Confined Dirac fermions in a constant magnetic field

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We obtain an exact solution of the Dirac equation in (2+1) dimensions in the presence of a constant magnetic field normal to the plane together with a two-dimensional Dirac-oscillator potential coupling. The solution space consists of positive- and negative-energy solutions, each of which splits into two disconnected subspaces depending on the sign of an azimuthal quantum number $k=0,\pm1,\pm2,\ldots$ and whether the cyclotron frequency is larger or smaller than the oscillator frequency. The spinor wave function is written in terms of the associated Laguerre polynomials. For negative k, the relativistic energy spectrum is infinitely degenerate due to the fact that it is independent of k. We compare our results with already published work and point out the relevance of these findings to a systematic formulation of the relativistic quantum Hall effect in a confining potential.

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I. INTRODUCTION

Recent technological advances in nanofabrication have created a great deal of interest in the study of low-dimensional quantum systems such as quantum wells, quantum wires, and quantum dots [1]. In particular, there has been considerable amount of work in recent years on semiconductor confined structures, which finds applications in electronic and optoelectronic devices. The application of a magnetic field perpendicular to the heterostructure plane quantizes the energy levels in the plane, drastically affecting the density of states giving rise to the famous quantum Hall effect (QHE) [2]. The latter remains as the most interesting phenomenon observed in physics because of its link to different theories and subjects.

The stationary state associated with the motion of electrons in a uniform magnetic field is a well-known textbook problem [3]. It results in a sequence of the quantized Landau energy levels and associated wave functions characterizing the dynamics in the two-dimensional (2D) plane normal to the applied magnetic field. This quantization has important consequences in condensed-matter physics ranging from the classical de Hass-van Alphen effect in metals [4] to QHE in semiconductors [2]. The relativistic extension of these models turned out to be of great importance in the description of 2D quantum phenomena such as QHE in graphene [5–8]. In fact, several condensed-matter phenomena point out to the existence of a (2+1)-dimensional energy spectrum determined by the relativistic Dirac equation [9]. For very recent works, one may consult Refs. [10–15] and for early works relevant to our subject we cite [16,17].

Motivated by different investigations on the Dirac fermions in (2+1) dimensions, we give an exact solution of a problem that has been studied at various levels by researchers dealing with different physical phenomena. We have done so by considering a relativistic particle subjected to an external magnetic field as well as to a confining potential. By introducing a similarity transformation, we show that the system can be diagonalized in a simple way. Solving the eigenvalue equation, we end up accounting for the full space of the eigenfunctions that include all cases related to different physical settings. More precisely, from the nature of the problem we get separate angular and radial solutions. The radial equation leads to the exact relationship between the two-spinor components. In fact, depending on the range of values of three physical quantities, the full solution space splits into eight disconnected subspaces as summarized in Table I in Sec. II B. This allowed us to obtain various solutions and emphasis on similarities to, and differences from, already published work elsewhere [18].

On the other hand, we give discussions of our results based on different physical settings. In fact, we show that for week and strong magnetic fields there is a symmetry that allows us to go from positive- to negative-energy solutions (states and spectrum). This can be done by interchanging the confinement frequency ω with the cyclotrons ω_c and vice versa. This suggests defining an effective magnetic field that produces the effective quantized Landau levels. In both cases, there is a degeneracy of the Landau levels where each quantum number n is k-times degenerate, in analogy with the nonrelativistic case [19]. For the intermediate magnetic field case, it is underlined that the degeneracy is possible. Finally, we compare our findings with those in a very significant work by Villalba and Maggiolo [18]. The full rich space of solutions suggested enabled us to carry out a deeper analysis in relation to various physical quantities. For instance, we obtained, as expected in the absence of an applied voltage, a null current density for both directions in the Cartesian representation. However, this is not the case in polar coordinate. In fact, we show that the radial current vanishes, whereas the angular component does not. It is dependent on various physical parameters in the problem. These results are sum-

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marized in Table III, showing clearly the dependence of these values on the given subspace. This may offer an alternative approach for a systematic study and understanding of the anomalous QHE [7,8]. Additionally, we discuss the nonrelativistic limit of the problem.

The paper is organized as follows. In Sec. II, we give the theoretical formulation of the problem where a similarity transformation is introduced to simplify the process for obtaining the solutions (spinor wave function and energy spectrum). We use the exact relationship between spinor components to obtain a second-order differential equation for one of the two-spinor components. The second spinor component is obtained from this using the exact relationship. The relativistic energy eigenvalues and the corresponding spinor wave functions are obtained as elements in the eight subspaces of the full and complete Hilbert space. In Sec. III, we discuss the physical meaning of our results and their potential application to QHE. To analyze the transport properties of the system, we determine the current density in Sec. IV and the nonrelativistic case in Sec. V. Finally, we conclude by discussing the main results and the possible extension of our work.

II. FORMULATION OF THE PROBLEM

We start by formulating the problem in terms of our language [20]. This is done by introducing a similarity transformation of the Dirac equation in polar coordinates. This will be convenient to handle the exact relationship between spinor components and therefore derive the full spectrum as a complete Hilbert space.

A. Hamiltonian system

The problem of a charged particle moving in a constant magnetic field $\vec{B}=B\hat{z}$ is a 2D problem in the plane normal to the field [the Cartesian (x,y) plane or the cylindrical (r,θ) plane]. In the relativistic units, $\hbar=c=1$, the Dirac equation in $(2+1)$ dimensions for a spinor of charge e and mass m in the electromagnetic potential $A_\mu=(A_0,\vec{A})$ reads as follows:

$$[i\gamma^\mu(\partial_\mu+ieA_\mu)-m]\psi=0, \quad \mu=0,1,2, \tag{1}$$

where the summation convention over repeated indices is used. $\gamma^\mu=(\gamma^0,\vec{\gamma})$ are three unimodular square matrices satisfying the anticommutation relation

$$\{\gamma^\mu,\gamma^\nu\}=\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2\mathcal{G}^{\mu\nu}, \tag{2}$$

where $\mathcal{G}$ is the metric of Minkowski space-time, which is equal to diag(+−−). A minimal irreducible matrix representation that satisfies this relation is taken as $\gamma^0=\sigma_3$, $\vec{\gamma}=i\vec{\sigma}$, where $\{\sigma_i\}_{i=1}^3$ are the 2×2 Hermitian Pauli spin matrices

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}, \quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}, \quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{3}$$

Equation (1) could be rewritten as

$$i\frac{\partial}{\partial t}\psi=(-i\vec{\alpha}\cdot\vec{\nabla}+e\vec{\alpha}\cdot\vec{A}+eA_0+m\beta)\psi, \tag{4}$$

where $\vec{\alpha}$ and β are the Hermitian matrices: $\vec{\alpha}=i\sigma_3\vec{\sigma}$, $\beta=\sigma_3$. We will see below that the symmetry of the problem is preserved even if we introduce an additional coupling to the 2D Dirac-oscillator potential. This coupling is introduced by the substitution $\vec{\nabla}\to\vec{\nabla}+m\omega\vec{r}\beta$, where ω is the oscillator frequency. For time-independent potentials, the two-component spinor wave function $\psi(t,r,\theta)$ is written as

$$\psi(t,r,\theta)=e^{-i\varepsilon t}\psi(r,\theta) \tag{5}$$

and Eq. (4) becomes the energy eigenvalue wave equation $(\mathcal{H}-\varepsilon)\psi=0$, where ε is the relativistic energy. The Dirac Hamiltonian $\mathcal{H}$ is the 2×2 matrix operator

$$\mathcal{H}=\mathcal{H}_0+i\sigma_3\vec{\sigma}\cdot\hat{r}\mathcal{H}_r+i\sigma_3\vec{\sigma}\cdot\hat{\theta}\mathcal{H}_\theta, \tag{6}$$

where $(\hat{r},\hat{\theta})$ are the unit vectors in cylindrical coordinates and

$$\mathcal{H}_0=eA_0+m\sigma_3,$$

$$\mathcal{H}_r=-i\partial_r+eA_r-im\omega r\sigma_3,$$

$$\mathcal{H}_\theta=-\frac{i}{r}\partial_\theta+eA_\theta. \tag{7}$$

For regular solutions of Eq. (4), square integrability (with respect to the measure $d^2\vec{r}=rdrd\theta$) and the boundary conditions require that $\psi(r,\theta)$ satisfies

$$\sqrt{r}\psi(r,\theta)\Big|_{r\to\infty}^{r=0}=0, \quad \psi(\theta+2\pi)=\psi(\theta). \tag{8}$$

To simplify the construction of the solution, we look for a local 2×2 similarity transformation $\Lambda(r,\theta)$ that maps the cylindrical projection of the Pauli matrices $(\vec{\sigma}\cdot\hat{r},\vec{\sigma}\cdot\hat{\theta})$ into their canonical Cartesian representation (σ_1,σ_2), respectively [21]. That is

$$\Lambda\vec{\sigma}\cdot\hat{r}\Lambda^{-1}=\sigma_1, \quad \Lambda\vec{\sigma}\cdot\hat{\theta}\Lambda^{-1}=\sigma_2. \tag{9}$$

A 2×2 matrix that satisfies this requirement is

$$\Lambda(r,\theta)=\lambda(r,\theta)e^{(i/2)\sigma_3\theta}, \tag{10}$$

where $\lambda(r,\theta)$ is a 1×1 real function and the exponential is a 2×2 unitary matrix. The Dirac Hamiltonian (6) gets mapped into

$$H=\Lambda\mathcal{H}\Lambda^{-1}=H_0-\sigma_2H_r+\sigma_1H_\theta, \tag{11}$$

where different operators are given by

$$H_0=\mathcal{H}_0,$$

$$H_r=-i\left(\partial_r-\frac{\lambda_r}{\lambda}\right)+ieA_r-im\omega r\sigma_3,$$

$$H_\theta=-\frac{i}{r}\left(\partial_\theta-\frac{\lambda_\theta}{\lambda}-\frac{i}{2}\sigma_3\right)+eA_\theta \tag{12}$$

with $\lambda_k=\partial_k\lambda$. Therefore, the 2×2 Dirac Hamiltonian becomes

$$H=\begin{pmatrix} m+eA_0 & \partial_r-\frac{\lambda_r}{\lambda}+\frac{1}{2r}+ieA_r-m\omega r-\frac{i}{r}\left(\partial_\theta-\frac{\lambda_\theta}{\lambda}\right)+eA_\theta \\ -\partial_r+\frac{\lambda_r}{\lambda}-\frac{1}{2r}-ieA_r-m\omega r-\frac{i}{r}\left(\partial_\theta-\frac{\lambda_\theta}{\lambda}\right)+eA_\theta & -m+eA_0 \end{pmatrix}. \tag{13}$$

Thus, Hermiticity of Eq. (13) requires that

$$\lambda_\theta=0,\quad \frac{\lambda_r}{\lambda}-\frac{1}{2r}=0 \tag{14}$$

and fixes the exact form of the modulus of similarity transformation to be $\lambda(r,\theta)=\sqrt{r}$. It is interesting to note that λ^2 turns out to be the integration measure in 2D cylindrical coordinates. We could have eliminated the λ factor in the definition of Λ in Eq. (10) by proposing that the new spinor wave function χ be replaced with $\frac{1}{\sqrt{r}}\chi(r,\theta)$. In that case, the transformation matrix Λ becomes simply $e^{(i/2)\sigma_3\theta}$, which is unitary. However, making the presentation as above gave us a good opportunity to show (in a different approach) why is it customarily to take the radial component of the wave function in 2D cylindrical coordinates to be proportional to $\frac{1}{\sqrt{r}}$. Finally, we obtain the (2+1)-dimensional Dirac equation $(H-\varepsilon)\chi=0$ for a charged spinor in static electromagnetic potential as

$$\begin{pmatrix} m+eA_0-\varepsilon & \partial_r+ieA_r-m\omega r-\frac{i}{r}\partial_\theta+eA_\theta \\ -\partial_r-ieA_r-m\omega r-\frac{i}{r}\partial_\theta+eA_\theta & -m+eA_0-\varepsilon \end{pmatrix}\begin{pmatrix}\chi_+(r,\theta)\\ \chi_-(r,\theta)\end{pmatrix}=0, \tag{15}$$

where $\chi_\pm$ are the components of the transformed wave function $|\chi\rangle=\Lambda|\psi\rangle$. This equation will be solved by choosing an appropriate gauge to end up with the full Hilbert space.

B. Eigenvalues and wave functions

Now, we specialize to the case where a constant magnetic field of strength B is applied at right angles to the (r,θ) plane, which is $\vec{B}=B\hat{z}$. Therefore, the electromagnetic potential has the time and space components as follows:

$$A_0=0,\quad \vec{A}(r,\theta)=\tfrac{1}{2}Br\hat{\theta}. \tag{16}$$

Consequently, Eq. (15) becomes completely separable and we can write the spinor wave function as

$$\chi_\pm(r,\theta)=\phi_\pm(r)\tau(\theta). \tag{17}$$

Thus, the angular component satisfies $-i\frac{d\tau}{d\theta}=\xi\tau$, where ξ is a real separation constant giving the function

$$\tau(\theta)=\frac{1}{\sqrt{2\pi}}e^{i\xi\theta}. \tag{18}$$

On the other hand, the boundary condition $\psi(\theta+2\pi)=\psi(\theta)$ requires that $e^{i2\pi\xi}e^{-i\sigma_3\pi}=+1$ which, in turn, demands that $e^{i2\pi\xi}=-1$ giving the following quantum number:

$$\xi=\tfrac{1}{2}\kappa,\quad \kappa=\pm1,\pm3,\pm5,\ldots. \tag{19}$$

Consequently, the Dirac equation for the two-component radial spinor is reduced to

$$\begin{pmatrix} m-\varepsilon & \frac{d}{dr}+\frac{\xi}{r}+Gr \\ -\frac{d}{dr}+\frac{\xi}{r}+Gr & -m-\varepsilon \end{pmatrix}\begin{pmatrix}\phi_+(r)\\ \phi_-(r)\end{pmatrix}=0, \tag{20}$$

where the physical constant G is given by $G=m(\omega_c-\omega)$ and ω_c is the cyclotron frequency $\omega_c=\frac{eB}{2m}$. Thus, the presence of the 2D Dirac-oscillator coupling did, in fact, maintain the symmetry of the problem as stated below Eq. (4). Moreover, its introduction is equivalent to changing the magnetic field as $eB\to eB-2m\omega$. As a result of the wave equation (20), the two radial spinor components satisfy the exact relationship

$$\phi_\mp(r)=\frac{1}{\varepsilon\pm m}\left[\mp\frac{d}{dr}+\frac{\xi}{r}+Gr\right]\phi_\pm(r), \tag{21}$$

where $\varepsilon\neq\pm m$. Therefore, the solution of the problem with the top (bottom) sign corresponds to the positive- (negative-) energy solution. Using the exact relationship (21) to eliminate one component in terms of the other in Eq. (20) results in the following Schrödinger-like differential equation for each spinor component:

$$\left\{-\frac{d^2}{dr^2}+\frac{\xi(\xi\mp1)}{r^2}+G^2r^2+[m^2-\varepsilon^2+G(2\xi\pm1)]\right\}\phi_\pm(r)=0. \tag{22}$$

Again, we stress that this equation gives only one radial spinor component. One must choose either the top or the bottom sign to obtain the component that corresponds to the positive- or the negative-energy solution, respectively. The second component is obtained by substituting this into the exact relationship (21). Nonetheless, we only need to find one solution (the positive- or the negative-energy solution), because the other is obtained by a simple map. For example, the following map takes the positive-energy solution into the negative-energy solution:

$$\varepsilon \to -\varepsilon, \quad \kappa \to -\kappa, \quad G \to -G, \quad \phi_\pm \to \phi_\mp, \tag{23}$$

which, in fact, is the $\mathcal{CPT}$ transformation. Here the charge conjugation C means that $e\to -e$ and $\omega \to -\omega$ or the exchange of ω and ω_c. It is easy to check that the above map (23) originates from the fact that the Dirac equation (20) is invariant under such transformation. Hence, we just need to solve for positive energies and use the above transformation to obtain the negative-energy solutions. The total spinor wave function reads as follows:

$$\psi(r,\theta) = \frac{1}{\sqrt{r}} e^{i\xi\theta} e^{-(i/2)\sigma_3\theta}\phi(r), \tag{24}$$

where $\phi(r)$ has two components, such as

$$\phi = \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix}. \tag{25}$$

Equation (22) looks like the nonrelativistic oscillator problem with a certain parameter map of the frequency, the angular momentum, and the energy. For regular solutions of Eq. (22), the bound states will be of the form

$$\phi_\pm \sim z^\mu e^{-z/2} L_n^\nu(z), \tag{26}$$

where $L_n^\nu(z)$ is the associated Laguerre polynomial of order $n=0,1,2,\ldots$ and $z=\rho^2 r^2$. The constants $\{\mu,\nu,\rho\}$ are real and related to the physical parameters B, ω, and ξ. Square integrability and the boundary conditions require that $2\mu \geq \frac{1}{2}$ and $\nu > -1$.

Substituting ansatz (26) into Eq. (22) and using the differential equation for the Laguerre polynomial shown in the Appendix, we obtain four equations. Three of them determine the parameters $\{\mu,\nu,\rho\}$ and one determines the energy spectrum. The first three are

$$2\mu = \nu + \tfrac{1}{2}, \quad \rho^2 = |G|,$$

$$\nu = \pm \begin{cases} \xi - \frac{1}{2}, & \varepsilon > 0 \\ \xi + \frac{1}{2}, & \varepsilon < 0. \end{cases} \tag{27}$$

For regular solutions of Eq. (22), the $\pm$ sign in the expression for ν corresponds to $\pm\xi > 0$. Now, the fourth equation gives the following (positive and negative) energy spectra:

$$\varepsilon_{n,\xi}^\pm = \pm m \sqrt{1 + \frac{2|G|}{m^2}\left[2n + 1 \pm \frac{s - s'}{2} + \xi(s + s')\right]} \tag{28}$$

where $s = \mathrm{sgn}(G) = \frac{|G|}{G}$ and $s' = \mathrm{sgn}(\xi)$. The sign of G depends on whether the oscillator frequency ω is larger or smaller than the cyclotron frequency ω_c. To compare our work with the frequently used notation in the literature, we can replace the quantum number ξ with $k+\frac{1}{2}$, where $k=0,\pm 1,\pm 2,\ldots$ and $\xi \to -\xi$ imply that $k \to -k-1$. In that case, one may write the energy spectrum as positive eigenvalues

$$\varepsilon_{n,k}^+ = m\sqrt{1 + \frac{2|G|}{m^2}[2n + 1 + s + k(s + s')]} \tag{29}$$

and as negative ones

$$\varepsilon_{n,k}^- = -m\sqrt{1 + \frac{2|G|}{m^2}[2n + 1 + s' + k(s + s')]}, \tag{30}$$

where $s' = +1$ for $k=0$. It is interesting to note that for $\xi G < 0$ the spectrum is infinitely degenerate because it is independent of ξ. However, for $\xi G > 0$ the degeneracy is finite and equal to $n+k+1$. Substituting the wave-function parameters given by Eq. (27) into ansatz (26) gives for $\varepsilon > 0$

$$\phi_+(r) = x^{|k+1/2|} e^{-(1/2)x^2} \begin{cases} A_{n,k}^{++} L_n^k(x^2), & k \geq 0 \\ A_{n,k}^{+-} x L_n^{-k}(x^2), & k < 0 \end{cases} \tag{31}$$

as well as for $\varepsilon < 0$

$$\phi_-(r) = x^{|k+1/2|} e^{-(1/2)x^2} \begin{cases} A_{n,k}^{-+} x L_n^{k+1}(x^2), & k \geq 0 \\ A_{n,k}^{--} L_n^{-k-1}(x^2), & k < 0, \end{cases} \tag{32}$$

where $x = r\sqrt{|G|}$ and $A_{n,k}^{ij}$ are normalization constants that depend on the physical quantities ω and ω_c. The lower component is obtained by substituting Eqs. (31) and (32) into the exact relationship (21). Doing so while exploiting the differential and the recursion properties of the Laguerre polynomials (see the Appendix), we obtain the following for $\varepsilon > 0$:

$$\phi_-(r) = \frac{\sqrt{|G|}}{\varepsilon_{n,k}^+ + m} x^{|k+1/2|} e^{-(1/2)x^2} \times \begin{cases} A_{n,k}^{++} x[(s-1)L_n^k(x^2) + 2L_n^{k+1}(x^2)], & k \geq 0 \\ A_{n,k}^{+-}[(s-1)(n-k)L_n^{-k-1}(x^2) - (s+1)(n+1)L_{n+1}^{-k-1}(x^2)], & k < 0. \end{cases} \tag{33}$$

On the other hand, repeating the same calculation for the upper component of the negative-energy solution gives the function

$$\phi_+(r) = \frac{\sqrt{|G|}}{\varepsilon_{n,k}^- - m} x^{|k+1/2|} e^{-(1/2)x^2} \times \begin{cases} A_{n,k}^{-+}[(1+s)(n+k+1)L_n^k(x^2) + (1-s)(n+1)L_{n+1}^k(x^2)], & k \geq 0 \\ A_{n,k}^{--} x[(1+s)L_n^{-k-1}(x^2) - 2L_n^{-k}(x^2)], & k < 0, \end{cases} \tag{34}$$

TABLE I. Full space solution.

Frequency	$\omega>\omega_c$	$\omega>\omega_c$	$\omega<\omega_c$	$\omega<\omega_c$
Energy	$\varepsilon>0$	$\varepsilon<0$	$\varepsilon>0$	$\varepsilon<0$
Azimuth	$k\geq 0$, $k<0$	$k\geq 0$, $k<0$	$k\geq 0$, $k<0$	$k\geq 0$, $k<0$

which could have also been obtained by applying the $\mathcal{CPT}$ map (23) to Eq. (33). Thus, the structure of the whole Hilbert-space solution consists of eight disconnected spaces that could be displayed in tabular form as shown in Table I.

Using the standard definition, we calculate all involved normalization constants in the above wave functions. These are summarized in the Table II, where as stated above, $s=\mathrm{sgn}(G)=|G|/G$.

III. DISCUSSIONS

It is worthwhile investigating the basic features of some limits of our results and their interesting underlying properties. We consider three different cases corresponding to the relative strength of the external magnetic field (cyclotron frequency) to the oscillator coupling (oscillator frequency). We also demonstrate the added value of our results as opposed to others in the literature, in particular the classic work of Villalba and Maggiolo [18].

A. Energy spectrum properties

To investigate the underlying symmetry of the system, one may study the properties of quantum number pairs (n,k). However, these may not provide simple hints on the ordering of the energy eigenvalues $\varepsilon_{n,k}^{\pm}$, with the exception of two limiting cases: the weak and the strong fields.

1. Weak-field case

Suppose that the cyclotron frequency is much smaller than the oscillator frequency. That is, $\omega_c \ll \omega$, $G\approx -m\omega$, or $s=-1$. Thus, one obtains the following positive

$$\varepsilon_{n,k}^{+}|_{\omega_c\ll\omega} \approx m\sqrt{1+\frac{2\omega}{m}[2n+k(s'-1)]} \tag{35}$$

and negative-energy spectrum

$$\varepsilon_{n,k}^{-}|_{\omega_c\ll\omega} \approx -m\sqrt{1+\frac{2\omega}{m}[2n+1+s'+k(s'-1)]}. \tag{36}$$

Consequently, for $k\geq 0$ (i.e., $s'=+1$) the energy spectrum is (semi)infinitely degenerate since it becomes independent of k. Moreover, the two spectra are related as

$$\varepsilon_{n,k}^{-}|_{\omega_c\ll\omega} = -\varepsilon_{n+1,k}^{+}|_{\omega_c\ll\omega}. \tag{37}$$

However, for $k<0$ $(s'=-1)$ we obtain

$$\varepsilon_{n,k}^{+}|_{\omega_c\ll\omega} \approx m\sqrt{1+\frac{4\omega}{m}(n-k)} = -\varepsilon_{n,k}^{-}|_{\omega_c\ll\omega}. \tag{38}$$

It is also interesting to note that for $k\geq 0$ there exits a positive-energy zero mode corresponding to $\varepsilon_0^{+}|_{\omega_c\ll\omega}=m$ with the following spinor wave function:

$$\psi_0(r,\theta) = \frac{A_k^{+}}{\sqrt{r}} e^{-(i/2)\sigma_3\theta}(\sqrt{m\omega}re^{i\theta})^{k+1/2}\exp\left[-\frac{1}{2}m\omega r^2\right]\begin{pmatrix}1\\0\end{pmatrix}, \tag{39}$$

where the normalization is

$$A_k^{+} = A_{0k}^{++} = \sqrt{2\left\{\pi k!\left[1+2\frac{\omega}{m}(k+1)\right]\right\}^{-1}}. \tag{40}$$

These results are in good agreement with those of Dirac fermions in the plane in the presence of a constant perpendicular magnetic field.

2. Strong-field case

Now, if the cyclotron frequency is much larger than the oscillator frequency then $G\approx m\omega_c$ and we obtain the positive relativistic energy spectrum

$$\varepsilon_{n,k}^{+}|_{\omega_c\gg\omega} \approx m\sqrt{1+\frac{2\omega_c}{m}[2(n+1)+k(1+s')]} \tag{41}$$

as well as the negative one

$$\varepsilon_{n,k}^{-}|_{\omega_c\gg\omega} \approx -m\sqrt{1+\frac{2\omega_c}{m}[2n+(k+1)(1+s')]}. \tag{42}$$

They are related to each other as

TABLE II. Normalization in terms of different physical quantities.

Energy	Azimuth	Normalization
$\varepsilon>0$	$k\geq 0$	$A_{n,k}^{++}=\sqrt{2n!\{\pi(n+k)![1+\frac{4\lvert G\rvert}{(\varepsilon_{n,k}^{+}+m)^2}\{2(n+k+1)+n(1-s)\}]\}^{-1}}$
$\varepsilon>0$	$k<0$	$A_{n,k}^{+-}=\sqrt{n!\{\pi(n-k)![1+\frac{2\lvert G\rvert}{(\varepsilon_{n,k}^{+}+m)^2}\{2(n+1)+(k+1)(s-1)\}]\}^{-1}}$
$\varepsilon<0$	$k\geq 0$	$A_{n,k}^{-+}=\sqrt{n!\{\pi(n+k+1)![1+\frac{2\lvert G\rvert}{(\varepsilon_{n,k}^{-}-m)^2}\{2(n+1)+k(s+1)\}]\}^{-1}}$
$\varepsilon<0$	$k<0$	$A_{n,k}^{--}=\sqrt{2n!\{\pi(n-k-1)![1+\frac{4\lvert G\rvert}{(\varepsilon_{n,k}^{-}-m)^2}\{2(n-k)+n(s+1)\}]\}^{-1}}$

$$\varepsilon^{+}_{n,k}|_{\omega_c\gg\omega}=\varepsilon^{-}_{n+1,k-1}|_{\omega_c\gg\omega}. \tag{43}$$

Thus, in this case the infinite degeneracy of the spectrum corresponds to negative values of the azimuthal quantum number (i.e., $s'=-1$) where $\varepsilon_n^+=\varepsilon_{n+1}^-$. Here, a negative-energy zero mode exits corresponding to $\varepsilon_0^-|_{\omega_c\gg\omega}=-m$ with the following spinor wave function:

$$\psi_0(r,\theta)=\frac{A_k^-}{\sqrt{r}}e^{-(i/2)\sigma_3\theta}(\sqrt{m\omega_c}re^{i\theta})^{-k-1/2}\exp\left[-\frac{1}{2}m\omega_c r^2\right]\begin{pmatrix}0\\1\end{pmatrix}, \tag{44}$$

where the normalization is

$$A_k^-=A_{0k}^{--}=\sqrt{2\left\{\pi(-k-1)!\left[1-2\frac{\omega_c}{m}k\right]\right\}^{-1}}. \tag{45}$$

Comparing the weak and the strong magnetic field limits, one can conclude that the dominant frequency that controls the physics of the problem is interchanged between the oscillator and the magnetic field as $\omega\leftrightarrow\omega_c$. More precisely, the k-independent infinitely degenerate energy spectra are related to each other as follows:

$$\varepsilon_n^+|_{\omega_c\ll\omega,k<0}=\varepsilon_{n+1}^+|_{\omega_c\gg\omega,k\geq 0},\quad \varepsilon_n^-|_{\omega_c\ll\omega,k\geq 0}=\varepsilon_{n+1}^-|_{\omega_c\gg\omega,k<0}, \tag{46}$$

where the quantum number n corresponds to the Landau-level index. The existence of a zero-mode energy is now very clear for positive (negative) energy with $k\geq 0$ ($k<0$), respectively.

3. Fine tuned case

If the oscillator frequency is tuned to resonate with the cyclotron frequency (i.e., $\omega\approx\omega_c$) then $G=m\Delta$, where $\Delta=\omega_c-\omega$ such that $|\Delta|\ll m$. In this case, the relativistic energy spectrum approaches the nonrelativistic energy limit

$$E=\frac{1}{2m}(\varepsilon^2-m^2) \tag{47}$$

giving the quantity

$$E_{nk}^{\pm}=|\Delta|\left[2n+1+\frac{s+s'}{2}\pm\frac{s-s'}{2}+k(s+s')\right]. \tag{48}$$

In this case, the energy spectrum degeneracy occurs when the quantum numbers associated with the two states ψ_1 and ψ_2 satisfy the relation

$$\frac{n_2-n_1}{k_2-k_1}=-\frac{s+s'}{2}. \tag{49}$$

That is, when the ratio of the shift in the principal quantum number is matched with the shift in the azimuthal number either up or down depending on the relative strength of the two frequencies and sign of k.

B. Comparisons with other studies

We compare our results with those in very similar studies found elsewhere in the literature, such as the classic work by Villalba and Maggiolo [18] and, in particular, the energy spectrum and the spinor wave function. As for the latter, ours is identified with theirs according to

$$\begin{pmatrix}\phi_+\\ \phi_-\end{pmatrix}\equiv\begin{pmatrix}\psi_1\\ i\psi_2\end{pmatrix}. \tag{50}$$

In what follows, we summarize our remarks regarding few points in [18]:

(1) The imaginary i is missing from some of the off-diagonal entries in the Dirac equation (11); however, it was later corrected in Eqs. (28) and (29).

(2) The relative strength of the cyclotron frequency ω_c to the oscillator frequency ω (i.e., whether ω_c is greater than or less than ω) is ignored.

(3) In addition, the negative-energy solutions were also ignored altogether. Thus, only one fourth of the regular solution space, which consists of eight subspaces and whose structure is shown in Table I, was obtained in [18]. The authors obtained only the two subspaces corresponding to $\varepsilon>0$ and $\omega_c>\omega$.

(4) The alternative signs in Eq. (31), which correspond to the sign of the energy, was confused with the independent signs for the physical parameter μ (k in our notation).

(5) The relative number of nodes of the top to the bottom spinor components for $k<0$ as given by Eqs. (33) and (34) is incompatible with the exact relationship between spinor components (21).

IV. DENSITY OF CURRENT

We examine the behavior of the present system by analyzing the electric current density. Indeed, from our findings we can show that

$$\vec{J}\sim\langle\vec{\alpha}\rangle=i\langle\sigma_3\vec{\sigma}\rangle. \tag{51}$$

For this calculation, we use the spinor wave function obtained above. This gives a null value in the Cartesian coordinates, which is $J_x=J_y=0$. This, of course, is expected since there is no net charge drift. As a reassuring exercise, we calculate the same current in cylindrical coordinates

$$J_r=\vec{J}\cdot\hat{r}=i\langle\sigma_3\vec{\sigma}\cdot\hat{r}\rangle,\quad J_\theta=\vec{J}\cdot\hat{\theta}=i\langle\sigma_3\vec{\sigma}\cdot\hat{\theta}\rangle. \tag{52}$$

In this calculation, we employ the similarity transformation (10). The calculation gives $J_r=0$; however, J_θ does not vanish having the components given in Table III. This is due to the fact that the physical problem in cylindrical coordinates is for a charged particle confined to a circular motion due to the constant magnetic field.

One could make a different analysis in terms of the physical quantities corresponding to different signs ($s=\pm$) and for all the eight different subspaces. All these analyses could be used to give an interesting description for the anomalous QHE.

V. NONRELATIVISTIC LIMIT

It is interesting to study the nonrelativistic limit of our work to reproduce results already known in the literature.

TABLE III. Density of current for four subspaces.

Energy	Azimuth	Angular current component
$\varepsilon>0$	$k\geq 0$	$J_\theta^{++}=\frac{8\sqrt{\lvert G\rvert}}{\varepsilon_{n,k}^{+}+m}[n+(k+1)\frac{s+1}{2}]\{1+\frac{4\lvert G\rvert}{(\varepsilon_{n,k}^{+}+m)^2}[2(n+k+1)+n(1-s)]\}^{-1}$
$\varepsilon>0$	$k<0$	$J_\theta^{+-}=\frac{2\sqrt{\lvert G\rvert}}{\varepsilon_{n,k}^{+}+m}(k+1)\frac{s-1}{2}\{1+\frac{4\lvert G\rvert}{(\varepsilon_{n,k}^{+}+m)^2}[2(n+k+1)+n(1-s)]\}^{-1}$
$\varepsilon<0$	$k\geq 0$	$J_\theta^{-+}=-\frac{2\sqrt{\lvert G\rvert}}{\varepsilon_{n,k}^{-}-m}k\frac{s+1}{2}\{1+\frac{4\lvert G\rvert}{(\varepsilon_{n,k}^{-}-m)^2}[2(n-k)+n(1+s)]\}^{-1}$
$\varepsilon<0$	$k<0$	$J_\theta^{--}=-\frac{8\sqrt{\lvert G\rvert}}{\varepsilon_{n,k}^{-}-m}(n+k\frac{s-1}{2})\{1+\frac{4\lvert G\rvert}{(\varepsilon_{n,k}^{-}-m)^2}[2(n-k)+n(1+s)]\}^{-1}$

This can be achieved by taking the limit $m\to\infty$ in the above findings. Now in the units $\hbar=c=1$, the nonrelativistic problem has already been worked (see, for example, [19]),

$$\mathcal{H}\Psi(r,\theta)=\left[-\frac{r^2}{2m}\left(\partial_r^2+\frac{1}{r}\partial_r+\frac{1}{r^2}\partial_\theta^2\right)-i\frac{1}{2}\omega_c\partial_\theta+\frac{m}{8}\omega^2r^2\right]\Psi(r,\theta)=E\Psi(r,\theta). \tag{53}$$

The wave functions are (where $\omega=0$ and $s=+1$)

$$\Psi_{n,\alpha}(r,\theta)=(-1)^n\frac{1}{\sqrt{\pi}l_0}\sqrt{\frac{n!}{(n+|\alpha|)!}}\exp\left(-\frac{r^2}{2l_0^2}\right)\left(\frac{r}{l_0}\right)^{|\alpha|}L_n^{(|\alpha|)}\times\left(\frac{r^2}{l_0^2}\right)e^{i\alpha\theta}, \tag{54}$$

where $n=0,1,2,\ldots$ is the principal quantum number, $\alpha=0,\pm1,\pm2,\ldots$ is the angular moment quantum number, and $l_0=\sqrt{\frac{1}{eB}}$ is the magnetic length. The corresponding energy eigenvalues are given by

$$E_{n,\alpha}=\Omega\left(n+\frac{|\alpha|+1}{2}\right)+\frac{\omega_c}{2}\alpha, \tag{55}$$

where Ω is the frequency $\Omega=\sqrt{\omega_c^2+4\omega^2}$.

To compare with the nonrelativistic limit of our work, we take the limit $m\to\infty$ and use the well-known nonrelativistic energy formula $E=(\varepsilon^2-m^2c^2)/2m$ giving

$$E_{n,k}^{\pm}=2\omega_c\begin{cases}n+k+1, & k\geq 0\\ n+\dfrac{1\pm1}{2}, & k<0.\end{cases} \tag{56}$$

The $\pm$ sign for $k<0$ is a remnant of the positive or negative energy spectrum of the relativistic theory that is exhibited as a zero-energy mode in the infinitely degenerate part of the spectrum.

VI. CONCLUSION

The present paper was devoted to give a complete solution to the confined Dirac fermion system in the presence of a perpendicular magnetic field. Indeed, using a similarity transformation, we have formulated our problem in terms of the polar coordinate representation that allows us to handle easily the exact relationship between spinor components. One spinor component was obtained by solving a second-order differential equation, while the other component was obtained using the exact relationship (21). It resulted in a full solution space made of eight subspaces, which suggests that it is necessary to include all components of this subspace in the computations of any physical quantity. A failure to do so will result in erroneous conclusions.

Our results were employed to discuss few important limiting cases: the weak, the strong, and the fine tuned magnetic field cases. In particular, we showed that there is a symmetry between the negative- and the positive-energy solutions. In the weak magnetic field case, the system was shown to behave like a two-dimensional Dirac system in the presence of an effective magnetic field controlled by the oscillator frequency ω. To support our analysis, we compared our findings favorable with those available in the literature and underlined the reason behind some of our differences.

On the other hand, we analyzed the transport properties of the present system in terms of the current density. As expected, we found a null current in the Cartesian coordinates; however, in polar coordinates the angular component of the current was nonvanishing. Finally, we studied the nonrelativistic limit where known results were recovered.

The emergence of the quantum Hall effect in graphene [7,8] opened a good opportunity not only for experimentalists but also for theorists as well. Because of the relativistic nature of the fermions in grapheme and due to some additional constraints, the appropriate mathematical system seems to be the massless Dirac fermions. However, our present work suggests that present theoretical investigations in the literature did not include adequately contributions from all solution parameter space [14] and hence will lead to incomplete, and sometimes erroneous, results. Extending our present analysis to the massless Dirac fermion system will be desirable to put the theoretical approach to grapheme systems on firm grounds.

Finally, we think that it will be appropriate to look for the irregular solutions of the present problem. The importance of this issue comes from the fact that it will help us to construct the two-point Green's function, which is very much needed in the calculation of many physical quantities and will enable us build the corresponding conformal theory.

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APPENDIX: PROPERTIES OF THE ASSOCIATED LAGUERRE POLYNOMIALS

The following are useful formulas and relations satisfied by the generalized orthogonal Laguerre polynomials $L_n^\nu(x)$ that are relevant to the developments carried out in this work. They are found in most textbooks on orthogonal polynomials [22]. We list them here for ease of reference.

The differential equation

$$\left[x\frac{d^2}{dx^2}+(\nu+1-x)\frac{d}{dx}+n\right]L_n^\nu(x)=0, \qquad (A1)$$

where $x\geq 0$, $\nu>-1$, and $n=0,1,2,\ldots$, could be expressed in terms of the confluent hypergeometric function as

$$L_n^\nu(x)=\frac{\Gamma(n+\nu+1)}{\Gamma(n+1)\Gamma(\nu+1)}\,{}_1F_1(-n;\nu+1;x). \qquad (A2)$$

The associated three-term recursion relation is

$$xL_n^\nu=(2n+\nu+1)L_n^\nu-(n+\nu)L_{n-1}^\nu-(n+1)L_{n+1}^\nu. \qquad (A3)$$

Other useful recurrence relations are

$$xL_n^\nu=(n+\nu)L_n^{\nu-1}-(n+1)L_{n+1}^{\nu-1}, \qquad (A4)$$

$$L_n^\nu=L_n^{\nu+1}-L_{n-1}^{\nu+1}, \qquad (A5)$$

the differential formula

$$x\frac{d}{dx}L_n^\nu=nL_n^\nu-(n+\nu)L_{n-1}^\nu, \qquad (A6)$$

and the orthogonality relation

$$\int_0^\infty \rho^\nu(x)L_n^\nu(x)L_m^\nu(x)dx=\frac{\Gamma(n+\nu+1)}{\Gamma(n+1)}\delta_{nm}, \qquad (A7)$$

where

$$\rho^\nu(x)=x^\nu e^{-x}. \qquad (A8)$$

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