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Barrier interaction time and the Salecker-Wigner quantum clock: Wave-packet approach

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The time-of-flight measurement approach of Peres based on the Salecker-Wigner quantum clock is applied to the one-dimensional scattering of a wave packet from a rectangular barrier. By directly evaluating the expectation value of the clock-time operator in the asymptotic states of wave packet long after the scattering process, we derive an average wave-packet clock time for the barrier interaction, which is expressed as an average of the stationary-state clock time over all possible initial scattering states of the wave packet. We show that the average wave-packet clock time is identical to the average dwell time of a wave packet.

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I. INTRODUCTION

The questions "how long does it take for a particle to tunnel through a potential barrier?" and "how much time does a particle spend in a potential barrier?" are still open problems. Although there have been large volume of theoretical literature [1–3] and some experimental works [4–7], the attempts to answer these questions raised many controversial definitions and a complete consensus has not been reached yet. One of the main theoretical approaches to these problems is to propose an operational definition of time using a physical clock. Many types of clocks have been proposed [8] to define and measure the time associated with tunneling (or scattering) of a particle. Among them, the Salecker-Wigner quantum clock [9] has been a subject of interest.

The application of the Salecker-Wigner quantum clock to the measurement of time was first proposed by Peres [10]. He used the quantum clock as playing a role of stopwatch to measure the time of flight of a free particle in a specified region. Subsequently, Davies [11] studied the same problem within relativistic regime. Later Leavens and McKinnon [12] applied the Peres' approach to the scattering of a particle from a one-dimensional potential barrier and showed that the quantum-clock approach can produce a dwell time for the one-dimensional scattering of a particle, but may have difficulties to give physical meanings of the separated transmission and reflection times. Recently, Davies [13] also applied the Peres' approach to study the tunneling times of a particle in simple models of potential step and barrier.

All of these studies, however, treated the scattering and tunneling problems within the stationary-state regime. For a realistic and rigorous investigation of the scattering of a particle from a potential barrier, it is of course necessary to adopt wave packets rather than stationary states [14]. Foden and Stevens [15] employed a wave packet to argue the validity of applying quantum clock to the measurement of tunneling time, but did not give a full account of wave-packet analysis for the quantum-clock approach to the tunneling time. In this paper, we exploit the Peres' approach based on the Salecker-Wigner quantum clock to study the scattering of a wave packet from a one-dimensional rectangular barrier.

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The plan of this paper is as follows. In Sec. II, we recapitulate the construction of the Salecker-Wigner quantum clock and Peres' application to the time-of-flight measurement for a free particle. We then extend the Peres' approach to the one-dimensional scattering of a particle from a rectangular barrier to find stationary-state solutions. Using these stationary-state solutions as the basis set for the expansion of a wave packet, we evaluate the expectation value of the clock-time operator in wave-packet states in Sec. III. By analyzing the asymptotic behaviors of the time-dependent wave packets long after the scattering process, we obtain an average wave-packet clock time for the interaction, which is expressed as the average of a stationary-state clock time over all possible scattering states. In Sec. IV, we derive an explicit expression of the stationary-state clock time for a particle in a particular scattering state, then compare the average wave-packet clock time to the average dwell time obtained from the dwell-time operator and discuss the similarity between them. Finally, there will be a brief summary of the present work in Sec. V

II. STATIONARY-STATE SOLUTIONS OF ONE-DIMENSIONAL SCATTERING

The Salecker-Wigner quantum clock modified by Peres [10] can be constructed from the following complete sets of orthonormal states:

$$|\chi_s\rangle = \frac{1}{\sqrt{N}}\sum_{n=-J}^{J} e^{-in\theta_s}|u_n\rangle, \tag{1}$$

where $N=2J+1$ with J being a positive integer represents the total number of clock states, $\theta_s=2\pi s/N$ $(s=0,\ldots,N-1)$, and $|u_n\rangle$ are the eigenstates of the angular-momentum operator $\hat{L}=-i\hbar\,\partial/\partial\theta$ satisfying $\hat{L}|u_n\rangle=n\hbar|u_n\rangle$ and $\langle u_n|u_{n'}\rangle=\delta_{nn'}$, from which the orthonormal relation $\langle\chi_s|\chi_{s'}\rangle=\delta_{ss'}$ can be deduced. In the angle representation [16,17], the corresponding wave functions are described by $u_n(\theta)=\langle\theta|u_n\rangle=(2\pi)^{-1/2}e^{in\theta}$ and $\chi_s(\theta)=\langle\theta|\chi_s\rangle=(2\pi N)^{-1/2}\sum_{n=-J}^{J}e^{in(\theta-\theta_s)}$, where the states $|\theta\rangle$ are the eigenstates of the operator $\hat{\Theta}$ satisfying $\hat{\Theta}|\theta\rangle=\theta|\theta\rangle$ and $\langle\theta|\theta'\rangle=\delta(\theta-\theta')$ and the eigenvalue θ is a continuous variable defined in the range $[0,2\pi]$. The eigenfunctions $u_n(\theta)$'s span a finite $(2J+1)$-dimensional

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Hilbert space and satisfy the periodic boundary condition $u_n(0)=u_n(2\pi)$. The clock wave function $\chi_s(\theta)$ displays a peak at $\theta=\theta_s$ with an accuracy of $2\pi/N$.

Introducing the clock Hamiltonian ($\hat{H}_c$) and the clock-time operator ($\hat{T}_c$),

$$\hat{H}_c \equiv \omega\hat{L} \quad (\omega \equiv 2\pi/N\tau), \tag{2}$$

$$\hat{T}_c \equiv \sum_{s=0}^{N-1} t_s|\chi_s\rangle\langle\chi_s| \quad (t_s = \tau s), \tag{3}$$

where τ is the time resolution of the clock, one can see that the states $|\chi_s\rangle$ and $|u_n\rangle$ satisfy eigenvalue equations

$$\hat{H}_c|u_n\rangle = \epsilon_n|u_n\rangle \quad (\epsilon_n = n\hbar\omega, n = 0, \pm 1, \ldots, \pm J), \tag{4}$$

$$\hat{T}_c|\chi_s\rangle = t_s|\chi_s\rangle \quad (s = 0, \ldots, N-1). \tag{5}$$

Note that the angle eigenvalue θ_s is related to the clock-time eigenvalue t_s: $\theta_s=2\pi s/N=\omega\tau s=\omega t_s$. Translations in time of the clock are then operated by the evolution operator $\hat{U}_c(t)=e^{-i\hat{H}_c t/\hbar}$, that is, $\hat{U}_c(t=t_s)|\chi_0\rangle=|\chi_s\rangle$.

Peres [10] applied the quantum clock to measure the time for a free particle to spend in a specified region, say $x_1<x<x_2$, provided that the clock runs only when the particle resides in that region. With the same assumption, the clock can be used to measure the time for the scattering of a particle from a potential barrier. Let us consider a particle with energy $E=\hbar^2k^2/2m$ incident on a one-dimensional rectangular potential barrier located in the region $[-d/2,d/2]$. We require that the clock runs only when the particle is inside the barrier region. The Hamiltonian for the particle plus clock is then given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \Theta(d/2-|x|)\hat{V}_0 + \Theta(d/2-|x|)\hat{H}_c, \tag{6}$$

where V_0 is the height of potential barrier and $\Theta(x)$ is the Heaviside step function. If we assume the particle and the clock are initially uncoupled, the initial particle plus clock states are expressed as $|\psi_k\rangle=|\varphi_k\rangle|\chi_0\rangle$, where $|\varphi_k\rangle$ and $|\chi_0\rangle$ are the initial particle and clock eigenstates, respectively. After scattering, the particle and the clock coordinates are coupled, so that the total eigenstates are given by

$$|\psi_k\rangle = \frac{1}{\sqrt{N}}\sum_{n=-J}^{J}|\varphi_{kn}\rangle|u_n\rangle, \tag{7}$$

where $|\varphi_{kn}\rangle$ are the particle eigenstates after the scattering. For a stationary state with energy E, the time development of $|\psi_k\rangle$ is $|\psi_k(t)\rangle=e^{-iEt/\hbar}|\psi_k\rangle$ and thus the Schrödinger equation for the state $|\psi_k\rangle$ becomes $\hat{H}|\psi_k\rangle=E|\psi_k\rangle$. Multiplying $\langle x|\langle u_n|$ on the left of the equation, applying the eigenvalue Eq. (4), and using the orthonormal property $\langle u_n|u_{n'}\rangle=\delta_{nn'}$, we can write the Schrödinger equation for a given n as

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \Theta(d/2-|x|)(V_0+\epsilon_n)\right]\varphi_{kn}(x) = E\varphi_{kn}(x). \tag{8}$$

Note that the eigenvalues ϵ_n can be incorporated into the potential barrier, so that the particle experiences an effective potential of $\Theta(d/2-|x|)(V_0+\epsilon_n)$. The stationary-state solutions $\varphi_{kn}(x)$ of the equation can be readily obtained by the continuity conditions at boundaries

$$\varphi_{kn}(x) = \frac{1}{\sqrt{2\pi}}\begin{cases} e^{ikx}+R_n(k)e^{-ikx} & x<-d/2 \\ B_n(k)e^{iq_nx}+C_n(k)e^{-iq_nx} & |x|\le d/2 \\ T_n(k)e^{ikx} & x>d/2, \end{cases} \tag{9}$$

where $q_n=\sqrt{k^2-K_{0n}^2}$, with $K_{0n}=\sqrt{2m(V_0+\epsilon_n)}/\hbar$ and $k=\sqrt{2mE}/\hbar$. The transmission and reflection amplitudes $T_n(k)$ and $R_n(k)$ are given by

$$T_n(k) = \frac{4kq_ne^{-ikd}}{Q_n(k)} = \frac{2ikq_n}{K_{0n}^2\sin(q_nd)}R_n(k),$$

$$Q_n(k) = (k+q_n)^2e^{-iq_nd} - (k-q_n)^2e^{iq_nd}. \tag{10}$$

In the above solutions, we have chosen δ-function normalized particle eigenstates so that $\langle\varphi_{kn}|\varphi_{k'n}\rangle=\delta(k-k')$, which gives the overall factor of $1/\sqrt{2\pi}$. In the next section, the eigenstates $|\psi_k\rangle$ with the stationary-state solutions of Eq. (9) will be employed as a basis set for the expansion of a wave packet associated with the scattering of a particle from the rectangular barrier with quantum clock. We then use the time-dependent form of the wave packet to evaluate the expectation value of the clock-time operator $\hat{T}_c$.

III. AVERAGE WAVE-PACKET CLOCK TIME OF INTERACTION

For wave-packet approach, we start with expressing a wave packet for the particle plus clock state as a superposition of eigenstates $|\psi_k\rangle$ of the Hamiltonian $\hat{H}$,

$$|\psi\rangle = \int_{-\infty}^{\infty}dk\,a(k)|\psi_k\rangle = \frac{1}{\sqrt{N}}\sum_{n=-J}^{J}\int_{-\infty}^{\infty}dk\,a(k)|\varphi_{kn}\rangle|u_n\rangle. \tag{11}$$

For our discussion, we choose a normalized Gaussian wave packet with average momentum of $\hbar k_0$ and position uncertainty of $1/2\Delta k$ and its center is initially located at $x=-x_0$ far left from the potential barrier. Thus a particle described by this packet initially moves toward the barrier from the left with mean energy $E_0=\hbar^2k_0^2/2m$. We also require that there are no reflected and transmitted wave packets at $t=0$. The initial momentum amplitude $a(k)$ corresponding to this wave packet is given by

$$a(k) = \left(\frac{1}{2\pi\Delta k^2}\right)^{1/4}e^{-(k-k_0)/4\Delta k^2}e^{ikx_0}. \tag{12}$$

The time-dependent form of the wave packet at time t is then

$$|\psi(t)\rangle = \frac{1}{\sqrt{N}}\sum_{n=-j}^{j}\int_{-\infty}^{\infty} dka(k)e^{-iE_k t/\hbar}|\varphi_{kn}\rangle|u_n\rangle, \quad (13)$$

where $E_k = \hbar^2k^2/2m$. From the orthonormal conditions $\langle\varphi_{kn}|\varphi_{k'n}\rangle = \delta(k-k')$ and $\langle u_n|u_{n'}\rangle = \delta_{nn'}$, one can see $\langle\psi(t)|\psi(t)\rangle = 1$.

In the present quantum-clock approach, the average barrier interaction time can be obtained by taking average over an ensemble of large number of identical replicas of the scattering experiment with quantum clock. Mathematically, this is equivalent to evaluate the expectation value of the clock-time operator $\hat{T}_c$ in the superposed states $|\psi(t)\rangle$,

$$\langle\tau_c\rangle = \langle\psi(t)|\hat{T}_c|\psi(t)\rangle = \sum_{s=0}^{N-1} t_s|\langle\chi_s|\psi(t)\rangle|^2. \quad (14)$$

Substituting Eqs. (1) and (13) into Eq. (14), using the property $\langle u_n|\chi_s\rangle = \langle\chi_s|u_n\rangle^* = e^{-in\omega t_s}/\sqrt{N}$, and inserting the closure relation $\int dx|x\rangle\langle x|$ where particle eigenstates $|\varphi_{kn}\rangle$ appear, we can write

$$\langle\tau_c\rangle = \frac{1}{N^2}\sum_{s=0}^{N-1} t_s \sum_{n,l=-J}^{J} e^{i(l-n)\omega t_s}\int\int dkdk' a^*(k)a(k')$$
$$\times e^{i(E_k - E_{k'})t/\hbar}\int_{-\infty}^{\infty} dx\varphi^*_{kn}(x)\varphi_{k'l}(x). \quad (15)$$

We now recall that the quantum clock runs only when the particle is inside the barrier region so that it retains permanent record of the time for the scattering of a particle. This implies that the recorded clock time of scattering can be read any time after the scattering process has completed. In the following, we shall consider wave packets in the long-time asymptotic limit after having finished the scattering process. For intermediate (or transient) times, both incident and reflected wave packets exist in the reflection region $[-\infty, -d/2]$ and hence the integrand of the x integral in Eq. (15) comprises four terms: incident term, two interference terms, and reflection term. For sufficiently large times, however, the packets associated with the incident and the two interference terms will eventually disappear [18]. Thus, in the limit of large times, we are left with two asymptotic wave packets: the transmitted and reflected packets. From this argument, we can perform the integration over x in Eq. (15) by retaining only the following particle eigenfunctions: $\varphi_{kn}(x) = R_n(k)e^{-ikx}/\sqrt{2\pi}$ for $x < -d/2$ and $\varphi_{kn}(x) = T_n(k)e^{ikx}/\sqrt{2\pi}$ for $x > d/2$, where $R_n(k)$ and $T_n(k)$ are given in Eq. (10). After arranging terms, we have

$$\langle\tau_c\rangle = \frac{1}{2\pi N^2}\sum_{s=0}^{N-1} t_s\sum_{n,l=-J}^{J} e^{i(l-n)\omega t_s}\int_{-\infty}^{\infty} dka^*(k)e^{iE_k t/\hbar}e^{-ikd/2}$$
$$\times[R^*_n(k)I_{Rl} + T^*_n(k)I_{Tl}], \quad (16)$$

where I_{Tl} and I_{Rl} are defined as

$$I_{Al} = \frac{i}{(2\pi\Delta k^2)^{1/4}}\int_{\mathcal{C}} dk'\frac{A_l(k')e^{\gamma(k')}}{k'-k+i0^+}, \quad (A = T,R), \quad (17)$$

with

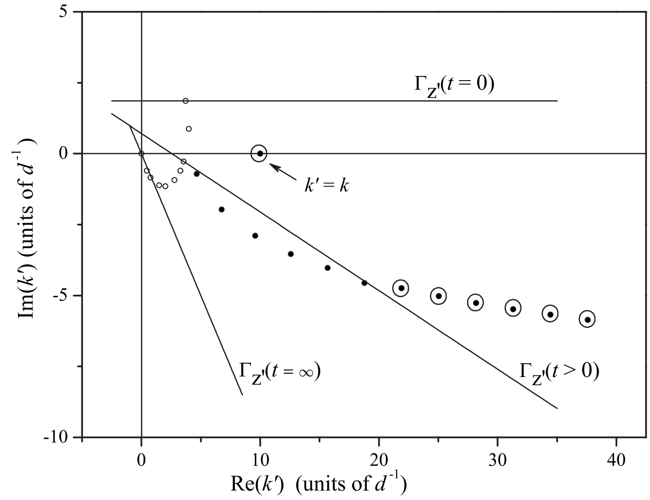

FIG. 1. Saddle points k_s (open circles), steepest-descent paths [straight lines denoted by $\Gamma_{z'}(t)$'s], and the poles k_j (solid points) of $T_l(k')$ and $R_l(k')$ for barrier with $V_0 = 0.5$ eV, $d = 1.12$ nm, and electron wave packet with $E_0 = 0.86\ V_0$, $x_0 = 25d$, and $\Delta k = 0.073k_0$ $(k_0 = \sqrt{2m_eE_0}/\hbar)$. The equation of line for $\Gamma_{z'}(t)$ is given by $k'_I = \tan\phi(k'_R - k_{sR}) + k_{sI}$ (units of d^{-1}). As time (units of $m_e/2\hbar\Delta k^2$) passes, the saddle point tends to zero and the steepest-descent path approaches the line passing the origin with slope -1. The residues (large circles with point centers) associated with the poles having been passed by $\Gamma_{z'}(t)$ contribute to the integral. The contour for each of the residues is in the clockwise direction.

$$\gamma(k') = -\frac{(k'-k_0)^2}{4\Delta k^2} - \frac{i\hbar t}{2m}k'^2 + i(x_0 + d/2)k'. \quad (18)$$

The integration contour $\mathcal{C}$ in Eq. (17) is from $-\infty$ to ∞ and closed with an infinite semicircle in the upper half of the complex k' plane, but excludes any poles from $T_l(k')$ [or $R_l(k')$] to ensure that initially, there are no transmitted and reflected packets [19]. The poles of $T_l(k')$ and $R_l(k')$ can be found by solving the algebraic equation $Q_n(k') = 0$ from Eq. (10), which gives infinite number of simple poles. All of the poles lie in the lower half of the complex k' plane (see Fig. 1).

The integral over k' in Eq. (17) can be carried out by the method of steepest descents [20]. For this, we first complete the square in Eq. (18) and change the variable as

$$z' = \frac{(1+\beta^2)^{1/4}}{2\Delta k}e^{-i\phi}(k' - k_s), \quad (19)$$

with

$$\phi = \tan^{-1}\left[\frac{1}{\beta}(1 - \sqrt{1+\beta^2})\right], \quad (20)$$

$$k_s = \frac{(1+\eta\beta) + i(\eta - \beta)}{1+\beta^2}k_0. \quad (21)$$

For convenience of notation, we have introduced dimensionless parameters for time (β) and position (η),

$$\beta \equiv \frac{2\hbar\Delta k^2}{m}t, \quad \eta \equiv \frac{2\Delta k^2}{k_0}(x_0 + d/2). \tag{22}$$

In the above equations, k_s is the saddle point at which the new variable z' is zero. The phase ϕ determines the slope (i.e., tan ϕ) of the steepest-descent path along which z' becomes real. Thus, by changing the variable from k' to z', the integration contour $\mathcal{C}$ is deformed into the real axis (the steepest-descent path) in the complex z' plane whose origin is at k_s. Note that z', ϕ, and k_s all depend on the time parameter β. Using the new variable, we can express Eq. (17) as

$$I_{Al} = \frac{e^{f_R + if_I}}{(2\pi\Delta k^2)^{1/4}} \int_{\Gamma_{z'}} dz' \frac{A_l(z')}{z' - z} e^{-z'^2} \quad (A = T,R), \tag{23}$$

where $\Gamma_{z'}$ is the deformed contour, f_R and f_I are defined as

$$f_R = -\frac{k_0^2(\eta - \beta)^2}{4\Delta k^2(1+\beta^2)}, \quad f_I = \frac{k_0^2[2\eta - (1-\eta^2)\beta]}{4\Delta k^2(1+\beta^2)}, \tag{24}$$

and z is the variable corresponding to the real value k given as

$$z = \frac{(1+\beta^2)^{1/4}}{2\Delta k} e^{-i\phi}(k - i0^+ - k_s). \tag{25}$$

In the following, we shall proceed our analysis with $A_l(k')$ unless it is necessary to express $T_l(z')$ and $R_l(z')$ explicitly. Since the integrand $A_l(z')(z'-z)^{-1}$, apart from the exponential term, has only simple poles as singularities, we can expand the function at each of the poles and express it as a series [21] such that

$$\frac{A_l(z')}{z'-z} = \frac{A_l(z)}{z'-z} + \frac{A_l(z)}{z} + \sum_{j=1}^{\infty}\left[\frac{r_j}{z'-z_j} + \frac{r_j}{z_j}\right] - \frac{A_l(0)}{z}, \tag{26}$$

where $z_j = (1+\beta^2)^{1/4}e^{-i\phi}(k_j - k_s)/2\Delta k$ with k_j being the poles of $T_l(k')$ and $R_l(k')$ and r_j are the residues associated with k_j given by

$$r_j = \frac{8\pi i z_j \sqrt{z_j^2 - z_{0n}^2}\, e^{-iz_j d}}{(z_j - z)Q_n'(z_j)} \quad \text{for} \quad T_l(z'), \tag{27}$$

$$r_j = \frac{2\pi z_{0n}^2 \sin(\sqrt{z_j^2 - z_{0n}^2}\, d) e^{-iz_j d}}{(z_j - z)Q_n'(z_j)} \quad \text{for} \quad R_l(z'), \tag{28}$$

with $z_{0n} = (1+\beta^2)^{1/4}e^{-i\phi}(K_{0n} - k_s)/2\Delta k$ and $Q_n'(z_j) = [dQ_n(z')/dz']_{z'=z_j}$. Note also that the last term in Eq. (26) is from $z'=0$, that is, the saddle point. Substitution of Eq. (26) into Eq. (23) yields

$$I_{Al} = \frac{e^{f_R + if_I}}{(2\pi\Delta k^2)^{1/4}}\left[I_z + \sum_j I_j + I_{sd}\right], \tag{29}$$

where

$$I_z = A_l(z)\left[\int_{\Gamma_{z'}} dz' \frac{e^{-z'^2}}{z'-z} + \frac{\sqrt{\pi}}{z}\right], \tag{30}$$

$$I_j = r_j\left[\int_{\Gamma_{z'}} dz' \frac{e^{-z'^2}}{z'-z_j} + \frac{\sqrt{\pi}}{z_j}\right], \tag{31}$$

$$I_{sd} = -A_l(z'=0)\frac{\sqrt{\pi}}{z} = -A_l(k_s)\frac{\sqrt{\pi}}{z}. \tag{32}$$

The contour $\Gamma_{z'}$ runs from $-\infty$ to $+\infty$ along the steepest-descent path where z' takes real values. The equation of line for the contour $\Gamma_{z'}$ in the complex k' plane is given by $k_I' = \tan\phi(k_R' - k_{sR}) + k_{sI}$, where k_R' and k_I' are the real and imaginary parts of the variable k' and k_{sR} and k_{sI} are the real and imaginary parts of the saddle point k_s, respectively. Because of the time dependences of ϕ and k_s, the line of the steepest-descent path changes with time. In Fig. 1, we illustrate some lines of $\Gamma_{z'}(t)$ for different times together with the saddle points. It starts with zero slope at $t=0$ and approaches the line with slope tan $\phi=-1$ as $t\to\infty$. Between the two limiting times, $\Gamma_{z'}(t)$ crosses the poles k_j as time passes, leaving residues associated with the poles in the upper half of the complex z' plane. Thus, each pole will contribute to the integral when it has been passed by the steepest-descent path. At short times, since the slope is very small, the steepest-descent path crosses only a small number of k_j's. At large times, the steepest-descent path will have passed more k_j's and eventually all of the k_j's, including the real pole at $k'=k$ (see Fig. 1), will be passed as $t\to\infty$. For the analysis of the pole contributions to the integral at large times, we write Eqs. (30) and (31) as

$$I_z = A_l(z)[-2\pi i e^{-z^2} + i\pi w(z) + \sqrt{\pi}/z], \tag{33}$$

$$I_j = r_j[-2\pi i e^{-z_j^2} + i\pi w(z_j) + \sqrt{\pi}/z_j], \tag{34}$$

where $w(u)$ is the *Faddeeva* function [22] defined as

$$w(u) = \frac{1}{i\pi}\int_{-\infty}^{\infty} ds \frac{e^{-s^2}}{s-u} = e^{-u^2}\text{erfc}(-iu) \quad (\text{Im}(u) > 0). \tag{35}$$

Since the function $w(u)$ is defined in the upper-half plane, the first terms in Eqs. (33) and (34) are due to the residues associated with the poles having been passed by the contour $\Gamma_{z'}$ [23]. As we shall see below, these exponential terms are important to the asymptotic behaviors of $I_{Al}(A=T,R)$ for large time.

We now examine the asymptotic behaviors of I_z, I_j, and I_{sd} as $t\to\infty$ (i.e., as $\beta\to\infty$). Preliminary to the investigation, we observe from Eq. (24) that $f_R \sim -k_0^2/4\Delta k^2$ and $f_I \sim O(\beta^{-1})$, which leads to $\exp(f_R + if_I) \sim \exp(-k_0^2/4\Delta k^2)$ in Eq. (29), so that it becomes a constant in time as $t\to\infty$. We also find the asymptotic forms of $\phi(\beta)$, $k_s(\beta)$, and $z(\beta)$ as $t\to\infty$: $\phi\sim-\pi/4$, $k_s \sim (\eta - i)k_0\beta^{-1}$, $z\sim(1+i)k\beta^{1/2}/\sqrt{8}\Delta k$. With these, let us look into the saddle-point contribution. Since k_s approaches zero as $t\to\infty$ (see also Fig. 1), it follows from Eq. (10) that $T_l(k_s)\to 0$ and $R_l(k_s)$ becomes constant. Then, taking account of the asymptotic form of z, we can find $I_{sd} = O(\beta^{-3/2}) = O(t^{-3/2})$ for $T_l(k_s)$ and $I_{sd} = O(\beta^{-1/2})$

$=O(t^{-1/2})$ for $R_l(k_s)$ as $t\to\infty$. Consequently, the saddle point has negligible contribution to I_{Al} at large times.

Next, to investigate contributions from the poles k_j, we first notice that z_j becomes large as $t\to\infty$ from the asymptotic form $z_j\sim(1+i)k_j\beta^{1/2}/\sqrt{8}\Delta k$. Then, using the asymptotic expansion of $w(z_j)$ for large z_j, we may write $w(z_j)\sim i/\sqrt{\pi}z_j$ for large time [24]. Substituting this expression into Eq. (34), we see that the term $i\pi w(z_j)$ cancels the third term, so that $I_j\sim-2\pi i r_j\exp(-z_j^2)$. To examine this remaining asymptotic form, we recall that all of the poles k_j are in the fourth quadrant (see Fig. 1) [25], which allows us to write $k_j=k_{jR}-ik_{jI}$, where k_{jR} and k_{jI} are positive real values. Using these values and from Eqs. (27) and (28), we find

$$\exp(-z_j^2)\sim\exp(-k_{jR}k_{jI}\beta/2\Delta k^2)$$

and

$$r_j\sim\beta^{-1/2}\exp[\pm O(\beta^{1/2})],$$

which reveals the exponential decay of I_j as $t\to\infty$. This result shows that contributions from the poles k_j of $T_l(k')$ and $R_l(k')$ are also negligible at large times.

Finally, for the pole at the real value $k'=k$, by the same analysis as in the case of I_j, we recognize that the second term $i\pi w(z)$ also cancels the third term in Eq. (33) for large z, so that $I_z\sim-2\pi iA_l(z)\exp(-z^2)$. Unlike the previous case, the term $\exp(-z^2)$ does not decay as $t\to\infty$ because $z^2\sim ik^2\beta/4\Delta k^2$ from Eq. (25) and hence the first term in Eq. (33) survives at large times.

Following above analysis, we neglect contributions from I_{sd} and I_j, keep only the first term in Eq. (33), and employ the expressions in Eqs. (24) and (25) to obtain the asymptotic forms of I_{Rl} and I_{Tl} as $t\to\infty$,

$$I_{Rl}\simeq-2\pi iR_l(k)a(k)e^{-iE_kt/\hbar}e^{ikd/2},$$

$$I_{Tl}\simeq-2\pi iT_l(k)a(k)e^{-iE_kt/\hbar}e^{ikd/2}, \tag{36}$$

where $a(k)$ is the momentum amplitude given in Eq. (12) and the time and the position variables have been recovered from Eq. (22). We now return to the original expression of $\langle\tau_c\rangle$ in Eq. (16) and replace I_{Rl} and I_{Tl} by their asymptotic forms described in Eq. (36). After arranging terms, we finally obtain the expectation value of the clock-time operator in the wave-packet states at sufficiently large times

$$\langle\tau_c\rangle=\int_{-\infty}^{\infty}dk|a(k)|^2\tau(k), \tag{37}$$

where

$$\tau(k)=\sum_{s=0}^{N-1}t_s[P_T(k,s)+P_R(k,s)], \tag{38}$$

with

$$P_A(k,s)=\frac{1}{N^2}\left|\sum_{n=-J}^{J}e^{in\omega t_s}A_n(k)\right|^2\quad(A=T,R). \tag{39}$$

This is our main result and interpretation of the terms in the equations is in order. First, $P_T(k,s)$ and $P_R(k,s)$ are the probabilities of finding the clock in a state $|\chi_s\rangle$ for transmitted and reflected particles in an eigenstate $|\varphi_{kn}\rangle$, respectively, and they satisfy $\Sigma_s[P_T(k,s)+P_R(k,s)]=1$. Thus, $\tau(k)$ is a total average clock time for particles interacting with the barrier in a stationary scattering state of $|\psi_k\rangle$ and it is expressed as a sum of two terms due to the probabilistic distribution of the scattered particles over which the recorded clock times are spread. In fact, the expression in Eq. (38) is just the expectation value of the clock-time operator evaluated in stationary-state scattering states $|\psi_k\rangle$, which was obtained in previous paper [12], and we call $\tau(k)$ a stationary-state clock time. As we shall see below, this is equivalent to the stationary-state dwell time. The expression of $\langle\tau_c\rangle$ in Eq. (37) is then an average of the stationary-state clock time $\tau(k)$ over all possible scattering states with probability distribution $|a(k)|^2$, which we call an average wave-packet clock time of barrier interaction. This is reminiscent of the average dwell time that can be obtained from the expectation value of the dwell-time operator $\hat{T}_D$ [see Eq. (47) below] in time-dependent wave-packet states. In the following section, we shall discuss the similarity between the present result and the average dwell time.

IV. COMPARISON TO AVERAGE DWELL TIME

To compare the average wave-packet clock time $\langle\tau_c\rangle$ to the average dwell time, we first derive the explicit expression of the stationary-state clock time $\tau(k)$ for the rectangular barrier. To find the explicit form of $\tau(k)$, it is convenient to write the transmission and reflection amplitudes in Eq. (10) as $T_n(k)=|T_n(k)|e^{i\Phi_{Tn}}$ and $R_n(k)=|R_n(k)|e^{i\Phi_{Rn}}$, where

$$\Phi_{Tn}=-kd+\delta(\epsilon_n)=\pi/2+\Phi_{Rn}, \tag{40}$$

$$\delta(\epsilon_n)=\tan^{-1}\left[\frac{k^2+q_n^2}{2kq_n}\tan(q_nd)\right]. \tag{41}$$

Note that the eigenvalues ϵ_n of the clock Hamiltonian have been treated as varying parameters and the dependence of the phases Φ_{Tn} and Φ_{Rn} on ϵ_n are through the barrier wave number q_n, not the free-particle wave number k [26]. To proceed, let us assume $\epsilon_n\ll E,|V_0-E|$ [27]. Expanding $T_n(k)$ and $R_n(k)$ to first order in ϵ_n to have $T_n(k)\simeq|T_0|e^{i\Phi_{T0}}e^{-in\omega\tau_T(k)}$ and $R_n(k)\simeq|R_0|e^{i\Phi_{R0}}e^{-in\omega\tau_R(k)}$, where

$$\tau_T(k)=-\hbar\frac{\partial\Phi_{Tn}}{\partial\epsilon_n}\bigg|_{\epsilon_n=0}=-\hbar\frac{\partial\delta}{\partial\epsilon_n}\bigg|_{\epsilon_n=0}=\tau_R(k), \tag{42}$$

and substituting them into $P_T(k,s)$ and $P_R(k,s)$, we have

$$P_A(k,s)\simeq\frac{|A_0|^2}{N^2}\left|\sum_{n=-J}^{J}e^{in\omega(t_s-\tau_A)}\right|^2\simeq|A_0|^2\delta_{t_s,\tau_A}(A=T,R). \tag{43}$$

In this equation, the second approximation is due to the fact that the probabilities peak at $t_s=\tau_T$ and τ_R. By substituting these into Eq. (38), we obtain the stationary-state clock time $\tau(k)$ as

$$\tau(k) = |T_0|^2 \tau_T(k) + |R_0|^2 \tau_R(k). \quad (44)$$

The explicit forms of $\tau_T(k)$ and $\tau_R(k)$ for the rectangular barrier considered here can be found from the definition in Eq. (42)

$$\tau_T(k) = \frac{m}{\hbar q} \frac{2kq(k^2+q^2)d - kK_0^2 \sin(2qd)}{4k^2q^2 + K_0^4 \sin^2(qd)} = \tau_R(k), \quad (45)$$

where $q=\sqrt{2m(E-V_0)}/\hbar$ and $K_0=\sqrt{2mV_0}/\hbar$. According to Büttiker, this expression is exactly the same as the local Larmor time corresponding to the in-plane spin rotation [28–30], defined as $\tau_{yT}=\tau_{yR}=-(m/\hbar q_n)\partial\delta/\partial q_n|_{q_n=q}$, with $\delta(q_n)$ being the same expression as Eq. (41).

The relation in Eq. (44) together with the definition in Eq. (42) is equivalent to the well-known identity [1,31] for the stationary-state dwell time $\tau_D(k)$ defined by [28]

$$\tau_D(k) = \frac{1}{j_{in}} \int_{-d/2}^{d/2} dx |\varphi_k(x)|^2, \quad (46)$$

where $j_{in}=Re[\varphi_k^*(x)(\hat{p}/m)\varphi_k(x)]=\hbar k/m$ is the incident probability current density for the component plane-wave state $\varphi_k(x)$ of a wave packet. Since $\tau_T(k)=\tau_R(k)$, the relation leads to $\tau(k)=\tau_T(k)=\tau_R(k)$, which is also a well-known result that has been verified for the dwell time in a symmetric barrier [28,32]. From this identification of $\tau(k)$ with $\tau_D(k)$, the stationary-state clock time can be interpreted as the stationary-state dwell time. It should be noted here that the relation in Eq. (44) is a consequence of the statistical nature of the wave packet describing a particle scattered off the barrier; it is a representative of a statistical ensemble of particles interacting with the barrier. As described in Eq. (38), the recorded clock times are distributed over particles having transmitted and been reflected with probabilities of $P_T(k,s)$ and $P_R(k,s)$, respectively. Since we have arrived at the result of Eq. (37) by considering the asymptotic wave packets long after the scattering event, there are no interference terms left, which led to the mutually exclusive relation between the two probabilities, $\Sigma_s P_T(k,s) + \Sigma_s P_R(k,s) = 1$. As pointed out in Ref. [1], the sum rule in Eq. (44) should be followed as a result of these mutually exclusive probabilities.

We now compare the result of Eq. (37) to the time-dependent case of the average dwell time. A number of authors have shown that the average dwell time for a time-dependent wave packet can be derived from the expectation value of the dwell-time operator $\hat{T}_D$ defined as [33]

$$\hat{T}_D = \int_{-\infty}^{\infty} dt e^{i\hat{\mathcal{H}}t/\hbar} \left[\int_{-d/2}^{d/2} dx |x\rangle\langle x| \right] e^{-i\tilde{\mathcal{H}}t/\hbar}. \quad (47)$$

The expectation value of this dwell-time operator in wave-packet states can be evaluated to be

$$\begin{aligned} \langle \tau_D \rangle &= \langle \psi(0)|\hat{T}_D|\psi(0)\rangle \\ &= \int_{-\infty}^{\infty} dt \int_{-d/2}^{d/2} dx |\psi(x,t)|^2 \\ &= \int dk |a(k)|^2 \tau_D(k), \end{aligned} \quad (48)$$

where $\tau_D(k)$ is the stationary-state dwell time given in Eq. (46) and $a(k)$ is the momentum amplitude of a wave packet [34]. Comparing this expression to that of the average wave-packet clock time $\langle\tau_c\rangle$ in Eq. (37), with the identification of $\tau(k)=\tau_D(k)$, we can see that $\langle\tau_c\rangle$ is the same as the average dwell time $\langle\tau_D\rangle$. Thus the average wave-packet clock time of a particle interacting with a barrier can be interpreted as the average dwell time of the barrier interaction.

That the two expressions of $\langle\tau_c\rangle$ and $\langle\tau_D\rangle$ are identical to each other may not be a surprising result because there are similarities between the present application of the Peres' time-of-flight approach with quantum clock to the barrier interaction time and the way of defining the average dwell time. The basic principle of the Peres' approach is that since the clock runs only when a particle is in the barrier region, it measures the time of duration of the particle being in the interaction region, not each of the absolute times of entrance to and escape from the barrier. Moreover, it does not distinguish whether the particle is transmitted or reflected; the clock only records the time of interaction without discerning the transmitting particles and the reflecting particles. These underlying properties in the quantum-clock approach to the measurement of barrier interaction time are compatible with the idea of defining the average dwell time, that is, the average time spent by particles in the interaction region regardless of being transmitted or reflected. In this sense, the present quantum-clock approach with wave packet may answer the second question in Sec. I and provide an operational definition of the dwell time. About the question of tunneling time, for which many controversial proposals exist, the present approach does not give a physically meaningful definition because the recorded clock times cannot be sorted into the transmission and reflection times, which was also pointed out in Ref. [12].

V. SUMMARY

We have evaluated the expectation value of the Peres' clock-time operator in the time-dependent wave-packet states scattered off a one-dimensional rectangular barrier to study the barrier interaction time. From the analysis of the asymptotic behaviors of the scattered wave packets long after having completed the scattering process, we have been able to derive an average wave-packet clock time. The resultant expression is shown to be the same as the average dwell time obtained from the expectation value of the dwell-time operator in time-dependent wave-packet states. Because of the statistical nature of a wave packet, the evaluated stationary-state clock time satisfies the well-known sum rule for the stationary-state dwell. The analogy between the average wave-packet clock time and the average dwell time should be anticipated because the definition of the dwell time is implicit in the quantum-clock approach to the measurement of a barrier interaction time.

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$$w(u) \sim \frac{i}{\sqrt{\pi}u}\left[1+\sum_{n=1}^{\infty}\frac{1\cdot 3\cdots(2n-1)}{(2u^2)^n}\right] \quad \left(|\arg u| < \frac{3\pi}{4}\right). \tag{49}$$

[25] There are also $k_j's$ in the third quadrant, located at symmetric positions about the imaginary k' axis, so that $k_j = -k_{jR} - ik_{jI}$. However, these poles do not contribute to the integral of Eq. (17) because they will never be passed by the steepest-descent line $\Gamma_{z'}$.
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