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Probability Distribution of Electric Fields in Thermal and Nonthermal Plasmas

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The probability distribution of electric fields, $P(E)$, in stable and unstable plasmas is investigated by numerical simulation techniques. The usefulness of this approach is demonstrated by comparing the present results with both new and previously published analytic calculations. The results have implications for experiments which use spectral line shapes [governed by $P(E)$] as a diagonostic tool to determine the plasma density and temperature.

I. INTRODUCTION

A number of plasma effects are related to the probability distribution $P(E)$ of electric fields in the plasma. For example, the shape and width of spectral lines produced by atomic radiators (a useful diagnostic tool since it can be employed as a remote and noninterfering density and temperature probe over a wide range of these parameters) is in many cases dominated by Stark broadening, which is directly related to $P(E)$.

Many theories have been proposed to deduce the probability distribution of electric fields in a thermal plasma. Holtsmark[1] simplified the problem by neglecting all interactions between particles. Baranger and Mozer[2,3] have included particle-particle correlations to various orders and have expressed the distribution as a series expansion which reduces to the Holtsmark result in the infinite temperature limit.

Baranger and Mozer also distinguished between two components of the electric field, a high-frequency and a low-frequency component. The total field is given as a sum of these two components. The high-frequency component is produced by the electrons, and its time variation is governed by the motion of the electrons. The low-frequency variation is governed by the motion of ions and their attendant shield clouds. This component is obtained by averaging the total field at a point in space over times which were long compared to electron relaxation times. Hooper[4,5] has also obtained an expansion of the distribution function for a higher-density lower-temperature plasma. Other attempts[6-10] at improving the Holtsmark result neglected correlations but modified the force between particles to include shielding effects.

All of these calculations assume that most of the electric fields come from nearest-neighbor particles, and that long-range collective oscillations can be neglected.

Ecker and Fischer[11] have calculated the effect of plasma oscillations on the high-frequency probability distribution and have shown that these collective fields have little effect on the distribution $P(E)$.

Ecker and Fischer's result does not imply that collective fields can never have a great effect on the total distribution function. Nonthermal situations can lead to an enhancement of wave energy such that the distribution of fields is altered. This, in fact, has been seen experimentally.[12,13] There is some discrepancy between these various analyses. In addition, the problem of $P(E)$ in a nonthermal or unstable plasma has been virtually tions along with various analytic calculations to resolve the discrepancies in previous thermal calculations of $P(E)$, to establish that computer simulation is a valid tool for determining the probability distribution, and to show how departure of the plasma from thermal equilibrium will alter the probability distribution. These results indicate to what extent the indiscriminate application of any of the existing calculations to nonthermal or unstable plasmas may lead to erroneous results in the calculation of line shapes and plasma parameters which depend on line shapes.

In Sec. II we describe the method of applying simulation technique to the study of $P(E)$; in Sec. III we summarize our results for low-frequency fields; in Sec. IV we present a calculation of wave field distributions, and in Sec. V compare our results with existing calculations.

II. FIELD DISTRIBUTIONS PRODUCED BY SIMULATION PARTICLES

In order to test some of the existing theories, we have carried out various computer simulations to determine the electric field distribution $P(E)$.

Owing to the size limitations of present-day computers, far fewer particles can be treated in the simulation than in the laboratory plasma. Consequently, fluctuations due to collisions are orders of magnitude greater in the simulation plasma.

In order to reduce these collisional fluctuations the force law between particles is smoothed at close range while keeping the long-range force unaltered, as suggested by Dawson.[14]

In particular, we spread the particles in x space by giving them a Gaussian charge density, $1/(2\pi a^2)^{n/2}\, e^{-x^2/2a^2}$, where a is the size of the cloud and n is the dimensionality of the system. This has the effect of spreading the particles in k space by $e^{-k^2a^2/2}$. Therefore, amplitudes of all modes such that $ka \gg 1$ are exponentially small. Since the short-wavelength modes produce collisions, the collisional effects are greatly reduced.

In order to compare the simulation results with the theoretical predictions, the probability distributions must be reevaluated for finite-size particles moving in a two-dimensional plasma, to separate effects likely to be found in the laboratory from effects due to the size of the simulated particles. To do this we use the electric field produced by a test particle in this plasma, as derived by Langdon,[15]

$$E(\vec{r}) = -\frac{\partial}{\partial \vec{r}}\frac{q_T}{2\pi}\int d\vec{k}\, s\cdot\frac{e^{-k^2a^2/2}e^{i\vec{k}\cdot\vec{r}}}{k^2+s^2/\lambda_D^2} = E_0\beta(\vec{r}), \tag{1}$$

where q_T is the charge of the test particle, $E_0 = 2q_T/\lambda_D$, s is the cloud shape, and λ_D is the Debye length.

Since shielding is a statistical effect, and the low-frequency component of the field is obtained by a short-time average over the total field, ions do not have time to shield other ions. Therefore λ_D is employed in the calculation of low-frequency fields, not $\lambda_D/\sqrt{2}$.

Using the results of Baranger and Mozer[2,3] to neglect two-body correlations, we will assume that the shielded particles are statistically independent, and include correlations only through shielding of the ions. Using this approximation the probability distribution reduces to

$$P(\vec{E}) = \int\cdots\int d\vec{r}_1\cdots d\vec{r}_N \frac{1}{A^N}\,\delta\Big(\vec{E}-\sum_1^N \vec{E}_i(\vec{r}-\vec{r}_i)\Big),$$

where $\vec{E}_i(\vec{r}-\vec{r}_i)$ is the field produced at $\vec{r}$ by particle i, and $1/A^N$ is the configurational distribution. A is the area of the system and N the total number of particles in the system. Expressing the δ function as an integral gives

$$P(\vec{E}) = [1/(2\pi)^2]\int e^{i\vec{\alpha}\cdot\vec{E}}F(\alpha)\,d\vec{\alpha},$$

where

$$F(\vec{\alpha}) = [(1/A)\int e^{-i\alpha\cdot\vec{E}(\vec{r})}\,d\vec{r}]^N.$$

For isotropic plasmas $\vec{E}(\vec{r}) = \hat{r}E(\vec{r})$. Letting $\vec{\alpha}$ be in the direction of the x axis and integrating over angles gives

$$F(\alpha) = [(2\pi/A)\int_0^R J_0(\alpha E(r))\,r\,dr]^N.$$

Letting N, $A\to\infty$ but keeping N/A = constant $= n_0$ and using the relation

$$\lim_{N\to\infty}(1-x/N)^N = e^{-x}$$

gives for F

$$F(\alpha) = \exp\Big(-n_0\lim_{R\to\infty}\,[\pi R^2 - 2\pi\int_0^R J_0(\alpha E(r))\,r\,dr]\Big).$$

The electric field of the shielded ion is

$$E(r) = E_0\beta(r),$$

where F_0 is $2q/\lambda_D$ and $\beta(r)$ is given in Eq. (1). Letting $\alpha' = \alpha E_0$, F becomes

$$F(\alpha') = \exp\Big(-n_0\pi\lambda_D^2\,[\lim_{R'\to\infty}(R')^2 - 2\int_0^{R'}J_0(\alpha'\beta(r'))\,r'\,dr']\Big), \tag{2}$$

where $R' = R/\lambda_D$ and $r' = r/\lambda_D$.

For a thermal plasma with no magnetic fields, P is isotropic, i.e.,

$$P(E)\,dE = 2\pi E P(\vec{E})\,dE.$$

This gives

$$P(E)\,dE = \frac{dE\,E}{2\pi E_0^2}\int d\vec{\alpha}'\,e^{i\vec{\alpha}'\cdot\vec{E}/E_0}F(\alpha').$$

If $\beta = E/E_0$ then P becomes for these dimensionless units

$$P(\beta)\,d\beta = d\beta\,\beta\int d\alpha'\,\alpha' J_0(\alpha'\beta)F(\alpha'),$$

where $F(\alpha')$ is given by Eq. (2). These integrals were evaluated numerically and the results for various values of cloud size and density appear in Fig. 1. If ions take part in the shielding then the denominator of Eq. (1) becomes $k^2 + 1/\lambda_i^2 + 1/\lambda_e^2$. Since $\lambda_e = \lambda_i$ for thermal plasmas, the denominator becomes $k^2 + 2/\lambda_e^2$. The results of this substitution are also indicated in Fig. 1. In a real situation the time scale of the measurement will make it obvious whether shielding due to all particles (or electrons only) is appropriate.

III. LOW-FREQUENCY SIMULATION RESULTS

The simulations were performed on Maryland's Univac 1108 and the code employed is similar to that described in Ref. 16. All runs were made on a system of 64×64 cells.

At each time step the electric field at each space point was averaged over a period of time, to obtain the low-frequency field. The distribution was calculated by forming a histogram of the time-averaged fields. The statistics were improved by repeating the time average over succeeding time intervals.

Although the theoretical analysis neglected ion motions and correlations between ions, a simulation must retain ion motions to reliably measure the low-frequency microfield probability. This is demonstrated in Figs. 2 and 3. Figure 2 shows a simulation with evenly spaced infinitely heavy ions. A short-time average removes the high-frequency fields, and gives probability distributions comparable to the theory. But even a relatively small increase in averaging time makes the comparison with theory worse! It is clear that this result is to be expected, since with fixed ions the electron microfields averaged over many plasma periods will vanish, and the distributions tend toward a δ function. A proper simulation of the low-frequency field must include ion motion, as shown in Fig. 3. Now after a few plasma periods the high-frequency fields are averaged out, and the simulation results compare more and more favorably with theory as an increasingly long averaging time is taken. Note that Fig. 3 gives good agreement with a time average of over a hundred electron plasma periods, while "fixed ion" simulation already disagreed with theory after six electron plasma periods. The horizontal error bars indicate the variance of an amount ΔE about E due to the histogramming, and the vertical

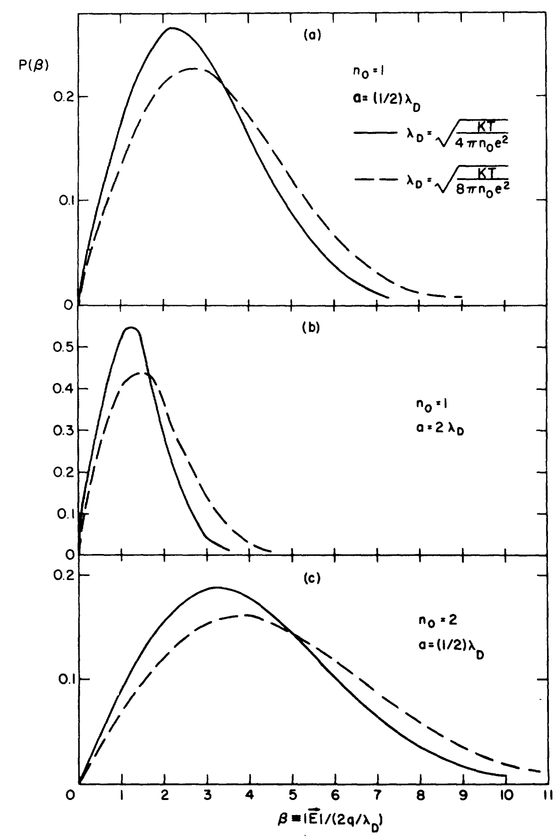

FIG. 1. Analytic calculations of the low-frequency microfield ($\beta \equiv E/E_0$) probability distribution in plasmas of finite-size particles moving on a two-dimensional grid, for various values of particle size a, and density n_0 (in relative units). The solid curves correspond to shielding of ions by electrons only; the dashed curves include electron and ion shielding.

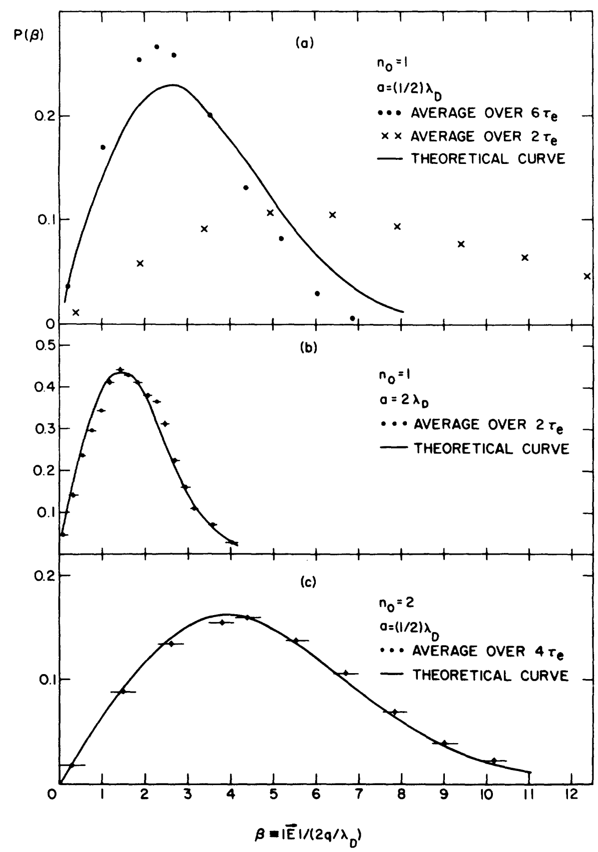

FIG. 2. Simulation results for the low-frequency microfield probability of a plasma with fixed ions, compared with analytic calculations. The microfields have been averaged over various periods of time, as would be appropriate in estimating the effect of the microfields on various line-emission problems [$\tau_e = (m/4\pi n e^2)^{1/2}$].

error bars indicate the statistical fluctuations arising from the finite number of data points obtained in the simulation. This result shows that fields averaged over this amount of time are little affected by the ion shielding of other ions.

Thus it is seen that for low-frequency fields, a moving-ion simulation agrees well with theoretical results.

We emphasize that in previous analytic calculations the same time averages are implicit in the approximations used, while in simulation the effects of time averaging must be dealt with directly.

IV. WAVE FIELD DISTRIBUTIONS

For the calculation of the contribution of plasma waves to the field distribution we use an alternate approach to the calculation of Ecker and Fischer[11] and also include the finite-size-particle effects. The present approach is more general in that it allows for a calculation of collective field distributions in nonthermal quasi-steady-state plasmas as well as thermal plasmas.

Consider a one-dimensional system with only a single wave given by

$$E_k = E_{0k} \sin kx .$$

The wave exists at all points in the system, and therefore has equal probability of being at any one point. The probability of finding E within dE is given by

$$P(E_k)\, dE_k = W(\theta)\, d\theta = c\, d\theta ,$$

where $\theta = kx$ and c is a constant. Therefore

$$P(E_k) = \frac{c}{dE_k/d\theta} = \frac{c}{(E_{0k}^2 - E_k^2)^{1/2}} .$$

Normalizing to unity gives

$$P(E_k) = \frac{1}{\pi} \frac{1}{(E_{0k}^2 - E_k^2)^{1/2}} .$$

If there are many waves in the system, the probability of measuring a particular electric field value E_0 is given by the sums of the probabilities of all possible combinations which give $E = E_0$. Generalizing to more than one dimension gives

$$P(\vec{E}) = \prod_{\vec{k}=1}^{M} \int \delta\left(\vec{E} - \sum_{\vec{k}=1}^{M} \vec{E}_{\vec{k}}\right) P_{\vec{k}}(\vec{E}_{\vec{k}})\, d\vec{E}_{\vec{k}} ,$$

where there are M modes in the system, the amplitude of mode $\vec{k}$ being $\vec{E}_{0\vec{k}}$.

If E is a traveling wave,

$$\vec{E}_{\vec{k}} = \vec{E}_{0\vec{k}_1} \cos(\vec{k}\cdot\vec{x} + \phi_k) + \vec{E}_{0\vec{k}_2} \sin(\vec{k}\cdot\vec{x} + \gamma_{\vec{k}}) ,$$

$$\vec{E}_{\vec{k}} = \vec{E}_{\vec{k}_1} + \vec{E}_{\vec{k}_2}$$

(where ϕ and γ are random phases), then by considering the two terms in the sum as independent the probability becomes

$$p(\vec{E}) = \prod_{\vec{k}} \int \delta\left(\vec{E} - \sum_{\vec{k}} \vec{E}_{\vec{k}}\right) p(\vec{E}_{\vec{k}_1}) p(\vec{E}_{\vec{k}_2})\, d\vec{E}_{\vec{k}_1}\, d\vec{E}_{\vec{k}_2} .$$

For a two-dimensional system, after expressing the δ function as an integral, P is given by

$$P(\vec{E}) = [1/(2\pi)^2] \int d\vec{q}\, e^{i\vec{q}\cdot\vec{E}} P(\vec{q}) ,$$

where

$$P(\vec{q}) = \left(\prod_{\vec{k}=1}^{M} \int d\vec{E}_{\vec{k}_1} \frac{\exp(i\vec{q}\cdot\vec{E}_{0\vec{k}_1}\cos\theta}{\pi(\vec{E}_{0\vec{k}_1}^2 - \vec{E}_{\vec{k}_1}^2)^{1/2}} \right)^2 ,$$

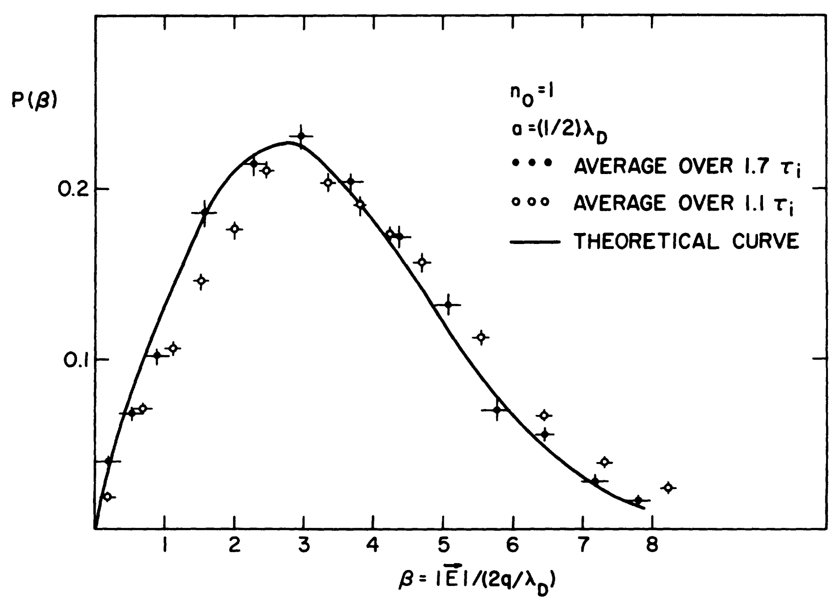

FIG. 3. Simulation of $p(\beta)$ including shielding by thermal ions, showing that an average over a few τ_i is sufficient to accurately measure $p(\beta)$ from a plasma including thermal ions.

and where we have used the fact that the integrals over $d\vec{E}_{\vec{k}_1}$ and $d\vec{E}_{\vec{k}_2}$ are identical.

The work required to create the fluctuation $\vec{E}_{\vec{k}_1}$ is $\vec{E}^2_{0\vec{k}_1}/8\pi$, and therefore the probability of finding its amplitude $\vec{E}_{0\vec{k}_1}$ in $d\vec{E}_{0\vec{k}_1}$ about $\vec{E}_{0\vec{k}_1}$ is

$$P(\vec{E}_{0\vec{k}_1}) = \frac{\exp(-\vec{E}^2_{0\vec{k}_1}/\langle \vec{E}^2_{0\vec{k}_1}\rangle)}{\int d\vec{E}_{0\vec{k}_1}\exp(-E^2_{0\vec{k}_1}/\langle E^2_{0\vec{k}_1}\rangle)}\,.$$

This probability must be included in the integrals comprising $P(\vec{q})$

$$P(\vec{q}) = \prod_{\vec{k}} \int d\vec{E}_{0\vec{k}_1} J_0(\vec{q}\cdot\vec{E}_{0\vec{k}_1}) \times \frac{\exp(-\vec{E}^2_{0\vec{k}_1}/\langle \vec{E}^2_{0\vec{k}_1}\rangle)}{\int d\vec{E}_{0\vec{k}_1}\exp(\vec{E}^2_{0\vec{k}_1}/\langle \vec{E}^2_{0\vec{k}_1}\rangle)}\,.$$

The calculation of $P(\vec{E})$ would involve a large amount of numerical integration. A simple quantity to calculate is $P(E_x)$ defined by

$$P(E_x) = \int dE_y P(\vec{E})\,.$$

Using the fact that

$$(1/2\pi)\int dE_y\, e^{iq_y E_y} = \delta(q_y)$$

gives

$$P(E_x) = (1/2\pi)\int dq_x\, e^{iq_x E_x} P(q_x)\,, \tag{3}$$

where

$$P(q_x) = \left(\prod_{\vec{k}} \exp(-\tfrac{1}{2}q_x^2\langle \vec{E}^2_{0\vec{k}}\rangle) I_0(\tfrac{1}{4}q_x^2\langle \vec{E}^2_{0\vec{k}}\rangle)\right)^2,$$

where I_0 is the Bessel function of imaginary argument.

For our simulation plasma $\langle \vec{E}^2_{0\vec{k}}\rangle$ is given by[14]

$$\langle E^2_{0k}\rangle = \frac{4\pi}{L^2}\frac{kT}{1+k^2\lambda_D^2\exp(k^2a^2)}\,.$$

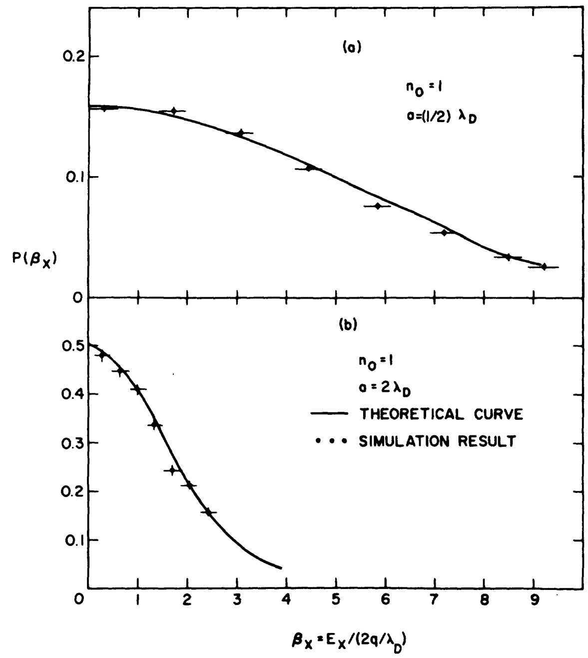

FIG. 4. Simulation and analytic calculations of the high-frequency value of the microfield probability distribution (no time averaging) in a plasma of finite-size particles, for two values of particle size.

Figure 4 gives the results of theoretical calculations for two cases of cloud radius obtained by numerical integration of Eq. (3). The number of modes employed in the calculation was such that the inclusion of additional short-wavelength modes had little effect on the result.

The above result for the wave field distribution is similar to Ecker and Fischer's. They have a term $\exp(\vec{Q}_{\vec{k}}\, e^{i\vec{k}\cdot\vec{r}})$ in their collective field distribution which they set equal to $e^{\vec{Q}_{\vec{k}}}$. This is equivalent to evaluating the probability for wave amplitudes only and does not consider the spatial

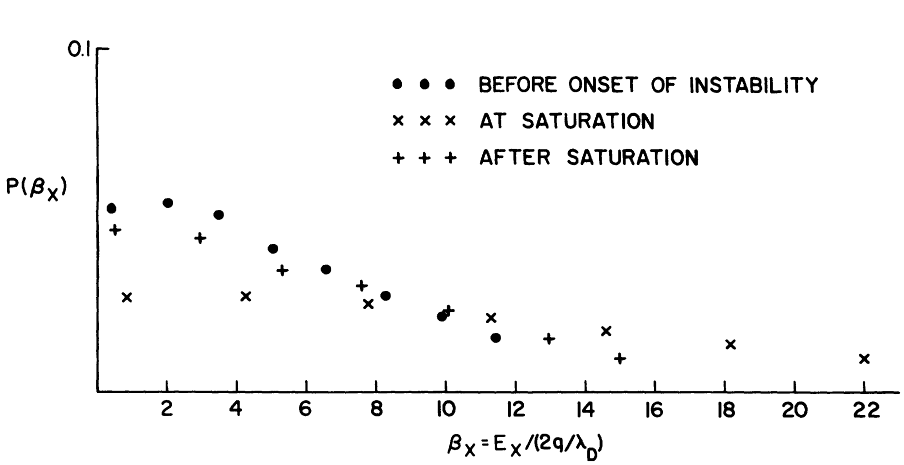

FIG. 5. Time history of the microfield probability distribution in a two-beam unstable plasma.

variation of the wave. Ecker and Fischer's calculation involves a cutoff between long-wavelength and short-wavelength modes and also an expansion in powers of kV/ω_p. Our procedure is superior in that it includes all modes of the electric field fluctuations, i.e., the electron plasma oscillations. (Ion waves are heavily damped in thermal equilibrium.) Also, if the plasma is in a nonthermal steady state, our method allows calculation of the wave field distribution for any known energy spectrum.

Such a spectrum might be known from an independent calculation of the asymptotic state of an unstable plasma. Since the wave fields for the thermal plasma consist for the most part of the electron plasma waves, the distribution of these fields was obtained by simulation of a gas of electrons with a neutralizing background. Since these are high-frequency fields, no time averaging was performed on the fields. The results of these simulations appear in Fig. 4 and are compared to the theoretical results.

For an unstable plasma there are modes that do grow to levels far in excess of their equilibrium values. Figure 5 gives the wave field distribution of electric fields in a particular unstable plasma as determined from plasma simulation. The instability is the two-beam instability and the figure shows the distributions before the onset of the instability, when the instability is fully developed, and after the fields and particles have come to an equilibrium after saturation. In the third case the plasma is in a stable, nonthermal state and the probability of measuring the large fields is larger at this stage than in a thermal plasma of the same density.

V. CONCLUSIONS

By comparing analytic calculations and simulation results for thermal plasmas, we have established simulation as a tool for the calculation of probability distribution of electric fields, and used it to calculate $P(E)$ for an unstable plasma. In addition, we have shown that, after suitable time averaging of the fields, theoretical predictions of the distribution of the fields due to shielded ions is borne out. This shows the real effect of the averaging procedure implied in the usual treatments (Baranger and Mozer) of the microfield distribution in warm plasmas.

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