the system being measured. However, we do not want our quantification of the cost of a measurement to depend on this initial state; we are trying to characterize the measurement itself, not the system on which it is being performed. So we assume that Alice and Bob are initially completely ignorant of the state of the particles they are measuring. That is, the state they initially assign to these particles is the completely mixed state. This is the state we will use in

computing any probabilities associated with the procedure.

Bringing together the above considerations, we now give the definition of the quantity we are investigating in this paper. Given a POVM M , let $\mathcal{P}(M)$ be the set of all LOCC procedures P such that (i) P uses pure entangled pairs, local operations, and classical communication and (ii) P realizes M exactly in the sense that for any initial state of the system to be measured, P yields classical outcomes with probabilities that agree with the probabilities given by M . Then $C(M)$, the entanglement cost of a measurement M , is defined to be

$$C(M) = \inf_{P \in \mathcal{P}(M)} \langle \mathcal{E}_{\text{initial}} - \mathcal{D}_{\text{final}} \rangle, \tag{2}$$

where $\mathcal{E}_{\text{initial}}$ is the total entanglement of all the resource states used in the procedure, $\mathcal{D}_{\text{final}}$ is the distillable entanglement of the state remaining at the end of the procedure [57,58], and $\langle \dots \rangle$ indicates an average over all the possible results of P when the system on which the measurement is being performed is initially in the completely mixed state [59]. (Though we allow and take into account the possibility of some residual entanglement $\mathcal{D}_{\text{final}}$, in all the procedures we consider explicitly in this paper, the entanglement in the resource states will, in fact, be used up completely.)

A different notion of the entanglement cost of a measurement is considered in Ref. [54], namely, the amount of entanglement needed to effect a Naimark extension of a given POVM. In that case the entanglement is between the system on which the POVM is to be performed and an ancillary system needed to make the measurement orthogonal. For any orthogonal measurement, and indeed for all the measurements considered in this paper, the entanglement cost in the sense of Ref. [54] is zero.

B. Measurements and unitary transformations

One way to perform a nonlocal orthogonal measurement on a bipartite system is to perform a nonlocal unitary transformation that takes the eigenstates of the desired measurement into the standard basis, so that the measurement can then be finished locally. (We will use this fact in Sec. II.) So one might wonder whether the problem we are investigating in this paper, at least for the case of orthogonal measurements, is equivalent to the problem of finding the cost of a nonlocal unitary transformation. A simple example shows that the two problems are distinct.

Suppose that Alice holds two qubits, labeled A' and A , and Bob holds a single qubit labeled B . They want to perform an orthogonal measurement having the following eight eigenstates:

$$\begin{aligned} (1/\sqrt{2})(|000\rangle + |011\rangle), & \quad |100\rangle, \\ (1/\sqrt{2})(|000\rangle - |011\rangle), & \quad |101\rangle, \\ (1/\sqrt{2})(|001\rangle + |010\rangle), & \quad |110\rangle, \\ (1/\sqrt{2})(|001\rangle - |010\rangle), & \quad |111\rangle. \end{aligned} \tag{3}$$

Here the order of the qubits in each ket is A' , A , B . Alice and Bob can carry out this measurement by the following

protocol: Alice measures qubit A' in the standard basis. If she gets the outcome $|1\rangle$, she and Bob can finish the measurement locally. If, on the other hand, she gets the outcome $|0\rangle$, she uses up 1 ebit to teleport the state of qubit A to Bob, who then finishes the measurement. The average cost of this protocol is $1/2$ ebit because the probability that Alice will need to use an entangled pair is $1/2$.

On the other hand, one can show that any unitary transformation that could change the above basis into the standard basis would be able to create 1 ebit of entanglement and must therefore consume at least 1 ebit. So the cost of the measurement in this case is strictly smaller than the cost of a corresponding unitary transformation.

The crucial difference is that when one does a unitary transformation, one can gain no information about the system being transformed. So there can be no averaging between easy cases and hard cases.

C. Two general bounds on the cost

There are two general bounds on $C(M)$, an upper bound and a lower bound, that apply to all complete bipartite measurements. These bounds are expressed in the following two propositions:

Proposition 1. Let M be a POVM on two objects A and B , having state spaces of dimensions d_A and d_B , respectively. Then $C(M) \leq \min\{\log d_A, \log d_B\}$.

Proof. Let Alice and Bob share, as a resource, a maximally entangled state of two d_A -dimensional objects. They can use this pair to teleport the state of A from Alice to Bob [60], who can then perform the measurement M locally. The entanglement of the resource pair is $\log d_A$. So $\log d_A$ ebits are sufficient to perform the measurement. Similarly, $\log d_B$ ebits would be sufficient to teleport the state of B to Alice. So the cost of M is no greater than $\min\{\log d_A, \log d_B\}$. ■

As we have mentioned, most of our lower bounds are obtained by considering the entanglement production capacity of our measurements. Specifically, we imagine that in addition to particles A and B , Alice and Bob hold, respectively, auxiliary particles C and D . We consider an initial state of the whole system such that the measurement M on AB collapses CD into a possibly entangled state [15,16,55]. The average amount by which the measurement increases the entanglement between Alice and Bob is then a lower bound on $C(M)$. That is,

$$C(M) \geq (\text{average final entanglement of } CD) - (\text{initial entanglement between } AC \text{ and } BD). \tag{4}$$

In the proof of the following proposition, the initial entanglement is zero.

Proposition 2. Let M be a bipartite POVM consisting of the operators $\alpha_i |\phi_i\rangle\langle\phi_i|$, where each $|\phi_i\rangle$ is a normalized state of particles A and B , each of which has a d -dimensional state space. Then $C(M)$ is at least as great as the average entanglement $\langle\mathcal{E}\rangle$ of the states $|\phi_i\rangle$. That is,

$$C(M) \geq \langle\mathcal{E}\rangle \equiv \frac{1}{d^2} \sum_i \alpha_i \mathcal{E}(|\phi_i\rangle). \tag{5}$$

Proof. Let the initial state of $ABCD$ be

$$|\Psi\rangle = \frac{1}{d} \sum_{kl} |kk\rangle_{AC} |ll\rangle_{BD}, \tag{6}$$

a tensor product of two maximally entangled states. Note that the reduced density matrix of particles A and B is the completely mixed state, in accordance with our definition of the problem. When the measurement yields the outcome i , its effect on $|\Psi\rangle$ can be expressed in the form [61]

$$|\Psi\rangle\langle\Psi| \rightarrow \sum_j (A_{ij} \otimes I_{CD}) |\Psi\rangle\langle\Psi| (A_{ij}^\dagger \otimes I_{CD}), \tag{7}$$

where I_{CD} is the identity on CD and the operators A_{ij} act on the state space of particles A and B , telling us what happens to the system when the i th outcome occurs. The trace of the right-hand side of Eq. (7) is not unity but is the probability of the i th outcome. (Note that A_{ij} may send states of AB to a different state space, including, for example, the state space of the system in which the classical record of the outcome is to be stored. The index j is needed because the final state of the system when outcome i occurs could be a mixed state.) The operators A_{ij} satisfy the condition

$$\sum_j A_{ij}^\dagger A_{ij} = \Pi_i = \alpha_i |\phi_i\rangle\langle\phi_i|. \tag{8}$$

Applying the operation of Eq. (7) to the state of Eq. (6) and then tracing out everything except particles C and D , one finds that these particles are left in the state

$$(|\phi_i\rangle\langle\phi_i|)^*, \tag{9}$$

where the asterisk indicates complex conjugation in the standard basis. This conjugation does not affect the entanglement, so, when outcome i occurs, particles C and D are left in a state with entanglement $\mathcal{E}(|\phi_i\rangle)$. The probability of this outcome is α_i/d^2 . So the average entanglement of CD after the measurement has been performed is the quantity $\langle\mathcal{E}\rangle$ of Eq. (5). But the average entanglement between the locations of Alice and Bob cannot have increased as long as Alice and Bob were restricted to local operations and classical communication. So in the process of performing the measurement, Alice and Bob must have used up an amount of entanglement equal to or exceeding $\langle\mathcal{E}\rangle$. ■

In Secs. II–IV we improve these two bounds for a specific measurement that we label M_a , an orthogonal measurement on two qubits with eigenstates given by

$$\begin{aligned} |\phi_a^+\rangle &= a|00\rangle + b|11\rangle, & |\phi_a^-\rangle &= b|00\rangle - a|11\rangle, \\ |\psi_a^+\rangle &= a|01\rangle + b|10\rangle, & |\psi_a^-\rangle &= b|01\rangle - a|10\rangle. \end{aligned} \tag{10}$$

Here a and b are non-negative real numbers with $a \geq b$ and $a^2 + b^2 = 1$. Section II presents an improved upper bound for this measurement, Sec. III derives a lower bound for a restricted class of procedures, and Sec. IV derives an absolute lower bound. We then consider a somewhat more general measurement in Sec. V.

In Sec. VI we exhibit a class of bipartite measurements, in dimension $d \times d$, for which we can find a procedure that achieves the lower bound of Eq. (5). As noted earlier, these

are the POVMs that are invariant under all local generalized Pauli operations.

II. UPPER BOUND FOR M_a

One way to perform the measurement M_a is to perform the following unitary transformation on the two qubits:

$$U = e^{i\alpha\sigma_y \otimes \sigma_x} = \begin{pmatrix} a & 0 & 0 & b \\ 0 & a & b & 0 \\ 0 & -b & a & 0 \\ -b & 0 & 0 & a \end{pmatrix}, \tag{11}$$

where $\cos \alpha = a$ and $\sin \alpha = b$; the matrix is written in the standard basis and the σ 's are the usual Pauli matrices,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{12}$$

Under this transformation, the four orthogonal states that define the measurement M_a are transformed into

$$\begin{aligned} |\phi_a^+\rangle &= a|00\rangle + b|11\rangle \rightarrow |00\rangle, \\ |\phi_a^-\rangle &= b|00\rangle - a|11\rangle \rightarrow -|11\rangle, \\ |\psi_a^+\rangle &= a|01\rangle + b|10\rangle \rightarrow |01\rangle, \\ |\psi_a^-\rangle &= b|01\rangle - a|10\rangle \rightarrow -|10\rangle. \end{aligned} \tag{13}$$

So once the transformation has been done, the measurement M_a can be completed locally; Alice and Bob both make the measurement $|0\rangle$ versus $|1\rangle$ and tell each other their results.

The transformation U is equivalent to one that has been analyzed in Refs. [39,40,44,48,52], all of which give procedures that are consistent with the rules we have set up for our problem; that is, the procedures can be used to perform the measurement once, rather than asymptotically, using arbitrary entangled states as resources. (Some of those papers consider the asymptotic problem, but their procedures also work in the setting we have adopted here.) It appears that the procedure presented by Berry in Ref. [40] is the most efficient one known so far. It is a multistage procedure, involving at each stage a measurement that determines whether another stage and another entangled pair are needed.

We now present a measurement-based protocol for performing M_a . The protocol can be derived from Berry's and yields the same upper bound on the cost, but we arrive at it in a different way that may have conceptual value in the analysis of other nonlocal measurements.

The construction of the protocol begins with the following observations. If Alice were to try to teleport her qubit to Bob using as a resource an incompletely entangled pair, she would cause a nonunitary distortion in its state. With his qubit and Alice's distorted qubit, Bob could, with some probability less than one, successfully complete the measurement. However, if he gets the wrong outcome, he will destroy the information necessary to complete the measurement. We require the measurement always to be completed, so this pro-

ocol fails. On the other hand, suppose Alice, again using a partially entangled pair, performs an *incomplete* teleportation, conveying to Bob only one rather than two classical bits, and suppose Bob similarly makes an incomplete measurement, extracting only one classical bit from his two qubits. In that case, if the incomplete measurements are chosen judiciously, a failure does not render the desired measurement impossible but only requires that Alice and Bob do a different nonlocal measurement on the qubits they now hold. In the following description of the protocol, we have incorporated the unitary transformations associated with teleportation into the measurements themselves, so that the whole procedure is a sequence of local projective measurements.

Like Berry's protocol, our protocol consists a series of rounds, beginning with what we will call "round one."

(1) Alice and Bob are given as a resource the entangled state $|\phi_x^+\rangle = x|00\rangle + y|11\rangle$, where the positive real numbers x and y (with $x^2 + y^2 = 1$) are to be determined by minimizing the eventual cost. Thus each participant holds two qubits: the qubit to be measured and a qubit that is part of the shared resource.

(2) Alice makes a binary measurement on her two qubits, defined by two orthogonal projection operators,

$$\begin{aligned} P &= |\Phi^+\rangle\langle\Phi^+| + |\Psi^-\rangle\langle\Psi^-|, \\ Q &= |\Phi^-\rangle\langle\Phi^-| + |\Psi^+\rangle\langle\Psi^+|. \end{aligned} \tag{14}$$

Here the Bell states $|\Phi^\pm\rangle$ and $|\Psi^\pm\rangle$ are defined by $|\Phi^\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$ and $|\Psi^\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$. Alice transmits (classically) the result of her measurement to Bob. (Here Alice is doing the incomplete teleportation. In a complete teleportation she would also distinguish $|\Phi^+\rangle$ from $|\Psi^-\rangle$ and $|\Phi^-\rangle$ from $|\Psi^+\rangle$.)

(3) If Alice gets the outcome P , Bob performs the following binary measurement on his two qubits:

$$\begin{aligned} P_1 &= |\phi_1^+\rangle\langle\phi_1^+| + |\psi_1^+\rangle\langle\psi_1^+|, \\ Q_1 &= |\phi_1^-\rangle\langle\phi_1^-| + |\psi_1^-\rangle\langle\psi_1^-|. \end{aligned} \tag{15}$$

Here $|\phi_1^+\rangle = A|00\rangle + B|11\rangle$, $|\phi_1^-\rangle = B|00\rangle - A|11\rangle$, $|\psi_1^+\rangle = B|01\rangle + A|10\rangle$, and $|\psi_1^-\rangle = A|01\rangle - B|10\rangle$, and the real coefficients A and B are obtained from (a, b) and (x, y) via the equation $Ax/a = By/b$, together with the normalization condition $A^2 + B^2 = 1$. (These values are chosen so as to undo the distortion caused by Alice's imperfect teleportation.) On the other hand, if Alice gets the outcome Q , Bob performs a different binary measurement,

$$\begin{aligned} P_2 &= |\phi_2^+\rangle\langle\phi_2^+| + |\psi_2^+\rangle\langle\psi_2^+|, \\ Q_2 &= |\phi_2^-\rangle\langle\phi_2^-| + |\psi_2^-\rangle\langle\psi_2^-|. \end{aligned} \tag{16}$$

Here $|\phi_2^+\rangle = B|00\rangle + A|11\rangle$, $|\phi_2^-\rangle = A|00\rangle - B|11\rangle$, $|\psi_2^+\rangle = A|01\rangle + B|10\rangle$, and $|\psi_2^-\rangle = B|01\rangle - A|10\rangle$.

(4) If Alice and Bob have obtained either of the outcomes $P \otimes P_1$ or $Q \otimes Q_2$, which we call the "good" outcomes, they can now finish the desired measurement M_a by making local measurements, with no further expenditure of entangled resources. For example, if they get the outcome $P \otimes P_1$, Alice

now distinguishes between $|\Phi^+\rangle$ and $|\Psi^-\rangle$ (which span the subspace picked out by P) and Bob distinguishes between $|\phi_1^+\rangle$ and $|\psi_1^+\rangle$ (which span the subspace picked out by P_1). The total probability of getting one of the two good outcomes is

$$(\text{probability}) = \frac{1}{(a/x)^2 + (b/y)^2}. \quad (17)$$

On the other hand, if they have obtained one of the other two outcomes, $P \otimes Q_1$ or $Q \otimes P_2$ —the “bad” outcomes—they find that in order to finish the measurement M_a on their *original* pair of qubits, they now have to perform a different measurement M_{a_2} on the system that they now hold. (Even though each participant started with two qubits, each of them has now distinguished a pair of two-dimensional subspaces, effectively removing one qubit’s worth of quantum information. So the remaining quantum information on each side can be held in a single qubit.) The measurement M_{a_2} has the same form as M_a but with new values a_2 and b_2 instead of a and b . The new values are determined by the equations

$$a_2 = \frac{(x^2 - y^2)ab}{\sqrt{x^4 b^2 + y^4 a^2}}, \quad b_2 = \sqrt{1 - a_2^2}. \quad (18)$$

In any case, Alice and Bob have now finished round one. If they have obtained one of the bad outcomes, they now have two choices: (i) begin again at step 1 but with the new values a_2 and b_2 or (ii) use up a whole ebit to teleport Alice’s system to Bob, who finishes the measurement locally. They choose the method that will ultimately be less costly in entanglement. If they choose option (i), we say that they have begun round two.

(5) This procedure is iterated until the measurement is finished or until L rounds have been completed, where L is an integer chosen in advance. In round j , the measurement parameter a_j is determined from the parameters a_{j-1} and x_{j-1} used in the preceding round according to Eq. (18) (with the appropriate substitutions). Here a_1 and x_1 are to be interpreted as the first-round values a and x .

(6) If L rounds are completed and the measurement is still unfinished, Alice teleports her system to Bob, who finishes the measurement locally.

The entanglement used in stage j of this procedure is $\mathcal{E}(|\phi_{x_j}^+\rangle) = h(x_j^2)$, where h is the binary entropy function $h(z) = -[z \log z + (1-z) \log(1-z)]$. From Eqs. (17) and (18), we therefore have the following upper bound on the cost of the measurement M_a .

Proposition 3. For each positive integer j , let x_j satisfy $0 < x_j < 1$. We define the functions $F(a, x)$ (failure probability) and $a'(a, x)$ (new value of the measurement parameter) as follows:

$$F(a, x) = 1 - \frac{1}{(a/x)^2 + (b/y)^2},$$

$$a'(a, x) = \frac{(x^2 - y^2)ab}{\sqrt{x^4 b^2 + y^4 a^2}}, \quad (19)$$

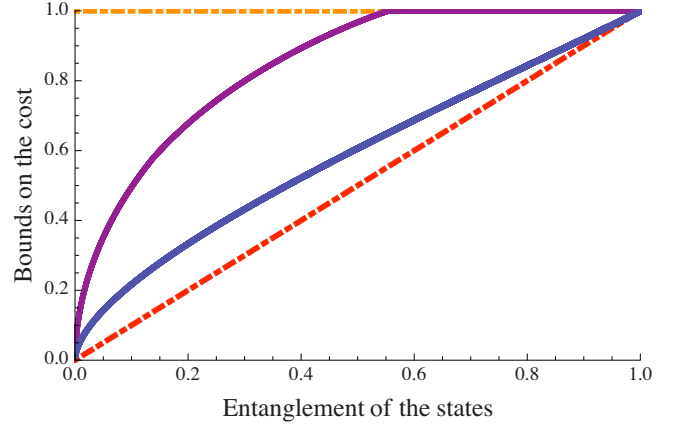


FIG. 1. (Color online) The solid curves are upper and lower bounds on the entanglement cost of the measurement M_a . (The derivation of the lower bound is in Sec. IV.) The diagonal dashed line (red) is the general lower bound defined by the entanglement of the states themselves and the horizontal dashed line (orange) is the general upper bound based on teleportation.

where $y = (1 - x^2)^{1/2}$ and $b = (1 - a^2)^{1/2}$. Let $B_1(a; x) = h(x^2) + F(a, x)$, and for each integer $n \geq 2$, let $B_n(a; x_1, \dots, x_n)$ be defined by

$$B_n(a; x_1, \dots, x_n) = h(x_1^2) + F(a, x_1) B_{n-1}(a'(a, x_1); x_2, \dots, x_n). \quad (20)$$

Then for each positive integer n , $B_n(a; x_1, \dots, x_n)$ is an upper bound on $C(M_a)$.

The protocol calls for minimizing the bound over the values of n and x_j . This optimization problem is exactly the problem analyzed by Berry. We present in Fig. 1 the minimal cost as obtained by a numerical optimization, plotted as a function of the entanglement of the eigenstates of the measurement. (In constructing the curve, we have limited Alice and Bob to two rounds. Additional rounds do not make a noticeable difference in the shape of the curve, given our choice of the axis variables.) We also show on the figure the lower bound to be derived in Sec. IV. We note that so far, for cases in which the entanglement of the eigenstates of M_a exceeds around 0.55 ebits, there is no known measurement strategy that does better than simple teleportation, with a cost of 1 ebit.

III. LIMITATION TO A SINGLE ROUND

As it happens, most of the savings in the above strategy—compared to the cost of simple teleportation—already appears in the first round. We now consider the single-round case in more detail. It turns out that, at least for small values of the entanglement of the eigenstates, we can determine quite precisely the minimal cost of the measurement M_a when Alice and Bob are restricted to a single round.

We begin by defining the class of measurement strategies we consider in this section. A “single-round procedure” is a measurement procedure of the following form. (i) Alice and Bob are given the state $x|00\rangle + y|11\rangle$ at first, with which they try to complete the measurement. (ii) If they use this re-

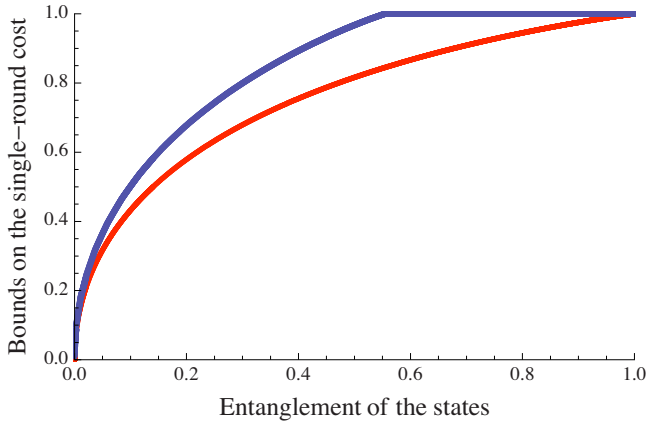


FIG. 2. (Color online) Upper and lower bounds on the entanglement cost of the measurement M_a when Alice and Bob are restricted to a single round before resorting to teleportation.

source but fail to carry out the measurement, Alice teleports a qubit to Bob, who finishes the measurement locally. (For the procedure outlined in Sec. II, this restriction amounts to setting L equal to 1.) We refer to the minimum entanglement cost entailed by any such procedure as the “single-round cost.” In this section we find upper and lower bounds on the single-round cost of M_a .

The minimal cost of the specific procedure outlined in Sec. II, when it is restricted to a single round, is given by

$$(\text{cost}) = h(x^2) + \left[1 - \frac{1}{(a/x)^2 + (b/y)^2} \right], \quad (21)$$

where the value of x is chosen so as to minimize the cost. (Here $x^2 + y^2 = 1$ as before.) The two terms of Eq. (21) are easy to interpret: the first term is the entanglement of the shared resource that is used up in any case, and the second term, obtained from Eq. (17), is the probability of failure (multiplied by the 1 ebit associated with the resulting teleportation). Numerically minimizing the cost over values of x , we obtain the upper curve in Fig. 2, which is thus an upper bound on the single-round cost of M_a . The same upper bound was obtained by Ye *et al.* [44] for performing the corresponding nonlocal unitary transformation.

We can also find a good lower bound for such procedures using a known upper bound on the probability of achieving a certain increase in the entanglement of a single copy through local operations and classical communication [62,63]. As in all our lower-bound arguments, we consider a state in which qubits A and B are initially entangled with auxiliary qubits C and D , which will not be involved in the measurement. (As before, Alice holds qubits A and C and Bob holds B and D .) For our present purpose, we choose the initial state to be

$$|\xi\rangle = \frac{1}{2} [|\phi_a^+\rangle_{AB} |\phi_c^+\rangle_{CD} + |\phi_a^-\rangle_{AB} |\phi_c^-\rangle_{CD} + |\psi_a^+\rangle_{AB} |\psi_c^+\rangle_{CD} + |\psi_a^-\rangle_{AB} |\psi_c^-\rangle_{CD}]. \quad (22)$$

Here the states with the index c are defined as in Eq. (10) but with c and d in place of a and b . We assume for definiteness that $1 > c > d > 0$, c and d to be determined later. Note that again the reduced state of qubits A and B , after tracing out

the auxiliary qubits, is the completely mixed state, as it must be to be consistent with our definition of the entanglement cost. One can show directly from Eq. (22) that the eigenvalues of the density matrix of Alice’s (or Bob’s) part of the system, that is, the squared Schmidt coefficients, are

$$(ac + bd)^2 \quad \text{and} \quad (ad - bc)^2. \quad (23)$$

In addition to these qubits, Alice and Bob hold their entangled resource, which we can take without loss of generality to be in the state

$$|\phi_x^+\rangle = x|00\rangle + y|11\rangle. \quad (24)$$

They now try to execute the measurement by using up this resource.

If Alice and Bob succeed in distinguishing the four states $\{|\phi_a^+\rangle, |\phi_a^-\rangle, |\psi_a^+\rangle, |\psi_a^-\rangle\}$, they will have collapsed qubits C and D into one of the four corresponding states represented in $|\xi\rangle$. Each of these states has Schmidt coefficients c^2 and d^2 . Using a result of Jonathan and Plenio [63], we can place an upper bound on the probability of achieving the transformation from the state $|\xi\rangle \otimes |\phi_x^+\rangle$ to one of the four desired final states of qubits C and D . This probability cannot be larger than

$$\frac{\sum_{j=\ell}^4 \alpha_j}{2} \cdot \frac{\sum_{j=\ell} \beta_j}{2}, \quad (25)$$

where α_j and β_j are, respectively, the squared Schmidt coefficients of the initial state and any of the desired final states in decreasing order. (There are at most four nonzero Schmidt coefficients in the initial state; hence the upper limit 4 in the numerator. Similarly, the upper limit 2 in the denominator reflects the fact that the final state, a state of C and D , has at most two nonzero Schmidt coefficients.) In general, ℓ can take any value from 1 to the number of nonzero Schmidt coefficients of the final state. In our problem there are only two values of ℓ to consider. The case $\ell=1$ tells us only that the probability does not exceed unity, so the only actual constraint comes from the case $\ell=2$, which tells us that

$$(\text{success probability}) \leq \frac{1 - (ac + bd)^2 x^2}{1 - c^2}. \quad (26)$$

The cost of any single-round procedure is therefore at least

$$(\text{cost}) \geq h(x^2) + \max \left\{ 0, \left[1 - \frac{1 - (ac + bd)^2 x^2}{1 - c^2} \right] \right\}, \quad (27)$$

since a failure will lead to a cost of 1 ebit for the teleportation.

Alice and Bob will choose their resource pair, that is, they will choose the value of x , so as to minimize the cost. So we want to find a value of x that minimizes the right-hand side of Eq. (27). Because the probability of failure cannot be less than zero, we can restrict our attention to values of x in the range

$$c/(ac + bd) \leq x \leq 1. \quad (28)$$

In this range, the cost is a concave function of x^2 , so the function achieves its minimum value at one of the two end points. We thus have the following lower bound on the cost of any single-round procedure:

$$(\text{cost}) \geq \min \left\{ h \left[\frac{c^2}{(ac + bd)^2} \right], \frac{(ac + bd)^2 - c^2}{1 - c^2} \right\}. \quad (29)$$

This bound holds for any value of c for which it is defined. To make the bound as strong as possible, we want to maximize it over all values of c . In the range $1/\sqrt{2} \leq c \leq a$, the first entry in Eq. (29) is a decreasing function of c , whereas the second entry is increasing. [For larger values of c , both functions are decreasing until they become undefined at $c = (ac + bd)$. Beyond this point we would violate Eq. (28).] Therefore, we achieve the strongest bound when the two entries are equal. That is, we have obtained the following result.

Proposition 4. The single-round cost of the measurement M_a is bounded below by the quantity

$$\frac{(ac + bd)^2 - c^2}{1 - c^2}, \quad (30)$$

where $d = (1 - c^2)^{1/2}$ and c is determined by the equation

$$h \left[\frac{c^2}{(ac + bd)^2} \right] = \frac{(ac + bd)^2 - c^2}{1 - c^2}. \quad (31)$$

We have solved this equation numerically for a range of values of a and have obtained the lower of the two curves in Fig. 2. For very weakly entangled eigenstates—that is, at the left-hand end of the graph where the parameter b is small—the single-round upper bound and the single-round lower bound shown in the figure are very close to each other. In fact, we find analytically that for small b , both the upper and lower bounds can be approximated by the function $2b\sqrt{\log(1/b)}$ in the sense that the ratio of each bound with this function approaches unity as b approaches zero (see the Appendix for the argument). In terms of the entanglement \mathcal{E} of the states, we can say that for small b , the single-round cost of the measurement is approximately equal to $\sqrt{2}\mathcal{E}$. Thus in this limit, we have a very good estimate of the cost of the measurement but only if we restrict Alice and Bob to a single round. We would prefer to have a lower bound that applies to any conceivable procedure and that is still better than the general lower bound we derived in Sec. I. We obtain such a bound in Sec. IV.

IV. ABSOLUTE LOWER BOUND FOR M_a

Again, we imagine a situation in which Alice and Bob hold two auxiliary qubits, C and D , which will not be involved in the measurement. We assume the same initial state as in Sec. III,

$$|\xi\rangle = \frac{1}{2} [|\phi_a^+\rangle_{AB} |\phi_c^+\rangle_{CD} + |\phi_a^-\rangle_{AB} |\phi_c^-\rangle_{CD} + |\psi_a^+\rangle_{AB} |\psi_c^+\rangle_{CD} + |\psi_a^-\rangle_{AB} |\psi_c^-\rangle_{CD}]. \quad (32)$$

As before, we are interested in the entanglement between Alice's part of the system and Bob's part, that is, between AC and BD . This entanglement is

$$\mathcal{E}_{\text{initial}} = h[(ac + bd)^2]. \quad (33)$$

If Alice and Bob perform the measurement M_a , the final entanglement of CD is

$$\mathcal{E}_{\text{final}} = h(c^2). \quad (34)$$

The quantity $\mathcal{E}_{\text{final}} - \mathcal{E}_{\text{initial}}$ is thus a lower bound on the cost of M_a , as expressed in the following proposition.

Proposition 5. Let c satisfy $0 \leq c \leq 1$ and let $d = (1 - c^2)^{1/2}$. Then $C(M_a) \geq h(c^2) - h[(ac + bd)^2]$.

By maximizing this quantity numerically over the parameter c , we get our best absolute lower bound on the entanglement cost $C(M_a)$. This bound is plotted in Fig. 1. What is most interesting about this bound is that, except at the extreme points where the eigenstates of the measurement are either all unentangled or all maximally entangled, the bound is strictly larger than the entanglement of the eigenstates themselves. This is another example, then, showing that the nonseparability of the measurement can exceed the nonseparability of the states that the measurement distinguishes.

Not only is our new lower bound absolute in the sense that it does not depend on the number of rounds used by Alice and Bob, it applies even asymptotically. Suppose, for example, that Alice and Bob are given n pairs of qubits and are asked to perform the same measurement M_a on each pair. It is conceivable that by using operations that involve all n pairs, Alice and Bob might achieve an efficiency not possible when they are performing the measurement only once. Even in this setting, the lower bound given in proposition 5 applies. That is, the cost of performing the measurement n times must be at least n times our single-copy lower bound. To see this, imagine that *each* of the given pairs of qubits is initially entangled with a pair of auxiliary qubits. Both the initial entanglement of the whole system (that is, the entanglement between Alice's side and Bob's side) and the final entanglement after the measurement are simply proportional to n , so that the original argument carries over to this case.

It is interesting to look at the behavior of the upper and lower bounds as the parameter b approaches zero, that is, as the eigenstates of the measurement approach product states. Berry has done this analysis for the upper bound and has found that for small b , the cost is proportional to b , with proportionality constant 5.6418. For our lower bound, it is a question of finding the value of c (with $c^2 + d^2 = 1$) that maximizes the difference

$$h(c^2) - h[(ac + bd)^2] \quad (35)$$

for small b . One finds that for small b , the optimal value of c approaches the constant value $c = 0.28848$ [the numerical solution to the equation $(d^2 - c^2)\ln(d/c) = 1$], for which the bound is approximately equal to $1.9123b$. Comparing this result with the upper bound in the limit of vanishingly small entanglement, $5.6418b$, we see that there is still a sizable gap between the two bounds.

The same limiting form, $1.9123b$, appears in Ref. [41] as the entanglement production capacity of the unitary transfor-

mation of Eq. (11) for small b . In fact, by extending the argument in Ref. [41] to noninfinitesimal transformations, one obtains the entire lower-bound curve in Fig. 1. Thus our lower bound for the cost of the measurement M_a is also a lower bound for the cost of the corresponding unitary transformation. We note, though, that the two optimization problems are not quite the same. To get a bound on the cost of the measurement, we maximized $h(c^2) - h[(ac+bd)^2]$. To find the entanglement production capacity of the unitary transformation, one maximizes $h[(ac+bd)^2] - h(c^2)$. Though the questions are different, it is not hard to show that maximum value is the same in both cases.

V. EIGENSTATES WITH UNEQUAL ENTANGLEMENTS

We now consider the following variation on the measurement M_a . It is an orthogonal measurement that we call $M_{a,c}$, with eigenstates

$$\begin{aligned} |\phi_a^+\rangle &= a|00\rangle + b|11\rangle, & |\phi_a^-\rangle &= b|00\rangle - a|11\rangle, \\ |\psi_c^+\rangle &= c|01\rangle + d|10\rangle, & |\psi_c^-\rangle &= d|01\rangle - c|10\rangle, \end{aligned} \quad (36)$$

where all the coefficients are real and non-negative and all the states are normalized. For this measurement we again use the entanglement production argument to get a lower bound. In this case we take the initial state of qubits $ABCD$ to be

$$\begin{aligned} |\eta\rangle &= \frac{1}{2} [|\phi_a^+\rangle_{AB} |\phi_{a'}^+\rangle_{CD} + |\phi_a^-\rangle_{AB} |\phi_{a'}^-\rangle_{CD} + |\psi_c^+\rangle_{AB} |\psi_{c'}^+\rangle_{CD} \\ &+ |\psi_c^-\rangle_{AB} |\psi_{c'}^-\rangle_{CD}], \end{aligned} \quad (37)$$

where the real parameters a' and c' are to be adjusted to achieve the most stringent lower bound. This initial state has an entanglement between Alice’s location and Bob’s location (that is, between AC and BD) that is equal to the Shannon entropy of the following four probabilities:

$$\begin{aligned} (aa' + bb' + cc' + dd')^2/4, & \quad (aa' + bb' - cc' - dd')^2/4, \\ (ab' - ba' + dc' - cd')^2/4, & \quad (ab' - ba' - dc' + cd')^2/4. \end{aligned} \quad (38)$$

Once the measurement is completed, the final entanglement of the CD system, on average, is

$$[h(a'^2) + h(c'^2)]/2. \quad (39)$$

The difference between the final entanglement and the initial entanglement is a lower bound on $C(M_{a,c})$, which we want to maximize by our choice of a' and c' . We have again done the maximization numerically for many values of the measurement parameters a and c , covering their domain quite densely. We plot the results in Fig. 3.

In almost every case, the resulting lower bound is higher than the average entanglement of the eigenstates of the measurement. The only exceptions we have found, besides the ones already mentioned in Sec. IV (in which all the states are maximally entangled or all are unentangled), are those for which two of the measurement eigenstates are maximally entangled and the other two are unentangled. That is, this

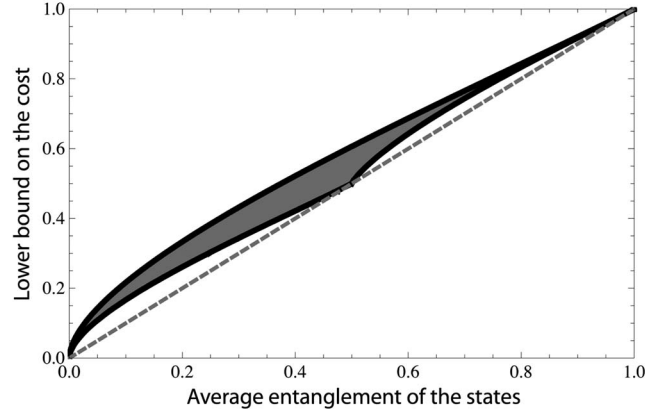


FIG. 3. Lower bound for the measurement $M_{a,c}$ as computed from the pure state given in Eq. (37), plotted against the average entanglement of the eigenstates (dashed line). Different values of the pair (a,c) can be associated with the same point on the horizontal axis but may yield different lower bounds, as indicated by the gray area. The point touching the dashed line in the middle of the graph represents a measurement with two Bell states and two product states as in Eq. (40).

method does not produce a better lower bound for the measurement with eigenstates

$$|\Phi^+\rangle, |\Phi^-\rangle, |01\rangle, |10\rangle \quad (40)$$

or for the analogous measurement with $|\Phi^\pm\rangle$ replaced by $|\Psi^\pm\rangle$ and with the product states suitably replaced to make the states mutually orthogonal. In all other cases the cost of the measurement is strictly greater than the average entanglement of the states.

The measurement $M_{a,c}$ has been considered in Ref. [15], whose results likewise give a lower bound on the cost: $C(M_{a,b}) \geq 1 - \log(a^2 + c^2)$ (where $a \geq b$ and $c \geq d$). This bound is weaker than the one we have obtained in part because we have followed the later paper (Ref. [16]) in assuming an initial pure state rather than a mixed state of $ABCD$.

VI. MEASUREMENTS FOR WHICH THE GENERAL LOWER BOUND CAN BE ACHIEVED

Here we consider a class of measurements for which the cost equals the average entanglement of the states associated with the POVM elements. We begin with another two-qubit measurement, which we then generalize to arbitrary dimension.

A. Eight-outcome measurement

A measurement closely related to M_a is the measurement $M_a^{(8)}$, which has eight outcomes, represented by a POVM whose elements $\Pi_i = \alpha_i |\phi_i\rangle\langle\phi_i|$ all have $\alpha_i = 1/2$, with the eight states $|\phi_i\rangle$ given by

$$\begin{aligned} |\phi_a^+\rangle &= a|00\rangle + b|11\rangle, & |\phi_a^-\rangle &= b|00\rangle - a|11\rangle, \\ |\psi_a^+\rangle &= a|01\rangle + b|10\rangle, & |\psi_a^-\rangle &= b|01\rangle - a|10\rangle, \end{aligned}$$

$$\begin{aligned} |\phi_b^+\rangle &= b|00\rangle + a|11\rangle, & |\phi_b^-\rangle &= a|00\rangle - b|11\rangle, \\ |\psi_b^+\rangle &= b|01\rangle + a|10\rangle, & |\psi_b^-\rangle &= a|01\rangle - b|10\rangle. \end{aligned} \quad (41)$$

That is, they are the same states as in M_a , plus the four states obtained by interchanging a and b . Thus, Alice and Bob could perform the measurement $M_a^{(8)}$ by flipping a fair coin to decide whether to perform M_a or M_b . This procedure yields the eight possible outcomes: there are two possible outcomes of the coin toss, and for each one, there are four possible outcomes of the chosen measurement. The coin toss requires no entanglement, so the cost of this procedure is equal to the cost of M_a (which is equal to that of M_b). We conclude that

$$C(M_a^{(8)}) \leq C(M_a). \quad (42)$$

As we will see shortly, the cost of $M_a^{(8)}$ is, in fact, strictly smaller for $0 < a < 1$.

The measurement $M_a^{(8)}$ is a nonorthogonal measurement, but any nonorthogonal measurement can be performed by preparing an auxiliary system in a known state and then performing a global orthogonal measurement on the combined system. We now show explicitly how to perform this particular measurement in a way that will allow us to determine the value of $C(M_a^{(8)})$. To do the measurement, Alice and Bob draw, from their store of entanglement, the entangled state $|\phi_a^+\rangle = a|00\rangle + b|11\rangle$ of qubits C and D . (As always, Alice holds C and Bob holds D .) Then each of them locally performs the Bell measurement $\{|\Phi^+\rangle, |\Phi^-\rangle, |\Psi^+\rangle, |\Psi^-\rangle\}$ on his or her pair of qubits.

The resulting 16-outcome orthogonal measurement on $ABCD$ defines a 16-outcome POVM on just the two qubits A and B . For each outcome k of the global orthogonal measurement, we can find the corresponding POVM element Π_k of the AB measurement as follows:

$$\Pi_k = \text{tr}_{CD}\{\pi_k[I_{AB} \otimes (|\phi_a^+\rangle\langle\phi_a^+|)_{CD}]\}, \quad (43)$$

where π_k is the k th POVM element of the global measurement. Less formally, we can achieve the same result by taking the ‘‘partial inner product’’ between the initial state $|\phi_a^+\rangle$ of the system CD and the k th eigenstate of the global measurement. For example, the eigenstate $|\Phi^+\rangle|\Phi^+\rangle$ yields the following partial inner product:

$$\langle\langle\phi_a^+|_{CD}\rangle\rangle(|\Phi^+\rangle_{AC}|\Phi^+\rangle_{BD}), \quad (44)$$

which works out to be $(1/2)|\phi_a^+\rangle_{AB}$. The corresponding POVM element on the AB system is $(1/4)|\phi_a^+\rangle\langle\phi_a^+|$. Continuing in this way, one finds the following correspondence between the 16 outcomes of the global measurement and the POVM elements of the AB measurement.

$$\begin{aligned} |\Phi^+\rangle|\Phi^+\rangle &\text{ or } |\Phi^-\rangle|\Phi^-\rangle \rightarrow \frac{1}{4}|\phi_a^+\rangle\langle\phi_a^+|, \\ |\Phi^+\rangle|\Phi^-\rangle &\text{ or } |\Phi^-\rangle|\Phi^+\rangle \rightarrow \frac{1}{4}|\phi_b^-\rangle\langle\phi_b^-|, \\ |\Phi^+\rangle|\Psi^+\rangle &\text{ or } |\Phi^-\rangle|\Psi^-\rangle \rightarrow \frac{1}{4}|\psi_a^+\rangle\langle\psi_a^+|, \\ |\Phi^+\rangle|\Psi^-\rangle &\text{ or } |\Phi^-\rangle|\Psi^+\rangle \rightarrow \frac{1}{4}|\psi_b^-\rangle\langle\psi_b^-|, \end{aligned}$$

$$|\Psi^+\rangle|\Psi^+\rangle \text{ or } |\Psi^-\rangle|\Psi^-\rangle \rightarrow \frac{1}{4}|\phi_b^+\rangle\langle\phi_b^+|,$$

$$|\Psi^+\rangle|\Psi^-\rangle \text{ or } |\Psi^-\rangle|\Psi^+\rangle \rightarrow \frac{1}{4}|\phi_a^-\rangle\langle\phi_a^-|,$$

$$|\Psi^+\rangle|\Phi^+\rangle \text{ or } |\Psi^-\rangle|\Phi^-\rangle \rightarrow \frac{1}{4}|\psi_b^+\rangle\langle\psi_b^+|,$$

$$|\Psi^+\rangle|\Phi^-\rangle \text{ or } |\Psi^-\rangle|\Phi^+\rangle \rightarrow \frac{1}{4}|\psi_a^-\rangle\langle\psi_a^-|. \quad (45)$$

Thus, even though there are formally 16 outcomes of the AB measurement, they are equal in pairs, so that there are only eight distinct outcomes, and they are indeed the outcomes of the measurement $M_a^{(8)}$.

The cost of this procedure is $\mathcal{E}(|\phi_a^+\rangle) = h(a^2)$. This is the same as the average entanglement of the eight states representing the outcomes of $M_a^{(8)}$, which we know is a lower bound on the cost. Thus the lower bound is achievable in this case, and we can conclude that $C(M_a^{(8)})$ is exactly equal to $h(a^2)$.

We note that the POVM $M_a^{(8)}$ is invariant under all local Pauli operations. This fact leads us to ask whether, more generally, invariance under such operations guarantees that the entanglement cost of the measurement is exactly equal to the average entanglement of the states associated with the POVM elements. Section VI B shows that this is indeed the case for complete POVMs.

B. Arbitrary complete POVM invariant under local generalized Pauli operations

We begin by considering a POVM on a bipartite system of dimension $d \times d$, generated by applying generalized Pauli operators to a single pure state $|\phi_0\rangle$. The POVM elements are of the form $(1/d^2)|\psi_{j_1 k_1 j_2 k_2}\rangle\langle\psi_{j_1 k_1 j_2 k_2}|$, where

$$|\psi_{j_1 k_1 j_2 k_2}\rangle = (Z^{j_1} X^{k_1} \otimes Z^{j_2} X^{k_2})|\phi_0\rangle \quad (46)$$

and each index runs from 0 to $d-1$. Here the generalized Pauli operators X and Z are defined by

$$X|m\rangle = |m+1\rangle, \quad Z|m\rangle = \omega^m|m\rangle, \quad m = 0, \dots, d-1, \quad (47)$$

with $\omega = \exp(2\pi i/d)$ and with the addition understood to be mod d . One can verify that the above construction generates a POVM for any choice of $|\phi_0\rangle$.

In order to carry out this POVM, Alice and Bob use, as a resource, particles C and D in the state $|\phi_0\rangle^*$, which has the same entanglement as $|\phi_0\rangle$. (As before, the asterisk indicates complex conjugation in the standard basis.) Alice performs on AC , and Bob performs on BD , the generalized Bell measurement whose eigenstates are

$$|B_{jk}\rangle = \frac{1}{\sqrt{d}} Z^j \otimes X^k \sum_{r=0}^{d-1} |r, r\rangle. \quad (48)$$

To see that this method does effect the desired POVM, we compute the partial inner products as in Sec. VI A,

$$\begin{aligned}
& \langle \langle \phi_0^*_{CD} | (|B_{j_1 k_1}\rangle_{AC} \otimes |B_{j_2 k_2}\rangle_{BD}) \\
&= \frac{1}{d} \sum_{r_1, r_2} \langle \phi_0^*_{CD} Z_A^{j_1} X_C^{k_1} Z_B^{j_2} X_D^{k_2} | r_1, r_1 \rangle_{AC} | r_2, r_2 \rangle_{BD} \\
&= \frac{1}{d} \sum_{r_1, r_2} Z_A^{j_1} Z_B^{j_2} | r_1, r_2 \rangle_{AB} \langle \phi_0^*_{CD} X_C^{k_1} X_D^{k_2} | r_1, r_2 \rangle_{CD} \\
&= \frac{1}{d} Z_A^{j_1} Z_B^{j_2} \sum_{r_1, r_2} | r_1, r_2 \rangle_{AB} \langle r_1, r_2 |_{CD} X_C^{k_1} X_D^{k_2} | \phi_0 \rangle_{CD} \\
&= \frac{1}{d} Z_A^{j_1} Z_B^{j_2} X_A^{k_1} X_B^{k_2} | \phi_0 \rangle_{AB} = \frac{1}{d} (Z^{j_1} X^{k_1} \otimes Z^{j_2} X^{k_2}) | \phi_0 \rangle_{AB}.
\end{aligned} \tag{49}$$

Thus the combination of Bell measurements yields the POVM defined by Eq. (46).

We now extend this example to obtain the following result.

Proposition 6. Let M be any complete POVM with a finite number of outcomes, acting on a pair of systems each having a d -dimensional state space, such that M is invariant under all local generalized Pauli operations, that is, under the group generated by $X \otimes I$, $Z \otimes I$, $I \otimes X$, and $I \otimes Z$. Then $C(M)$ is equal to the average entanglement of the states associated with the outcomes of M , as expressed in Eq. (5).

Proof. The most general such POVM is similar to the one we have just considered, except that instead of a single starting state $|\phi_0\rangle$, there may be an ensemble of states $|\phi_s\rangle$ with weights p_s , $s=1, \dots, m$, such that $\sum p_s=1$. The POVM elements (of which there are a total of md^4) are $(p_s/d^2)|\psi_{j_1 k_1 j_2 k_2; s}\rangle\langle\psi_{j_1 k_1 j_2 k_2; s}|$, where

$$|\psi_{j_1 k_1 j_2 k_2; s}\rangle = (Z^{j_1} X^{k_1} \otimes Z^{j_2} X^{k_2}) |\phi_s\rangle. \tag{50}$$

[So p_s/d^2 plays the role of α_i in Eq. (5).] In order to perform this measurement, Alice and Bob first make a random choice of the value of s , using the weights p_s . They then use, as a resource, particles C and D in the state $|\phi_s\rangle^*$ and perform Bell measurements as above. The cost of this procedure is the average entanglement of the resource states, which is

$$(\text{cost}) = \sum_s p_s \mathcal{E}(|\psi_s\rangle) = \frac{1}{d^2} \sum_{j_1 k_1 j_2 k_2 s} \frac{p_s}{d^2} \mathcal{E}(|\phi_{j_1 k_1 j_2 k_2; s}\rangle) = \langle \mathcal{E} \rangle. \tag{51}$$

But we know that $\langle \mathcal{E} \rangle$ is a lower bound on $C(M)$. Since the above procedure achieves this bound, we have that $C(M) = \langle \mathcal{E} \rangle$. ■

VII. DISCUSSION

As we discussed in Sec. I, a general lower bound on the entanglement cost of a complete measurement is the average entanglement of the pure states associated with the measurement's outcomes. Perhaps the most interesting result of this paper is that, for almost all the orthogonal measurements we considered, the actual cost is strictly greater than this lower bound. The same is true in the examples of “nonlocality without entanglement,” in which the average entanglement is

zero but the cost is strictly positive. However, whereas those earlier examples may have seemed special because of their intricate construction, the examples given here are quite simple. The fact that the cost in these simple cases exceeds the average entanglement of the states suggests that this feature may be a generic property of bipartite measurements. If this is true, then in this sense the nonlocality of a measurement is generically a distinct property from the nonlocality of the eigenstates. (A related fact is that there exists a large class of separable two-qubit operations that cannot be carried out locally [10]. It is also interesting in this connection that for certain questions of distinguishability of generic bipartite states, the presence or absence of entanglement seems to be completely irrelevant [31].)

We have also found a class of measurements for which the entanglement cost is *equal* to the average entanglement of the corresponding states. These measurements have a high degree of symmetry in that they are invariant under all local generalized Pauli operations.

What is it that causes some measurements to be “more nonlocal” than the states associated with their outcomes? Evidently the answer must have to do with the *relationships* among the states. In the original nonlocality without entanglement measurement, the crucial role of these relationships is clear: in order to separate any eigenstate $|v\rangle$ from any other eigenstate $|w\rangle$ by a local measurement, the observer must disturb some of the other states in such a way as to render them indistinguishable. One would like to have a similar understanding of the “interactions” among states when the eigenstates are entangled. Some recent papers have quantified relational properties of ensembles of bipartite states [35,64]. Perhaps one of these approaches, or a different approach yet to be developed, will capture the aspect of these relationships that determines the cost of the measurement.

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APPENDIX: THE ONE-ROUND COST IN THE LIMIT OF SMALL ENTANGLEMENT

1. Lower bound

Our lower bound on the one-round cost is given by Eqs. (30) and (31), which we rewrite here in an equivalent form,

$$(\text{cost}) \geq \frac{(ac + bd)^2 - c^2}{d^2}, \tag{A1}$$

where d is determined by the equation

$$h\left[\frac{(ac+bd)^2-c^2}{(ac+bd)^2}\right] = \frac{(ac+bd)^2-c^2}{d^2}. \quad (\text{A2})$$

For a small value of the parameter b , we would like to obtain an approximation to the value of d that solves Eq. (A2). As discussed in Sec. III, we are looking for a solution in the range $b \leq d \leq 1/\sqrt{2}$ and the forms of the functions in Eq. (A2) guarantee that there will be a unique solution in this range. One can show that within this range, the right-hand side of Eq. (A2) satisfies the inequalities

$$\frac{b}{d} \leq \frac{(ac+bd)^2-c^2}{d^2} \leq \frac{2b}{d}. \quad (\text{A3})$$

Applying these same inequalities to the argument of the function h on the left-hand side of Eq. (A2), we have

$$\frac{bd}{(ac+bd)^2} \leq \frac{(ac+bd)^2-c^2}{(ac+bd)^2} \leq \frac{2bd}{(ac+bd)^2}. \quad (\text{A4})$$

For sufficiently small b , the function h evaluated at the values appearing in Eq. (A4) is an increasing function, so we can write

$$h\left[\frac{bd}{(ac+bd)^2}\right] \leq h\left[\frac{(ac+bd)^2-c^2}{(ac+bd)^2}\right] \leq h\left[\frac{2bd}{(ac+bd)^2}\right]. \quad (\text{A5})$$

We can bound the entropies to obtain

$$-bd \log b \leq h\left[\frac{(ac+bd)^2-c^2}{(ac+bd)^2}\right] \leq -16bd \log b. \quad (\text{A6})$$

Combining Eqs. (A2), (A3), and (A6), we get

$$-\frac{1}{2} \log b \leq \frac{1}{d^2} \leq -16 \log b. \quad (\text{A7})$$

Thus d goes to zero as b goes to zero, but it does so much more slowly.

We now use this observation to approximate each side of Eq. (A2). First, in the entropy function $h(x) = -x \log x - (1-x) \log(1-x)$, for very small x we can ignore the second term, so that Eq. (A2) can be simplified to

$$-\log\left[\frac{(ac+bd)^2-c^2}{(ac+bd)^2}\right] \approx \frac{(ac+bd)^2-c^2}{d^2}. \quad (\text{A8})$$

Now, with b very small and d of order $1/\sqrt{\log(1/b)}$, we can approximate $(ac+bd)^2-c^2$ as

$$(ac+bd)^2-c^2 \approx 2bd. \quad (\text{A9})$$

So the equation becomes $-\log(2bd) \approx \frac{1}{d^2}$, but since $-\log 2d$ becomes negligible compared to $-\log b$, we can just as well write

$$-\log b \approx \frac{1}{d^2}. \quad (\text{A10})$$

Finally, the lower bound given by Eq. (A1) becomes

$$(\text{lower bound}) \approx \frac{2bd}{d^2} \approx 2b\sqrt{\log(1/b)}. \quad (\text{A11})$$

All of our approximations have been such that the ratio between the approximating function and the exact function approaches unity as b approaches zero. So the same is true of the approximate expression $2b\sqrt{\log(1/b)}$ relative to the exact lower bound.

2. Upper bound

Our upper bound for the single-round cost [Eq. (21)] is the minimum over q in the range $0 \leq q \leq 1/2$ of the function

$$f(q) = h(q) + g(q), \quad (\text{A12})$$

where

$$g(q) = 1 - \frac{1}{[a^2/(1-q)] + (b^2/q)}. \quad (\text{A13})$$

[Here q is playing the role of y^2 in Eq. (21).] The function $g(q)$ decreases monotonically from the value 1 at $q=0$ to its minimum value $2ab/(1+2ab)$ at $q=b/(a+b)$. Thus the minimum value of $g(q)$ approaches zero for small b and is attained arbitrarily close to $q=0$. Therefore for sufficiently small b , the function $g(q)$, as it falls to its minimum value, falls farther than $h(q)$ rises and the minimum value of $f(q)$ is less than 1. This minimum value is attained at some value of q —call it q_0 —which is less than $b/(a+b)$. [Beyond that point both $h(q)$ and $g(q)$ are increasing for $q < 1/2$.] More simply, $q_0 < b$. So we can limit our attention to values of q less than b .

With this limitation, for small b we can approximate the function $f(q)$ as

$$f(q) \approx -q \log q + \frac{q^2 + b^2}{q + b^2}. \quad (\text{A14})$$

Setting the derivative of this function equal to zero, we find that q_0 can be made arbitrarily close (in the sense that the fractional error can be made arbitrarily small) to a solution of

$$\frac{b^2}{(q+b^2)^2} = -\log q. \quad (\text{A15})$$

For small b there are two solutions to this equation with $q < b$. The smaller one, with q of order $\exp(-1/b^2)$, corresponds to a local *maximum* of $f(q)$, reflecting the fact that the slope of $h(q)$ approaches positive infinity as q approaches zero, whereas the competing negative slope of $g(q)$ is finite at $q=0$. The other solution, with q approximately equal to $b/\sqrt{\log(1/b)}$, is therefore the one we want. At this value we have $f(q) \approx 2b\sqrt{\log(1/b)}$. Again, the approximation is such that the ratio of the exact upper bound to this approximate value approaches unity as b approaches zero.

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