$$Q(R,\phi,t)=\sum_{n,m} A_{nm} e^{-\lambda_{n,m}t+im\phi} U_{nm}(R) \tag{42}$$

where n and m are integers, and $U_{n,m}(R)$ is the eigenfunction corresponding to $\lambda_{n,m}$. From (42), we can identify the separation constants $\lambda_{m,0}$ and $\lambda_{0,1}$ with our eigenvalues of Sec. II.

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[1]M. O. Scully and W. E. Lamb, Jr., Phys. Rev. **159**, 208 (1967), Paper I of a series of papers on the quantum theory of a laser.

[2]See, for example, A. Ralston and H. Wilf, *Mathematical Methods for Digital Computer* (Wiley, New York, 1962), Vol. I, p. 110.

[3]M. Sargent, III, M. O. Scully, and W. E. Lamb, Jr., Appl. Opt. **2**, 2423 (1970).

[4]J. P. Gordon and W. W. Aslaksen, IEEE J. Quantum Electron. **QE-6**, 428 (1970).

[5]R. J. Glauber, Phys. Rev. **131**, 2766 (1963).

[6]Equation (39) is an approximation, valid only near threshold, the complete form correct to order g^4 can be found in, Y. K. Wang, Ph.D. thesis (Yale University, 1971) (unpublished), which can be obtained from University Microfilm, Ann Arbor, Mich.

[7]R. D. Hempstead and M. Lax, Phys. Rev. **161**, 350 (1967).

[8]H. Risken, Prog. Opt. **8**, 240 (1970).

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Theory of Some Laser Noise Effects*†

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Using semiclassical theory, we investigate the shot- and thermal-noise effects on the behavior of a laser. The Fokker-Planck equations for the probability distribution of the laser field are derived. These equations are approximately solved, using a Gaussian function, from which we calculate the spectral profile of the laser field. The width constant for the thermal noise is related to the temperature of the cavity.

I. INTRODUCTION

The basic paper on laser theory[1] was semiclassical, in that quantum-mechanical atoms were coupled to a classical electromagnetic field. It gave a satisfactory discussion of phenomena such as the threshold condition, power output, frequency pulling and pushing, mode competition, frequency locking, etc., but omitted any consideration of fluctuation phenomena of the laser. Later, one of us[2] extended the semiclassical method to consider the phase diffusion caused by thermal fluctuations and found the corresponding width of the Lorentzian spectral profile of the laser radiation. The development of a fully quantum-mechanical laser theory by Scully and Lamb[3] made possible[4] a calculation of both thermal and spontaneous emission contributions to the spectral profile. With this as a guide, a simple change in the noise polarization of Ref. 2 leads to the correct linewidth.

Many other papers have been written on laser noise phenomena. Very complete bibliographies have been given by Lax[5] and by Haken.[6] With few exceptions, the emphasis of these papers has been on noise phenomena, and the underlying laser theory has been rather schematic and not as well adapted for a discussion of the actual operating characteristics of a laser, somewhat above threshold, as the semiclassical theory of Ref. 1. The present work applies a simple version of the semiclassical theory to shot effect, and also extends the previous consideration[2] of thermal noise to allow for amplitude fluctuations.

As in Ref. 1, the laser is considered to be a lossy cavity of the Fabry-Perot type in single-mode operation with circular frequency ν driven by an inverted population of active atoms. The electric field is taken to be transverse to the cavity axis:

$$E(z,t)=E(t)\cos[\nu t+\varphi(t)]\sin Kz, \tag{1}$$

where z is the distance measured along the cavity axis and K is the wave number $K=n\pi/L$, with L being the length of the cavity and the mode number

n being a large integer.

The amplitude $E(t)$ and phase $\varphi(t)$ are assumed to be slowly varying functions and can be shown[1] to satisfy the following self-consistency equations:

$$(\nu - \Omega + \dot{\varphi})E = -\tfrac{1}{2}(\nu/\epsilon_0)C(t), \tag{2}$$

$$\dot{E} + \tfrac{1}{2}(\nu/Q)E = -\tfrac{1}{2}(\nu/\epsilon_0)S(t), \tag{3}$$

where Ω is the cavity eigenfrequency, Q is the cavity quality factor, $S(t)$ and $C(t)$ are the sine and cosine components of the polarization function $P(t)$ defined by

$$P(t) = (2/L)\int_0^L dz\, P(z, t)\sin Kz, \tag{4}$$

and $P(z, t)$ is the macroscopic polarization.

The active atoms are taken to have two excited energy levels W_a and W_b separated by a resonant frequency ω, between which the laser action takes place. The levels a and b are assumed phenomenologically to decay to lower levels at rates γ_a and γ_b, respectively. Considering only the electric dipole interaction between the field and the atoms, the functions $S(t)$ and $C(t)$ are calculated in Ref. 1 by a perturbation technique, and Eq. (3) is shown to be in the form

$$\dot{E} = \alpha E - \beta E^3, \tag{5}$$

where α and β are constants determined by the various parameters of the laser. We shall consider the effect of shot noise in Sec. II using the above semiclassical model of the laser. In Sec. III, we shall apply the same method to the thermal noise, and finally, a numerical example is given in Sec. IV.

II. SHOT-NOISE EFFECTS

A. Fokker-Planck Equation

For simplicity, we consider a laser in single-mode operation and neglect atomic motion. The active atoms are assumed to be excited only to the upper energy level at random times with average rate λ.

In third-order perturbation theory, the amplitude $E(t)$ of (3) satisfies an equation

$$\dot{E}(t) = -\tfrac{1}{2}(\nu/Q)E(t) - \tfrac{1}{2}(\nu/\epsilon_0)[S^{(1)}(t) + S^{(3)}(t)], \tag{6}$$

where $S^{(1)}(t)$ and $S^{(3)}(t)$ are the first- and third-order terms in the out-of-phase part of the macroscopic polarization, respectively.

Consider one active atom excited at time t_k. The polarization contributed by this atom can be written as

$$s(t, t_k) = s^{(1)}(t, t_k) + s^{(3)}(t, t_k). \tag{7}$$

Here, again $s^{(1)}(t, t_k)$ and $s^{(3)}(t, t_k)$ are the first- and third-order terms. Since the total polarization is equal to the sum of polarizations contributed by each atom, we have

$$\dot{E}(t) = -\tfrac{1}{2}(\nu/Q)E(t) - \tfrac{1}{2}(\nu/\epsilon_0)\sum_k [s^{(1)}(t, t_k) + s^{(3)}(t, t_k)], \tag{8}$$

where the sum is taken over all the active atoms.

The atoms in the laser have lifetimes γ_a^{-1} and γ_b^{-1} typically of the order of 10^{-8} sec, which are very short compared to the time in which $E(t)$ changes significantly ($\approx 10^{-5}$ sec). Hence, we may treat $s^{(1)}(t, t_k)$ and $s^{(3)}(t, t_k)$ as similar to δ functions $\delta(t - t_k)$. Since $s^{(1)}(t, t_k)$ is proportional to $E(t)$, we may write

$$s^{(1)}(t, t_k) = \sigma_1 E(t)\delta(t - t_k). \tag{9}$$

Similarly,

$$s^{(3)}(t, t_k) = \sigma_3 E(t)^3\delta(t - t_k). \tag{10}$$

The constants σ_1 and σ_3 can be found by carrying out detailed calculations of $s^{(1)}(t, t_k)$ and $s^{(3)}(t, t_k)$. In terms of σ_1 and σ_3, we have

$$\dot{E} = -\tfrac{1}{2}(\nu/Q)E - \tfrac{1}{2}(\nu/\epsilon_0)(\sigma_1 E + \sigma_3 E^3)\sum_k \delta(t - t_k). \tag{11}$$

To solve the stochastic differential equation (11), we multiply by $2E$ on both sides:

$$\frac{d}{dt}(E^2) = -\left(\frac{\nu}{Q}\right)E^2 - \left(\frac{\nu}{\epsilon_0}\right)(\sigma_1 E^2 + \sigma_3 E^4)\sum_k \delta(t - t_k) \tag{12}$$

or

$$\frac{d(E^2)/dt}{E^2 + (\sigma_3/\sigma_1)E^4} = -\left(\frac{\nu}{Q}\right)\frac{E^2}{E^2 + (\sigma_3/\sigma_1)E^4} - \left(\frac{\nu\sigma_1}{\epsilon_0}\right)\sum_k \delta(t - t_k). \tag{13}$$

Introducing a new variable x such that

$$dx = [E^2 + (\sigma_3/\sigma_1)E^4]^{-1}d(E^2)$$

or

$$x = \ln\{E^2[1 + (\sigma_3/\sigma_1)E^2]^{-1}\}, \tag{14}$$

Eq. (13) becomes

$$\frac{dx}{dt} = -(\nu/Q)G(x) + (\nu\sigma_1/\epsilon_0)\sum_k \delta(t - t_k), \tag{15}$$

where

$$G(x) = 1 - (\sigma_3/\sigma_1)e^x.$$

Let Δt be an interval of time which is short compared to the time scale of $E(t)$, but long enough for many active atoms to be excited. The existence of this Δt is equivalent to the assumption that Eq. (15) is a Markoffian stochastic equation.[7] The number of atoms excited in the laser between t and $t + \Delta t$ is

$$N(t, \Delta t) = \int_t^{t+\Delta t} dt \sum_k \delta(t - t_k). \tag{16}$$

The probability distribution $\tau(N)$ of $N(t, \Delta t)$ will be a Poisson distribution which in the limit of large N goes to a Gaussian distribution:

$$\tau(N) = (2\pi\lambda\Delta t)^{-1/2} \exp\left(-\frac{(N - \lambda\Delta t)^2}{2\lambda\Delta t}\right), \tag{17}$$

where λ is the average rate of excitation at t.

From Eq. (17) it can be shown, using random-variable analysis,[7] that the probability distribution $p(x, t)$ of the variable x of Eq. (15) satisfies the Fokker-Planck equation

$$\frac{\partial p(x, t)}{\partial t} = \frac{\partial}{\partial x}\left[\left(\frac{\nu}{Q}\right)G(x)p - \left(\frac{\lambda\sigma_1\nu}{\epsilon_0}\right)p\right] + \tfrac{1}{2}\lambda\left(\frac{\nu\sigma_1}{\epsilon_0}\right)^2\left(\frac{\partial^2 p}{\partial x^2}\right). \tag{18}$$

To obtain the equation for the probability distribution $\omega(E, t)$ of finding for the field amplitude $E(t)$ a value E at time t, we note that

$$\omega(E, t)dE = p(x, t)dx. \tag{19}$$

Using (14) and (19), we can transform (18) into an equation for $\omega(E, t)$:

$$\frac{\partial \omega(E, t)}{\partial t} = \frac{\partial}{\partial E}\left[\frac{1}{2}\left(\frac{\nu}{Q}\right)E\omega - \left(\frac{\lambda\nu}{2\epsilon_0}\right)(\sigma_1 E + \sigma_3 E^3)\omega\right] + \tfrac{1}{2}\lambda\left(\frac{\nu}{2\epsilon_0}\right)^2\left[\frac{\partial^2}{\partial E^2}[(\sigma_1 E + \sigma_3 E^3)^2\omega] - \frac{\partial}{\partial E}[(\sigma_1 E + \sigma_3 E^3)(\sigma_1 + 3\sigma_3 E^2)\omega]\right], \tag{20}$$

which is the Fokker-Planck equation describing shot noise in a laser.

From (20), it is easy to obtain the equation for the average amplitude of the field $\langle E\rangle$ defined by

$$\langle E\rangle = \int_{-\infty}^{\infty} dE\, E\omega(E, t). \tag{21}$$

Differentiating (21) with respect to t, we have

$$\frac{d\langle E\rangle}{dt} = \int_{-\infty}^{\infty} dE\, E\, \frac{\partial\omega(E, t)}{\partial t}$$
$$= \int_{-\infty}^{\infty} dE\, E\left[\frac{\partial}{\partial E}[\tfrac{1}{2}(\nu/Q)E\omega - (\lambda\nu/2\epsilon_0)(\sigma_1 E + \sigma_3 E^3)\omega] + \tfrac{1}{2}\lambda(\nu/2\epsilon_0)^2\left(\frac{\partial^2}{\partial E^2}[(\sigma_1 E + \sigma_3 E^3)^2\omega] - \frac{\partial}{\partial E}[(\sigma_1 E + \sigma_3 E^3)(\sigma_1 + 3\sigma_3 E^2)\omega]\right)\right].$$

Integrating by parts we have

$$\frac{d\langle E\rangle}{dt} = -\tfrac{1}{2}(\nu/Q)\langle E\rangle + (\lambda\nu/2\epsilon_0)[\sigma_1\langle E\rangle + \sigma_3\langle E^3\rangle] + \tfrac{1}{2}\lambda(\nu/2\epsilon_0)^2[\sigma_1^2\langle E\rangle + 4\sigma_1\sigma_3\langle E^3\rangle + 3\sigma_3^2\langle E^5\rangle]. \tag{22}$$

Equation (22) generalizes the amplitude equation (5) to allow for shot effect. We can reduce (22) to (5) by neglecting the shot noise, i.e., letting $\lambda \to \infty$, $\sigma_1, \sigma_3 \to 0$ such that $\lambda\sigma_1$ and $\lambda\sigma_3$ remain finite, then $\langle E\rangle$, $\langle E^3\rangle$ reduce to $E(t)$ and $E(t)^3$ in (5), respectively, and (22) becomes

$$\frac{dE(t)}{dt} = -\tfrac{1}{2}(\nu/Q)E(t) + (\lambda\nu/2\epsilon_0)[\sigma_1 E(t) + \sigma_3 E(t)^3].$$

Hence we can relate the coefficients α and β of Eq. (5) to σ_1 and σ_3 by

$$\alpha = -\tfrac{1}{2}(\nu/Q) + (\lambda\nu/2\epsilon_0)\sigma_1, \qquad \beta = -(\lambda\nu/2\epsilon_0)\sigma_3. \tag{23}$$

The Fokker-Planck equation can then be written in the form

$$\frac{\partial\omega(E, t)}{\partial t} = \frac{\partial}{\partial E}[-(\alpha E - \beta E^3)\omega] + \left(\frac{1}{2\lambda}\right)\left(\frac{\partial^2}{\partial E^2}\left\{\left[\left(\alpha + \frac{\nu}{2Q}\right)E - \beta E^3\right]^2\omega\right\} - \frac{\partial}{\partial E}\left\{\left[\left(\alpha + \frac{\nu}{2Q}\right)E - \beta E^3\right]\left[\left(\alpha + \frac{\nu}{2Q}\right) - 3\beta E^2\right]\omega\right\}\right). \tag{24}$$

The first two field amplitude averages satisfy the following equations:

$$\frac{d\langle E\rangle}{dt} = \alpha\langle E\rangle - \beta\langle E^3\rangle - (1/2\lambda)[(\alpha + \nu/2Q)^2\langle E\rangle - 4\beta(\alpha + \nu/2Q)\langle E^3\rangle + 3\beta^2\langle E^5\rangle], \tag{25}$$

$$\frac{d\langle E^2\rangle}{dt} = 2\alpha\langle E^2\rangle - 2\beta\langle E^4\rangle + (1/\lambda)\{\langle 2[(\alpha + \nu/2Q)^2 E - \beta E^3][(\alpha + \nu/2Q)E - 2\beta E^3]\rangle\}. \tag{26}$$

B. Simple Solutions of Fokker-Planck Equation

Let us consider some approximate solutions of this Fokker-Planck equation for an initial condition

$$\omega(E, 0)=\delta(E-E_0) \quad \text{at } t=0, \tag{27}$$

where E_0 is the electric field amplitude at $t=0$.

If λ is very large, and we can neglect the second term on the right-hand side of the Fokker-Planck equation (24), then

$$\frac{\partial \omega}{\partial t}=-\frac{\partial}{\partial E}[(\alpha E-\beta E^3)\omega]. \tag{28}$$

The solution in this case is trivially

$$\omega(E, t)=\delta(E-\mathcal{E}(t)), \tag{29}$$

where $\mathcal{E}(t)$ is the average field satisfying

$$\dot{\mathcal{E}}(t)=\alpha\mathcal{E}(t)-\beta\mathcal{E}(t)^3, \quad \mathcal{E}(0)=E_0. \tag{30}$$

The physical meaning of this solution is the following. For $\lambda\Delta t\to\infty$, $\tau(N)$ of Eq. (17) behaves like a δ function $\delta(N-\lambda\Delta t)$, and the shot fluctuations are neglected. The probability distribution function $\omega(E, t)$ in this case will also be a δ function.

To consider shot noise, we must keep the second term on the right-hand side of (24). As $\lambda\Delta t\gg 1$ is our basic assumption, we expect that $\omega(E, t)$ should be a sharply peaked function, and try a normalized Gaussian function for $\omega(E, t)$:

$$\omega(E, t)=\frac{(2\pi)^{-1/2}}{\sigma(t)}\exp\left(-\frac{[E-\mathcal{E}(t)]^2}{2\sigma(t)^2}\right), \tag{31}$$

where $\sigma(t)^2\ll\mathcal{E}(t)^2$. The amplitude averages for this $\omega(E, t)$ are

$$\begin{aligned}\langle E\rangle&=\mathcal{E},\\ \langle E^2\rangle&=\mathcal{E}^2+\sigma^2,\\ \langle E^3\rangle&=\mathcal{E}^3+3\mathcal{E}\sigma^2, \quad \text{etc.}\end{aligned} \tag{32}$$

Putting (32) into (25) and (26), we have

$$\frac{d}{dt}(\mathcal{E})=\alpha\mathcal{E}-\beta\mathcal{E}^3+O(\beta\sigma^2\mathcal{E}), \tag{33}$$

$$\frac{d}{dt}(\sigma^2)=2(\alpha-3\beta\mathcal{E}^2)\sigma^2+(1/\lambda)[(\alpha+\nu/2Q)\mathcal{E}-\beta\mathcal{E}^3]^2+O(\beta\sigma^4). \tag{34}$$

As we assume $\sigma^2\ll\mathcal{E}^2$, we can neglect $O(\beta\sigma^2\mathcal{E})$ in (33) and $O(\beta\sigma^4)$ in (34). The solution of (33) is then

$$\mathcal{E}(t)^2=\langle E(t)\rangle^2=(\alpha/\beta)\{1+[(\alpha/\beta E_0^2)-1]e^{-2\alpha t}\}^{-1}. \tag{35}$$

We note that

$$\begin{aligned}\langle E\rangle&\to(\alpha/\beta)^{1/2} && \text{as } t\to\infty\\ &\to E_0 && \text{as } t\to 0.\end{aligned}$$

The steady-state value of $\sigma(t)$ can be found from (34) by putting $d\sigma^2/dt=0$, giving

$$\sigma^2=(1/16\beta\lambda)(\nu/Q)^2 \text{ as } t\to\infty. \tag{36}$$

C. Spectral Profile

We may apply this approximate solution to calculate the laser spectral profile due to shot noise at steady state for a perfectly tuned laser. Consider a laser cavity with a transverse electric field given by

$$E(z, t)=A(t)\sin Kz.$$

In the Appendix, we find that the spectral function $I(\omega)$, i.e., the average energy of the electromagnetic field per unit volume per unit frequency range, is

$$I(\omega)=\frac{\epsilon_0}{2\pi}\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}dt\int_{-T/2}^{T/2}dt'\langle\langle A(t)A(t')\rangle\rangle e^{-i\omega(t-t')}, \tag{37}$$

where the brackets denote an ensemble average and only positive frequencies ω are considered.

For a perfectly tuned laser, we have $\Omega=\nu=\omega$; hence[1]

$$C^{(1)}(t)=0, \quad C^{(3)}(t)=0.$$

The phase equation becomes

$$\dot{\varphi}=0 \text{ or } \varphi=\text{const}$$

and for the choice $\varphi=0$, the steady-state field becomes

$$E(z, t)=E(t)\cos\nu t\sin Kz.$$

The spectral function is then

$$I(\omega)=\lim_{T\to\infty}\frac{\epsilon_0}{2\pi T}\iint_{-T/2}^{T/2}dt\,dt'\times\langle\langle E(t)E(t')\rangle\rangle\cos\nu t\cos\nu t' e^{-i\omega(t-t')}. \tag{38}$$

The correlation function $\langle\langle E(t)E(t')\rangle\rangle$ can be obtained from $\omega(E, t)$:

$$\langle\langle E(t)E(t')\rangle\rangle=\int_{-\infty}^{\infty}dE\int_{-\infty}^{\infty}dE'EE'\omega(E,\infty)\omega(E', |t-t'|), \tag{39}$$

where $\omega(E', |t-t'|)$ is subject to the condition that at $t=t'$, $\omega(E', |t-t'|)=\delta(E'-E)$. Using (31) and (35), we have

$$R(t-t') \equiv \langle E(t)E(t')\rangle = (\alpha/\beta)^{1/2}\frac{(2\pi)^{-1/2}}{\sigma}\int_{-\infty}^{\infty} dE \frac{E^2 \exp[-(E-\sqrt{\alpha}/\sqrt{\beta})^2/2\sigma^2]}{[(1-e^{-2\alpha|\tau|})E^2+(\alpha/\beta)e^{-2\alpha|\tau|}]^{1/2}}, \tag{40}$$

with

$$\sigma^2 = (1/16\beta\lambda)(\nu/Q)^2, \quad \tau = t - t'.$$

To integrate (38), we change variables form t, t' to σ, τ, where $\tau = t-t'$ and $\sigma = t+t'$, and have

$$\int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt' \cdots \to \tfrac{1}{2}\iint_D d\sigma\, d\tau \cdots, \tag{41}$$

where D is the domain in the $\sigma\tau$ plane shown in Fig. 1. Hence

$$I(\omega) = \lim_{T\to\infty} \frac{\epsilon_0}{4\pi T}\iint_D d\sigma\, d\tau\, R(\tau)e^{-i\omega\tau} \times \tfrac{1}{4}[e^{i\nu\sigma}+e^{-i\nu\sigma}+e^{-i\nu\tau}+e^{i\nu\tau}].$$

Only the last term in the above expression contributes to $I(\omega)$ as $T\to\infty$. Therefore

$$I(\omega) = \lim_{T\to\infty}\frac{\epsilon_0}{16\pi T}\iint_D d\sigma\, d\tau\, R(\tau)e^{-i(\omega-\nu)\tau}. \tag{42}$$

Carrying out the σ integration, and letting $T\to\infty$, we have

$$I(\omega) = \frac{\epsilon_0}{8\pi}\int_{-\infty}^{\infty} d\tau\, R(\tau)e^{-i(\omega-\nu)\tau}$$
$$= \frac{\epsilon_0}{8\pi}(\alpha/\beta)^{1/2}\frac{(2\pi)^{-1/2}}{\sigma}\int_{-\infty}^{\infty} d\tau \times \int_{-\infty}^{\infty} dE \frac{E^2\exp[-(E-\sqrt{\alpha}/\sqrt{\beta})^2/2\sigma^2 - i(\omega-\nu)\tau]}{[E^2(1-e^{-2\alpha|\tau|})+(\alpha/\beta)e^{-2\alpha|\tau|}]^{1/2}}. \tag{43}$$

The integrand in (43) is significant only for E near $\sqrt{\alpha}/\sqrt{\beta}$; hence we may expand the integrand into a power series in $(E-\sqrt{\alpha}/\sqrt{\beta})$. The integral (43) can then be carried out easily to yield

$$I(\omega) = (\epsilon_0/8\pi)(\alpha/\beta)\big(2\pi\delta(\omega-\nu) + (1/8\lambda)(\nu/Q)^2 \times\{6[16\alpha^2+(\omega-\nu)^2]^{-1} - [4\alpha^2+(\omega-\nu)^2]^{-1}\} + O(\lambda^{-2})\big), \tag{44}$$

which describes the spectral profile of the laser field. This consists of a monochromatic component superposed on a broad two-peaked distribution of full-width at half-height (FWHH) ~15.6α. The energy of the diffuse part of the spectrum is $(1/16\lambda\alpha)(\nu/Q)^2$ of that monochromatic component.

III. THERMAL NOISE

A passive laser cavity contains blackbody radiation emitted by the walls of the cavity. The average energy of the radiation is given by Planck's law

$$W = \hbar\nu[(e^{\hbar\nu/kT}-1)^{-1}+\tfrac{1}{2}], \tag{45}$$

where ν is the circular laser frequency. A semiclassical treatment of the effect of this radiation on the laser medium can be obtained by putting a stochastic polarization $P^{(0)}(z,t)$ into Maxwell's equations:

$$P^{(0)}(z,t) = \{C^{(0)}(t)\cos[\nu t+\varphi(t)] + S^{(0)}(t)\sin[\nu t+\varphi(t)]\}\sin Kz, \tag{46}$$

where $C^{(0)}(t)$ and $S^{(0)}(t)$ are assumed to be random functions fluctuating rather slowly compared with $e^{i\nu t}$.

We have seen that for a perfectly tuned laser shot noise influenced only the amplitude $E(t)$ of the field. This is not the case for thermal fluctuations. The amplitude and phase equations for a perfectly tuned laser satisfy

$$\dot{\varphi}E = -\tfrac{1}{2}(\nu/\epsilon_0)C(t) = -\tfrac{1}{2}(\nu/\epsilon_0)C^{(0)}(t), \tag{47}$$

$$\dot{E} = \alpha E - \beta E^3 - \tfrac{1}{2}(\nu/\epsilon_0)S^{(0)}(t). \tag{48}$$

We will consider the coupled stochastic equations (47) and (48) later. Let us now approximate $E(t)$ in (47) by its average value $\langle E(t)\rangle$ so that

$$\dot{\varphi} = -\tfrac{1}{2}(\nu/\epsilon_0\langle E\rangle)C^{(0)}(t). \tag{49}$$

Again we assume that $C^{(0)}(t)$ is a Markovian random function with an average zero and, therefore, that there exists an interval of time Δt short compared to the time for appreciable change of $\varphi(t)$ but long compared to that for $C^{(0)}(t)$. If we denote $C(t,\Delta t)$ by

$$C(t,\Delta t) = \int_t^{t+\Delta t} dt\, C^{(0)}(t),$$

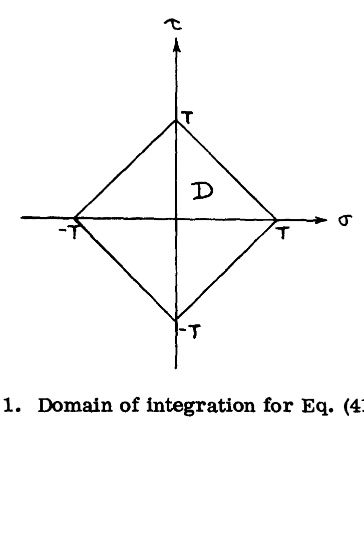

FIG. 1. Domain of integration for Eq. (41).

the probability distribution $\tau(C)$ of C is a Gaussian

$$\tau(C)=(2\pi d\Delta t)^{-1/2}\exp(-C^2/2d\Delta t), \tag{50}$$

where d is a constant to be determined later.

The Fokker-Planck equation for the probability distribution $\omega(\varphi, t)$ for the phase of Eq. (49) is

$$\frac{\partial\omega(\varphi,t)}{\partial t}=\tfrac{1}{2}D\frac{\partial^2\omega}{\partial\varphi^2}, \tag{51}$$

with

$$D=\nu^2 d/4\epsilon_0^2\langle E\rangle^2. \tag{52}$$

At steady state, D is a constant and the solution of Eq. (51) with the initial condition that

$$\omega(\varphi,0)=\delta(\varphi-\varphi_0)\ \text{at}\ t=0 \tag{53}$$

is

$$\omega(\varphi,t)=(2\pi Dt)^{-1/2}\exp\left(-\frac{(\varphi-\varphi_0)^2}{2Dt}\right). \tag{54}$$

It is easy to find the following averages:

$$\begin{aligned}&\langle\varphi\rangle=\varphi_0,\\&\langle\varphi^2\rangle=\varphi_0^2+D|t|,\\&\langle\varphi^2\rangle-\langle\varphi\rangle^2=D|t|.\end{aligned} \tag{55}$$

The amplitude equation (48) can be solved similarly by assuming the probability distribution of $S(t,\Delta t)$ defined by

$$S(t,\Delta t)=\int_t^{t+\Delta t}dt\,S^{(0)}(t)$$

to be a Gaussian

$$\tau(S)=(2\pi d\Delta t)^{-1/2}\exp\left(-\frac{S^2}{2d\Delta t}\right). \tag{56}$$

Note that the constant d in (56) is the same as that in (50) since $S^{(0)}(t)$ and $C^{(0)}(t)$ are merely the amplitude for the sine and cosine components of the random polarization $P^{(0)}(z,t)$. The Fokker-Planck equation for $\omega(E,t)$, i.e., the probability distribution for the amplitude $E(t)$, is

$$\frac{\partial\omega}{\partial t}=-\frac{\partial}{\partial E}[(\alpha E-\beta E^3)\omega]+\tfrac{1}{2}d\left(\frac{\nu}{2\epsilon_0}\right)^2\frac{\partial^2\omega}{\partial E^2}. \tag{57}$$

The field averages satisfy the following equations:

$$\frac{d\langle E\rangle}{dt}=\alpha\langle E\rangle-\beta\langle E^3\rangle, \tag{58}$$

$$\frac{d\langle E^2\rangle}{dt}=d(\nu/2\epsilon_0)^2+2\alpha\langle E^2\rangle-2\beta\langle E^4\rangle. \tag{59}$$

The approximate solution for $\omega(E,t)$ with a δ-function initial condition can be found as before:

$$\omega(E,t)=\frac{(2\pi)^{-1/2}}{\sigma(t)}\exp\left(-\frac{[E-\mathcal{E}(t)]^2}{2\sigma(t)^2}\right), \tag{60}$$

with

$$\dot{\mathcal{E}}=\alpha\mathcal{E}-\beta\mathcal{E}^3, \tag{61}$$

$$\frac{d}{dt}(\sigma^2)=d(\nu/2\epsilon_0)^2+2(\alpha-3\beta\mathcal{E}^2)\sigma^2. \tag{62}$$

In particular, at steady state,

$$\mathcal{E}=(\alpha/\beta)^{1/2}, \tag{63a}$$

$$\sigma^2=(d/16\alpha)(\nu/\epsilon_0)^2. \tag{63b}$$

We shall now calculate the spectral function $I(\omega)$ of a perfectly tuned laser at steady state due to thermal noise. The spectral function in the approximation used in (49), where the amplitude $E(t)$ and the phase $\varphi(t)$ are uncorrelated, is

$$I(\omega)=\frac{\epsilon_0}{8\pi}\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}dt\int_{-T/2}^{T/2}dt'\langle E(t)E(t')\rangle \times\langle\exp\{i[\varphi(t)-\varphi(t')]\}\rangle\exp[-i(\omega-\nu)(t-t')]. \tag{64}$$

The amplitude-correlation function $\langle E(t)E(t')\rangle$ can be found as before, while the phase-correlation function can be evaluated by using (55):

$$\begin{aligned}\langle\exp\{i[\varphi(t)-\varphi(t')]\}\rangle&=\langle\exp\{i[\varphi(0)-\varphi(|t-t'|)]\}\rangle\\&=1+i\langle\varphi(0)-\varphi(|t'-t|)\rangle\\&\quad-\tfrac{1}{2}\langle[\varphi(0)-\varphi(|t-t'|)]^2\rangle\\&\quad+\cdots\\&=1-\tfrac{1}{2}D|t-t'|+\cdots\\&\approx\exp(-\tfrac{1}{2}D|t-t'|).\end{aligned} \tag{65}$$

Carrying out the integration of (64), in the approximation $D\ll\alpha$, we find

$$\begin{aligned}I(\omega)=d(\epsilon_0/8\pi)(\nu/2\epsilon_0)^2\big([D^2/4+(\omega-\nu)^2]^{-1}&\\+\tfrac{1}{2}\{6[16^2_\alpha+(\omega-\nu)^2]^{-1}&\\-[4\alpha^2+(\omega-\nu)^2]^{-1}\}\big)+O(d^2).&\end{aligned} \tag{66}$$

Thus the spectral profile of the laser field due to thermal noise consists of a very sharp Lorentzian FWHH D and a broad two-peaked structure of FWHH ~15.6α. The energy content of the broad distribution is $D/4\alpha$ of that for the sharp one.

We shall now express the width constant d in terms of the temperature of the cavity. If we turn the laser action off, the field remaining in the cavity is then due to the blackbody radiation from the walls only. This requires the following changes in Eq. (57):

$$\alpha\to-(\nu/2Q),\quad\beta\to0. \tag{67}$$

Hence the Fokker-Planck equation for the thermal radiation becomes

$$\frac{\partial\omega}{\partial t}=\frac{\partial}{\partial E}\left[\left(\frac{\nu}{2Q}\right)E\omega\right]+d\left(\frac{\nu}{2\epsilon_0}\right)^2\frac{\partial^2\omega}{\partial E^2}. \tag{68}$$

The steady-state solution can be found exactly to be

$$\omega(E,\infty)=\frac{(2\pi)^{-1/2}}{\sigma}\exp\left(-\frac{E^2}{2\sigma^2}\right), \tag{69}$$

where $\sigma^2=Q\nu d/4\epsilon_0^2$. The electric field in the cavity is

$$E(z,t)=E(t)\cos[\nu t+\varphi(t)]\sin Kz,$$

which contributes to an average energy of

$$W=\frac{\epsilon_0}{2}\int_{\text{cavity}}2\langle[E(z,t)^2]_{\text{av}}\rangle, \tag{70}$$

where the factor 2 in the integrand comes from magnetic energy, and the subscript av means time average. Evaluating the integral in (70), we have

$$\begin{aligned}W&=\tfrac{1}{2}\epsilon_0\int d\tau\,2[\langle E(t)^2\rangle\tfrac{1}{2}\sin^2 Kz]\\&=\tfrac{1}{2}\epsilon_0\langle E(t)^2\rangle\int d\tau\sin^2 Kz\\&=d(Q\nu/8\epsilon_0)(V/2),\end{aligned} \tag{71}$$

where V is the volume of the cavity. Equating this average energy with the right-hand side of (45), we have

$$d=\frac{4\epsilon_0\hbar}{QV}\left(\frac{e^{\hbar\nu/kT}+1}{e^{\hbar\nu/kT}-1}\right). \tag{72}$$

Our results are in complete agreement with a previous calculation[2] where the stochastic phase equation was solved by a Fourier-transform method.

The above argument follows the approximation (49) in which $E(t)$ is replaced by its average value $\langle E(t)\rangle$. In general, we should consider the probability distribution $\omega(E,\varphi,t)$ that the amplitude and the phase will be E and φ at t. By a similar argument we obtain the Fokker-Planck equation for $\omega(E,\varphi,t)$:

$$\frac{\partial\omega}{\partial t}=-\frac{\partial}{\partial E}[(\alpha E-\beta E^3)\omega]+\tfrac{1}{2}d\left(\frac{\nu}{2\epsilon_0}\right)^2\left(\frac{\partial^2\omega}{\partial E^2}+\frac{1}{E^2}\frac{\partial^2\omega}{\partial\varphi^2}\right), \tag{73}$$

but we will not consider the solution in the present paper.

IV. NUMERICAL EXAMPLE

For a typical laser, the various parameters may be given the following values: $\nu\sim10^{15}$ sec^{-1}, $Q\sim10^8$, $\alpha\sim10^6$ sec^{-1}, $\beta\sim10^2$ m^{2}V^{-2} sec^{-1}, volume of the cavity $V\sim10^{-4}$m^3, and power generated ~1 μW.

The excitation rate λ can be obtained from

$$\hbar\nu\lambda=\text{power}=1\ \mu\text{W}$$

or (74)

$$\lambda\sim10^{12}\ \text{atoms/sec}.$$

At steady state the shot effect gives a distribution

$$\omega(E,\infty)\propto\exp[-(E-\mathcal{E})^2/2\sigma^2],$$

with

$$|\sigma/\mathcal{E}|=(1/4\sqrt{\beta}\sqrt{\lambda})(\nu/Q)(\sqrt{\beta}/\sqrt{\alpha})\sim10^{-3}.$$

The FWHH of the broad spectral profile is 15.6 $\times10^6$ sec^{-1}, and it shares about 10^{-5} of the total energy.

The thermal noise is determined by the temperature T. At room temperature, $T\sim300$ °K, therefore,

$$d=\frac{4\epsilon_0\hbar}{VQ}\left(\frac{e^{\hbar\nu/kT}-1}{e^{\hbar\nu/kT}+1}\right)\simeq\frac{4\epsilon_0\hbar}{VQ}. \tag{75}$$

The steady-state distribution for amplitude fluctuation is

$$\omega(E,\infty)\propto\exp[-(E-\mathcal{E})^2/2\sigma^2],$$

with

$$|\sigma/\mathcal{E}|=d(\nu/2\epsilon_0)(1/4\alpha)(\beta/\alpha)^{1/2}\sim10^{-5}.$$

The width constant D of the spectral function is in the order of

$$D=(\nu/2\epsilon_0)^2(d/\mathcal{E}^2)\sim\frac{\nu}{Q}(\hbar\nu/\epsilon_0V\mathcal{E}^2)\sim10^{-1}\ \text{sec}^{-1}.$$

V. CONCLUSION

We have shown, using relatively simple mathematics, two effects of noise on the behavior of a laser. The shot-noise effect influences mainly the amplitude of the laser field, and gives a spectral profile consisting of a monochromatic component superposed on a broad background. The thermal noise is characterized by a temperature-dependent d. It contributes to the spectral profile as a very sharp Lorentzian on a broad background.

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APPENDIX: SPECTRAL FUNCTION

The spectral function given in (37) can be derived as follows. Consider a laser cavity with a transverse electric field

$$E(z,t)=A(t)\sin Kz. \tag{A1}$$

Expanding $A(t)$ into a Fourier series with periodicity T, we have

$$A(t)=\sum_{r=-\infty}^{\infty} a_r e^{i2\pi rt/T}, \tag{A2}$$

where

$$a_r=\frac{1}{T}\int_{-T/2}^{T/2} dt A(t) e^{-i2\pi rt/T}. \tag{A3}$$

The average energy $\bar{\epsilon}_r$ associated with the component of the field with frequency $\omega=\pm 2\pi r/T$ is given by

$$\bar{\epsilon}_r=\epsilon_0 V\langle |a_r|^2\rangle, \tag{A4}$$

where the average means ensemble average, and V is the volume of the cavity.

As $T\to\infty$, the spectrum of the field approximates a continuum. The energy of the field associated with a frequency ranging from $\omega=2\pi r/T$ to $\omega+\Delta\omega$, with $\Delta\omega=2\pi\Delta r/T$, is given by

$$\lim_{T\to\infty}\sum_{r}^{r+\Delta r}\bar{\epsilon}_r=\epsilon_0 V \lim_{T\to\infty}\Delta r\langle|a_r|^2\rangle$$

$$=\Delta\omega(\epsilon_0 V/2\pi)\lim_{T\to\infty}\frac{1}{T}$$

$$\times\left\langle\left|\int_{-T/2}^{T/2} dt A(t) e^{-i\omega t}\right|^2\right\rangle. \tag{A5}$$

The spectral function $I(\omega)$, equal to the energy of the field per unit volume per unit frequency range, is

$$I(\omega)=\frac{\epsilon_0}{2\pi}\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} dt\int_{-T/2}^{T/2} dt'$$

$$\times\langle A(t)A(t')\rangle e^{-i\omega(t-t')}, \tag{A6}$$

where only positive ω are considered.

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[1]W. E. Lamb, Jr., Phys. Rev. **134**, 1429 (1964).

[2]W. E. Lamb, Jr., *Quantum Optics and Electronics* (Gordon and Breach, New York, 1965), p. 377.

[3]M. O. Scully and W. E. Lamb, Jr., Phys. Rev. **159**, 208 (1967).

[4]M. O. Scully, D. M. Kim, and W. E. Lamb, Jr., Phys. Rev. A **2**, 2529 (1970).

[5]R. D. Hempstead and M. Lax, Phys. Rev. **161**, 350 (1967); M. Lax, Phys. Rev. 157, 213 (1967).

[6]See the article by H. Haken in *Handbuch der Physik,* edited by S. Flügge (Springer-Verlag, Berlin, 1970), Vol. XXV/2C; see also W. Brunner *et al.,* Ann. Phys. (Leipz.) **27**, 67 (1971); Ann. Phys. (Leipz.) **27**, 91 (1971); Ann. Phys. (Leipz.) **27**, 199 (1971); P. Richter and S. Grossmann, Z. Phys. **248**, 244 (1971); H. Risken and K. Seybold, Phys. Lett. A **38**, 63 (1972).

[7]S. Chandrasekhar, Rev. Mod. Phys. **15**, 1 (1943).