Quantum Theory of an Optical Maser. VI. Transient Behavior*†

Y. K. Wang† and W. E. Lamb, Jr.
Department of Physics, Yale University, New Haven, Connecticut 06520
(Received 11 August 1972)

The transient behavior of a laser is discussed using the quantum theory as did Scully and Lamb. The formal solution of the density-matrix equation is expressed in terms of exponentially decaying eigenmodes. Some of the lower decay constants are obtained numerically. The equations for the moments of the density matrix are then derived and solved by a truncation method. The equations of motion are integrated numerically for the case where the average number of photons in a laser cavity has the realistically large value 1.3×10^5. An alternative Fokker-Planck-equation approach is discussed.

I. INTRODUCTION

In the quantum theory of a laser by Scully and Lamb,[1] the laser is considered to be a lossy cavity of the Fabry-Perot type driven by an inverted population of active atoms. The electromagnetic field is described in the interaction picture by a reduced density matrix ρ of the system which, in the n representation for single-mode operation, obeys the following equations of motion:

$$\dot{\rho}_{nn'} = -[(n+1)R_{nn'} + (n'+1)R^*_{n'n}]\rho_{nn'} + [R_{n-1,n'-1} + R^*_{n'-1,n-1}]n^{1/2}(n')^{1/2}\rho_{n-1,n'-1} - \tfrac{1}{2}C(n+n')\rho_{nn'} + C(n+1)^{1/2}(n'+1)^{1/2}\rho_{n+1,n'+1}\,. \tag{1}$$

The constant C is the cavity bandwidth ν/Q, where ν is the laser frequency and Q is the cavity quality factor. The coefficients $R_{nn'}$ are given by

$$R_{nn'} = r_a g^2 \frac{\gamma_b(\gamma_{ab} + i\Delta) + g^2(n-n')}{\gamma_a\gamma_b(\gamma_{ab}^2 + \Delta^2) + 2\gamma_{ab}^2 g^2(n+1+n'+1) + g^2(n-n')[g^2(n'-n) + i\Delta(\gamma_a - \gamma_b)]}\,, \tag{2}$$

where the detuning $\Delta = \omega - \nu$, g is a coupling constant between the field and active atoms, and the γ's are atomic decay constants.

Equations (1) describe the transient behavior of laser action. It is the purpose of this paper to investigate the solutions of these equations.

II. FORMAL SOLUTION OF THE EQUATION OF MOTION: EIGENVALUES

Let us, for simplicity, consider a perfectly tuned laser, i.e., $\Delta = \omega - \nu = 0$, then

$$R_{nn'} = \frac{\tfrac{1}{2}A + D(n-n')}{1 + \tfrac{1}{2}(B/A)(n+1+n'+1) + (g^4/\gamma_a\gamma_b)(n-n')^4}\,, \tag{3}$$

where

$$A = 2r_a(g^2/\gamma_a\gamma_{ab})\,,$$
$$B = 8r_a(g^2/\gamma_a\gamma_{ab})(g^2/\gamma_a\gamma_b)\,, \tag{4}$$
$$D = r_a(g^2/\gamma_a\gamma_{ab})(g^2/\gamma_a\gamma_{ab})\,.$$

The steady-state solution $\rho_{nn}^{(0)}$ of the diagonal density-matrix equation (1) can be readily found to be

$$\rho_{nn}^{(0)} = \rho_{00}^{(0)} \prod_{k=1}^{n} (A/C)(1 + Bk/A)^{-1}\,. \tag{5}$$

It is by no means trivial to obtain transient solutions of (1). We may, however, simplify the problem by expanding $R_{nn'}$ into powers of g^2 (small-signal theory). To order g^4, we have

$$R_{nn'} \simeq \tfrac{1}{2}[A - \tfrac{1}{2}B(n+1+n'+1)] + D(n-n') \tag{6}$$

and (1) becomes

$$\dot{\rho}_{nn'} = -[A - \tfrac{1}{2}B(n+1+n'+1)]\tfrac{1}{2}(n+1+n'+1)\rho_{nn'} - D(n-n')^2\rho_{nn'} + [A - \tfrac{1}{2}B(n+n')](nn')^{1/2}\rho_{n-1,n'-1} - \tfrac{1}{2}C(n+n')\rho_{nn'} + C(n+1)^{1/2}(n'+1)^{1/2}\rho_{n+1,n'+1}\,. \tag{7}$$

Let us first consider Eq. (7) for the diagonal elements

$$\dot{\rho}_{nn} = -[A - B(n+1)](n+1)\rho_{nn} - Cn\rho_{nn} + [A - Bn]n\rho_{n-1,n-1} + C(n+1)\rho_{n+1,n+1}. \quad (8)$$

A particular solution of (8) has the form

$$\rho_{nn}(t) = \rho_n e^{-\lambda t} \quad (9)$$

if the ρ_n obey the difference equations

$$c_n\rho_{n-1} + (a_n - \lambda)\rho_n + b_{n+1}\rho_{n+1} = 0, \quad n = 0, 1, 2, \cdots \quad (10)$$

where

$$a_n = [A - B(n+1)](n+1) + Cn,$$
$$b_n = -Cn, \quad c_n = -[A - Bn]n. \quad (11)$$

Equation (10) can be written in the form $M\rho = \lambda\rho$, where ρ is a column vector and M is the matrix

$$M = \begin{pmatrix} a_0 & b_1 & 0 & 0 & \cdots \\ c_1 & a_1 & b_2 & 0 & \cdots \\ 0 & c_2 & a_2 & b_3 & \cdots \\ 0 & 0 & c_3 & a_3 & \cdots \\ & & \cdots & & \end{pmatrix}. \quad (12)$$

We note that the decay constants λ are the eigenvalues of the matrix M, and satisfy the secular equation

$$D_0 \equiv \begin{vmatrix} a_0 - \lambda & b_1 & 0 & 0 & \cdots \\ c_1 & a_1 - \lambda & b_2 & 0 & \cdots \\ 0 & c_2 & a_2 - \lambda & b_3 & \cdots \\ & & \cdots & & \end{vmatrix} = 0. \quad (13)$$

It is easily seen that $\lambda = 0$ is one of the eigenvalues, corresponding to the steady-state solution of (8). Proof: If we add every element in each column to the first element of that column, (13) becomes

$$D_0 = \begin{vmatrix} -\lambda & -\lambda & -\lambda & -\lambda & \cdots \\ c_1 & a_1 - \lambda & b_2 & 0 & \cdots \\ 0 & c_2 & a_2 - \lambda & b_3 & \cdots \\ & & \cdots & & \end{vmatrix} = 0.$$

It is seen that $\lambda = 0$ satisfies (13). Q.E.D.

The steady state of the density matrix corresponding to $\lambda = 0$ can be obtained from (10) to be

$$\rho_n = \rho_0 \prod_{k=1}^{n} (A - Bk)/C, \quad (14)$$

where ρ_0 is determined by normalization.

We note from (14) that for n sufficiently large, ρ_n in general reverses sign and increases in magnitude with each increment of n. This corresponds to an unphysical situation. However, comparing with the exact steady-state solution (5), we see that this difficulty arises from our expansion of $R_{nn'}$ into powers of g^2. This expansion is obviously invalid for very large n.

We note that from (5) that for $n > A/B$ the exact steady state $\rho_{nn}^{(0)}$ is well beyond its peak at $n_p = A(A-C)/BC$ and tends to zero exponentially, while if n^* is an integer, the approximate solution (14) is identically zero for $n > n^* = A/B$, and does not have the oscillatory divergence. From now on, we shall take $n^* = A/B$ = an integer to avoid this oscillatory divergence.

We now derive some properties of the eigenvalues that will be useful for our further discussion. One helpful fact is that all eigenvalues are real. Proof: It is easy to show that the eigenvalue equation $M\rho = \lambda\rho$ can be transformed into Hermitian form, i.e., $H\sigma = \lambda\sigma$ where H is a Hermitian matrix. The eigenvalues of a Hermitian matrix are real. Q.E.D.

To proceed further, let us define the determinants

$$D_k = \begin{vmatrix} a_k - \lambda & b_{k+1} & 0 & 0 & \cdots \\ c_{k+1} & a_{k+1} - \lambda & b_{k+2} & 0 & \cdots \\ 0 & c_{k+2} & a_{k+2} - \lambda & b_{k+3} & \cdots \\ & & \cdots & & \end{vmatrix}. \quad (15)$$

By expanding D_k in minors, we find the recursion relations

$$D_0 = (a_0 - \lambda)D_1 - b_1c_1D_2 = 0,$$
$$D_1 = (a_1 - \lambda)D_2 - b_2c_2D_3,$$
$$\vdots$$
$$D_k = (a_k - \lambda)D_{k+1} - b_{k+1}c_{k+1}D_{k+2}, \quad (16)$$
$$\vdots$$

where by (13), D_0 has been set equal to zero. These equations lead to a continued fraction equation

$$a_0 - \lambda = b_1c_1(D_2/D_1) = \frac{b_1c_1}{D_1/D_2} = \cfrac{b_1c_1}{a_1 - \lambda - \cfrac{b_2c_2}{D_2/D_3}} = \cfrac{b_1c_1}{a_1 - \lambda - \cfrac{b_2c_2}{a_2 - \lambda - \cdots - \cfrac{b_{n^*-1}c_{n^*-1}}{a_{n^*-1} - \lambda}}}. \quad (17)$$

We may prove that all roots of (17) are greater or equal to zero. Proof: Define

$$f_k(\lambda)=\frac{b_kc_k}{a_k-\lambda-\dfrac{b_{k+1}c_{k+1}}{a_{k+1}-\lambda-\cdots}}\ . \tag{18}$$

Suppose $\lambda<0$, then proceeding from $k=n^*-1$, n^*-2, ...2, 1, we can show by mathematical induction that

$$f_k(\lambda)<f_k(0) \text{ for } k=n^*-1,\ n^*-2,\ \ldots 2,1\,.$$

In particular, for $k=1$, we have

$$a_0-\lambda=f_1(\lambda)<f_1(0)=a_0$$

which implies $\lambda>0$ in contradicting our supposition $\lambda<0$. Therefore, λ must be non-negative. Q.E.D.

We may write down the formal solution of (8) for the density matrix in the form

$$\rho_{nn}(t)=\rho_n^{(0)}+\sum_{k=1}^{\infty}\rho_n^{(k)}e^{-\lambda_k t}\,, \tag{19}$$

where $\rho_n^{(k)}$ are the nth components of the eigenvector corresponding to the eigenvalue λ_k. Since all eigenvalues are real and positive, the steady state of the density matrix is stable and equal to $\rho_n^{(0)}$ as given in (14).

Aside from the eigenvalue $\lambda_0=0$, other eigenvalues can be found only by numerical methods. In certain cases, we may use the continued fraction (17) to find the eigenvalues by an iteration procedure. Let us consider the following set of parameters: $n^*=1600$, $C=1.0\ \mu\text{sec}^{-1}$, while A ranges between 0.8 μsec^{-1}, which is under threshold, and 1.2 μsec^{-1}, which is well above threshold. The numerical procedure can be outlined as follows.

(i) We find the eigenvalues approximately by plotting numerically the continued fraction $f_1(\lambda)$ on the right-hand side of (17), and compare it with the function $a_0-\lambda$.

(ii) We pick an approximate eigenvalue, say Λ_0. For those values of λ near Λ_0, we fit $f_k(\lambda)$ for $k=10$ by a hyperbolic function of the form $\alpha+\beta/(\lambda-\gamma)$. The constants α, β, and γ are determined from the values of $f_{10}(\Lambda_0-\Delta)$, $f_{10}(\Lambda_0)$, and $f_{10}(\Lambda_0+\Delta)$ for an appropriate Δ.

(iii) Substituting the extrapolated hyperbolic function for $f_k(\lambda)$ into (17), we solve (17) then by Newton's method. A better approximate eigenvalue Λ_1 can thus be obtained.

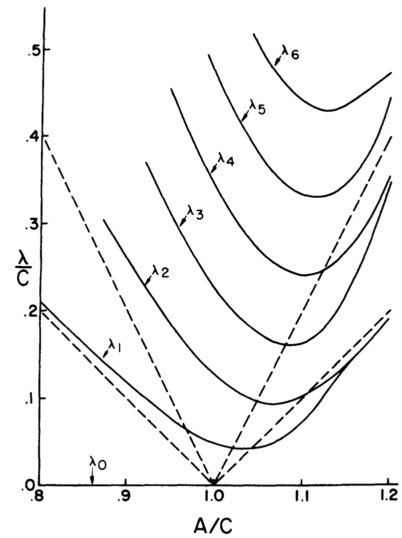

Fig. 1. Lowest eigenvalues of the diagonal-density-matrix equation as a function of the pumping parameter ratio A/C calculated by the continued fraction method when $n^*=1600$. The dashed lines represent eigenvalues for the linear case $B=0$.

(iv) We replace Λ_0 by Λ_1 in step (2), and repeat steps (2) and (3). We then obtain Λ_2 as a better approximation than Λ_1. The process is continued until $|\Lambda_{n+1}-\Lambda_n|$ is reduced to a preassigned value, say $10^{-6}\Lambda_0$.

In this way, we find the A dependence of the seven lowest eigenvalues. The results are shown in Fig. 1. We observe that the nonzero eigenvalues are nearly pairwise degenerate when A is well above the threshold value. In particular, it is seen from Fig. 1 that λ_1 and λ_2 are closely degenerate for $A\gtrsim 1.2C$. The eigenvector $\rho_n^{(1)}$ corresponding to λ_1 has only one node, and $\rho_n^{(2)}$ of λ_2 has two nodes. As λ_1 and λ_2 approach degeneracy we find that $\rho_n^{(1)}$ and $\rho_n^{(2)}$ become equal to each other except in a small region of n where both $|\rho_n^{(1)}|$ and $|\rho_n^{(2)}|$ are very small compared to their peak values. In that region $\rho_n^{(2)}$ has an extra node point. The actual symmetry corresponding to the degeneracy is not known to us.

The off-diagonal elements of the density matrix can be written from (7) as

$$\dot\rho_{n,n+k}=-[A-B(n+1+\tfrac{1}{2}k)](n+1+\tfrac{1}{2}k)\rho_{n,n+k}-Dk^2\rho_{n,n+k}-C(n+\tfrac{1}{2}k)\rho_{n,n+k}+[A-B(n+\tfrac{1}{2}k)]n^{1/2}(n+k)^{1/2}\rho_{n-1,n-1+k}$$
$$+C(n+1)^{1/2}(n+1+k)^{1/2}\rho_{n+1,n+1+k}\,. \tag{20}$$

We note that the off-diagonal elements of the density matrix in the form $\rho_{n,n+k}$ are coupled only with themselves. A particular solution of (20) is

$$\rho_{n,n+k}(t)=Q_ne^{-\mu t} \tag{21}$$

if the coefficients Q_n obey the difference equations

$$c_n Q_{n-1} + (a_n - \mu) Q_n + b_{n+1} Q_{n+1} = 0\,, \quad n = 0, 1, 2, \cdots \tag{22}$$

where

$$\begin{aligned} a_n &= [A - B(n + 1 + \tfrac{1}{2}k)](n + 1 + \tfrac{1}{2}k) + Dk^2 + C(n + \tfrac{1}{2}k)\,, \\ b_n &= -Cn^{1/2}(n+k)^{1/2}\,, \\ c_n &= -[A - B(n + \tfrac{1}{2}k)]\, n^{1/2}(n+k)^{1/2}\,. \end{aligned} \tag{23}$$

A procedure similar to that used before can be applied to find the eigenvalues and eigenvectors of the off-diagonal elements. We plot the lowest eigenvalue of the $\rho_{n,n+1}$ equations for $n^* = 1600$ in Fig. 2. For a laser well-above threshold, the lowest eigenvalues $\mu_1^{(k)}$ of the $\rho_{n,n+k}$ equations are found in Ref. 1 to be approximately equal to $\mu_1^{(k)} = \frac{1}{4}k^2 CB/(A - C)$. We find that our results are very close to this approximation for $A > 1.1C$.

III. MOMENT EQUATIONS

A. Moment Equations and Their Solutions

The diagonal elements of the density matrix $\rho_{nn}(t)$ are equal to the probability of finding n photons in the laser cavity. They can be used to calculate the moments of the photon distribution $\langle n\rangle$, $\langle n^2\rangle$, etc. defined by

$$\langle n\rangle = \sum_{n=0}^{\infty} n\rho_{n,n}(t)\,,$$
$$\langle n^2\rangle = \sum_{n=0}^{\infty} n^2\rho_{n,n}(t)\,. \tag{24}$$

The derivative of $\langle n\rangle$ with respect to t is

$$\begin{aligned} \frac{d\langle n\rangle}{dt} &= \sum_n n\dot{\rho}_{nn}(t) \\ &= \sum_n n\{-[A - B(n+1)](n+1)\rho_{nn} - Cn\rho_{nn} \\ &\qquad + [A - Bn]n\rho_{n-1,n-1} + C(n+1)\rho_{n+1,n+1}\} \\ &= (A - B) + (A - 2B - C)\langle n\rangle - B\langle n^2\rangle\,. \end{aligned} \tag{25}$$

For higher moments, we have, in general,

$$\begin{aligned} \frac{d\langle n^k\rangle}{dt} &= (A - B) + (B - A)\langle n^k\rangle \\ &\quad + \sum_{j=1}^{k}\{[AC_{k+1}^j - BC_{k+2}^j + (-1)^{k-j+1}CC_k^{j-1}]\langle n^j\rangle\} \\ &\quad - kB\langle n^{k+1}\rangle\,, \end{aligned} \tag{26}$$

where $C_k^j = k!/[j!(k-j)!]$ is the combinatorial factor.

Equations (26) are coupled differential equations which describe the transient behavior of the moments. A systematic approximation procedure has been found for solving them.

Let $\xi_k = \langle n^k\rangle^{1/k}$. We assume that $\rho_{nn}(t)$ have the property such that ξ_k is a smoothly varying function of k. Then, by polynomial extrapolation, we may approximate ξ_{k+1} in terms of $\xi_1, \xi_2, \ldots, \xi_k$:

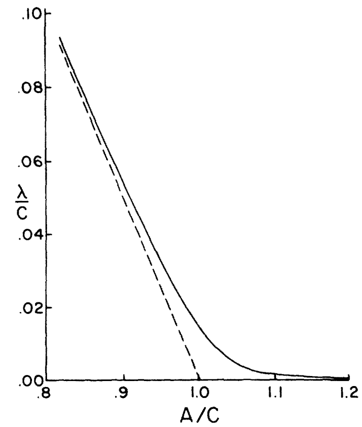

FIG. 2. Lowest eigenvalue for the $\rho_{n,n+1}$ equations as a function of the pumping parameter ratio A/C when $n^* = 1600$. The dashed line indicates the corresponding eigenvalue for the linear case $B = 0$.

$$\xi_{k+1} = \sum_{j=0}^{k-1}(-1)^j C_k^{j+1}\xi_{k-j}\,; \tag{27}$$

the first k moment differential equations become self-contained, and a numerical procedure, for instance, the Runge-Kutta method,[2] can be applied to integrate them.

Let us first take only the first moment equation and extrapolate to $\langle n^2\rangle^{1/2}$ by $\langle n\rangle$, or $\langle n^2\rangle \approx \langle n\rangle^2$. We have

$$\begin{aligned} \frac{d}{dt}\langle n\rangle &= (A - B) + (A - 2B - C)\langle n\rangle - B\langle n\rangle^2 \\ &= -B(\langle n\rangle - n_1)(\langle n\rangle - n_2)\,, \end{aligned} \tag{28}$$

where

$$\begin{aligned} n_1 &= (1/2B)\{(A - 2B - C) + [(A - 2B - C)^2 + 4B(A - B)]^{1/2}\}\,, \\ n_2 &= (1/2B)\{(A - 2B - C) - [(A - 2B - C)^2 + 4B(A - B)]^{1/2}\}\,. \end{aligned} \tag{29}$$

The solution of (30) with the initial condition

$\langle n\rangle(0) = n_0$ at $t = 0$

is

$$\frac{\langle n\rangle - n_1}{\langle n\rangle - n_2} = \frac{n_0 - n_1}{n_0 - n_2}\, e^{-B(n_1 - n_2)t}\,. \tag{30}$$

If we consider higher-order moment equations, we need a higher-order polynomial extrapolation and have more differential equations to integrate. We applied a fourth-order Runge-Kutta method to integrate the four and six extrapolated moment equations for the case of $n^* = 1600$, $A = 1.15$ μsec^{-1}, and $C = 1.0$ μsec^{-1}. The results are plotted in Fig. 3. Use of this procedure in higher-order extrapolation seems to give rapid convergence.

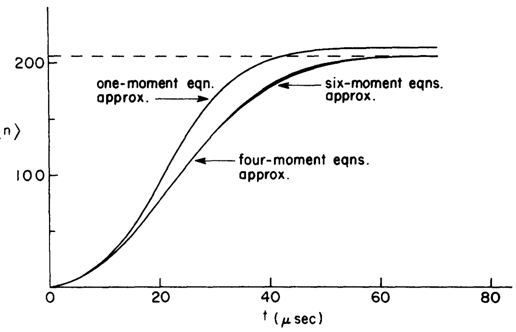

FIG. 3. Time-dependent moment $\langle n\rangle(t)$ obtained by considering one-, four-, and six-moment equations. The parameters used are $A = 1.15\ \mu\text{sec}^{-1}$, $C = 1.0\ \mu\text{sec}^{-1}$, and $n^* = 1600$. The initial condition is $\langle n\rangle(0) = 0$.

B. Moment Equations and the Eigenvalues

The success of the extrapolation procedure for the moment equations suggests a similar method for the calculation of the eigenvalues of Sec. II. We note that

$$\langle n\rangle = \sum_n n\rho_{nn}(t) = \sum_n n\left(\rho_n^{(0)} + \sum_{k=1}^{\infty} \rho_n^{(k)} e^{-\lambda_k t}\right)$$

$$= \langle n\rangle_0 + \sum_{k=1}^{\infty} \langle n\rangle_k e^{-\lambda_k t}, \tag{31}$$

where $\langle n\rangle_k = \sum_n n\rho_n^{(k)}$ are the moments corresponding to the eigenvector $\rho_n^{(k)}$ belonging to the eigenvalue λ_k. Near steady state, the second term $\sum_k \langle n\rangle_k e^{-\lambda_k t}$ in (31) is small compared to $\langle n\rangle_0$ and can be put approximately equal to $\langle n\rangle_1 e^{-\lambda_1 t}$, with λ_1 being the smallest eigenvalue other than 0. Let

$$\delta_1 = \sum_{k=1}^{\infty} \langle n\rangle_k e^{-\lambda_k t} \simeq \langle n\rangle_1 e^{-\lambda_1 t}$$

and substitute it into (28):

$$\frac{d}{dt}(\langle n\rangle_0 + \delta_1) = (A - B) + (A - 2B - C)(\langle n\rangle_0 + \delta_1) - B(\langle n\rangle_0 + \delta_1)^2 .$$

To first order of δ_1, we have

$$\frac{d\delta_1}{dt} = (A - 2B - C - 2B\langle n\rangle_0)\delta_1 ; \tag{32}$$

the solution is obviously

$$\delta_1 = \langle n\rangle_1 e^{-\lambda_1 t},$$

where

$$\lambda_1 = (C - A + 2B) + 2B\langle n\rangle_0 . \tag{33}$$

By considering m of the moment equations with the truncation (27) for ξ_{m+1}, we can introduce

$$\langle n^k\rangle = \langle n^k\rangle_0 + \delta_k , \quad k = 1, 2, \ldots, m$$

TABLE I. Comparison of the eigenvalues calculated by the continued-fraction method (CFM) and by the moment-equations method for various values of the ratio A/C. The value of n^* is 1600.

	λ_1	λ_2 (in units of C)	λ_3	λ_4
		$A = C$		
CFM	0.049 825	0.127 16	0.225 31	0.3404
$m = 2$	0.048 21	0.1616	···	···
$m = 5$	0.049 627	0.129 26	0.240 38	0.442 98
$m = 8$	0.049 807	0.127 39	0.225 81	0.364 29
$m = 11$	0.049 823	0.127 19	0.225 09	0.345 94
$m = 14$	0.049 824	0.127 16	0.225 25	0.341 60
		$A = 1.1\,C$		
CFM	0.073 72	0.102 11	0.164 61	0.2417
$m = 2$	0.096 02	0.2634	···	···
$m = 7$	0.083 01	0.1560	0.2859	0.4827
$m = 12$	0.079 38	0.1297	0.2247	0.3578
		$A = 1.2\,C$		
CFM	0.190 88	0.190 97	0.3446	0.3534
$m = 2$	0.1942	0.4398	···	···
$m = 5$	0.1911	0.3614	0.5711	0.9051
$m = 8$	0.1909	0.3524	0.5065	0.7207

where $\langle n^k\rangle_0 = \sum_n n^k \rho_n^{(0)}$. To first order in δ's, we get a matrix equation of the form

$$\frac{d}{dt}\begin{pmatrix}\delta_1\\ \vdots\\ \delta_m\end{pmatrix} = \begin{pmatrix}\alpha_{11}\cdots\alpha_{1m}\\ \vdots\\ \alpha_{m1}\cdots\alpha_{mm}\end{pmatrix}\begin{pmatrix}\delta_1\\ \vdots\\ \delta_m\end{pmatrix} . \tag{34}$$

Let $\delta_k \propto e^{-\lambda t}$, and substitute into (34). We hope to obtain the lower eigenvalues λ's from the secular equation.

In Table I, we compare the eigenvalues calculated by the moment-equations method and that by the continued-fraction method of Sec. II. We note that in the region where the eigenvalues are well separated, the moment-equations method gives results in excellent agreement with our previous calculation. When two eigenvalues are close together, the method is only partially successful. The reason is probably that the extrapolation method is not sufficiently accurate to account for the fine structures of the eigenvalues in the case of near degeneracy.

IV. TRANSIENT BEHAVIOR OF THE DENSITY MATRIX

We shall try to integrate the diagonal equations (8) in this section. First, we note that the diagonal elements of the density matrix have the conservation of probability property

$$\sum_n \rho_{nn}(t) = 1 . \quad (35)$$

Proof: Using (8), we have

$$\frac{d}{dt}\sum_n \rho_{nn}(t) = \sum_n \{-[A - B(n+1)](n+1)\rho_{nn} - Cn\rho_{nn} + (A - Bn)n\rho_{n-1,n-1} + C(n+1)\rho_{n+1,n+1}\} = 0 .$$

Hence, $\sum_n \rho_{nn}(t) = \text{const}$. The constant should be taken to be unity for normalization. Q.E.D.

Furthermore, if we initially set

$$\rho_{nn}(0) = 0 \quad \text{for } n \geq n^* \text{ at } t = 0 ,$$

then it follows that

$$\rho_{nn}(t) = 0 \quad \text{for } n \geq n^* \text{ at } t \geq 0$$

and only n^* equations in (8) need consideration.

For a realistic laser, n^* is a very large integer, and it is not feasible to numerically integrate the n^* coupled differential equations (8). We avoid this difficulty by grouping the ρ_{nn}'s into "clumps" and considering the average values of ρ_{nn}'s in each clump. The number of differential equations for these average values is thus considerably reduced and a numerical method can be applied to integrate them.

We first note that as n^* increases the $\rho_{nn}(t)$ becomes a more slowly varying function of n. We may expand $\rho_{n-1,n-1}(t)$ and $\rho_{n+1,n+1}(t)$ into the Taylor series

$$\rho_{n-1,n-1}(t) \simeq \rho_{n,n}(t) - \frac{\partial}{\partial n}\rho_{nn}(t) + \frac{1}{2}\frac{\partial^2}{\partial n^2}\rho_{nn}(t) ,$$

$$\rho_{n+1,n+1}(t) \simeq \rho_{n,n}(t) + \frac{\partial}{\partial n}\rho_{nn}(t) + \frac{1}{2}\frac{\partial^2}{\partial n^2}\rho_{nn}(t) .$$

Considering n as a continuous variable, and writing $\rho_{nn}(t) = \rho(n,t)$ we get the following equation:

$$\frac{\partial \rho(n,t)}{\partial t} = [-A + C + B(2n+1)]\rho(n,t) + [C(n+1) - (A - Bn)n]\frac{\partial \rho(n,t)}{\partial n} + \tfrac{1}{2}[C(n+1) + (A - Bn)n]\frac{\partial^2 \rho(n,t)}{\partial n^2} . \quad (36)$$

The function $S(n,t)$ defined by $\rho(n,t) = e^{S(n,t)}$ will be an even more slowly varying function of n. Numerically it will be more effective to solve for $S(n,t)$ instead of $\rho(n,t)$. From (36), we have

$$\frac{\partial S(n,t)}{\partial t} = [-A + C + B(2n+1)] + [C(n+1) - (A - Bn)n]\frac{\partial S(n,t)}{\partial n} + \tfrac{1}{2}[C(n+1) + (A - Bn)n]\left[\frac{\partial^2 S}{\partial n^2} + \left(\frac{\partial S}{\partial n}\right)^2\right] . \quad (37)$$

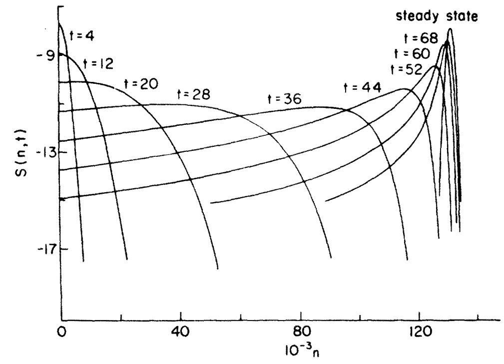

FIG. 4 Plots of the function $S(n,t)$ of (37) at times 4, 12, 20, 28, 36, 44, 52, 60, and 68 μsec and at steady state, obtained by the method described in Sec. IV with $\Delta n = 10^3$. The initial condition is $\rho_{nn}(0) = \rho_0 e^{-n^2/2n^*}$. The parameters are $A = 1.15\ \mu\text{sec}^{-1}$, $C = 1.0\ \mu\text{sec}^{-1}$, and $n^* = 10^6$. The time increment of the fourth-order Runge-Kutta integration is chosen to be 0.02 μsec.

Let us take a net over the n domain, defined by

$$n_k = k\Delta n , \quad k = 1, 2, \ldots, n^*/\Delta n ,$$

where Δn is some integral divisor of n^*. Then

$$\left.\frac{\partial S}{\partial n}\right|_{n=n_k} \simeq \frac{S(n_{k+1},t) - S(n_{k-1},t)}{2(\Delta n)} , \quad (38)$$

$$\left.\frac{\partial^2 S}{\partial n^2}\right|_{n=n_k} \simeq \frac{S(n_{k+1},t) - 2S(n_k,t) + S(n_{k-1},t)}{(\Delta n)^2} .$$

Substituting (38) into (37), we get a set of differential equations for $S(n_k,t)$ that can be integrated numerically.

We consider a case with the following parameters $A = 1.15\ \mu\text{sec}^{-1}$, $C = 1.0\ \mu\text{sec}^{-1}$, and $n^* = 10^6$.

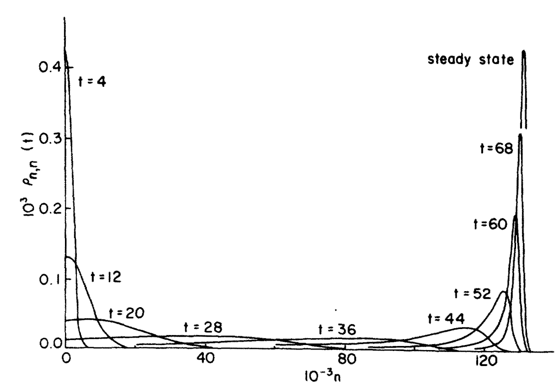

FIG. 5. Plots of the normalized $\rho_{nn}(t)$ at times 4, 12, 20, 28, 36, 44, 52, 60, and 68 μsec and at steady state, by the method described in Sec. IV. $A = 1.15\ \mu\text{sec}^{-1}$, $C = 1.0\ \mu\text{sec}^{-1}$, and $n^* = 10^6$.

TABLE II. Comparison of the time dependence of the moment $\langle n\rangle$ calculated by the moment equations method of Sec. III and by integrating the density matrix equations. Also tabulated is the trace of the density matrix $\sum_n \rho_{nn}(t)$. The parameters used are $A/C=1.15$ and $n^*=10^6$. Time is measured in units of C^{-1} which typically might be 1 μsec.

Ct	$\langle n\rangle$ (moment equations)	$\langle n\rangle$ (density matrix)	Trρ
0	0.007 978 8	0.007 978 8	1
4	0.014 487	0.014 482	1.000 34
8	0.026 088	0.026 087	1.0010
12	0.046 415	0.046 430	1.0016
16	0.081 029	0.081 074	1.0021
20	0.137 32	0.137 42	1.0024
24	0.222 80	0.222 99	1.0027
28	0.341 18	0.341 46	1.0030
32	0.487 90	0.488 12	1.0034
36	0.649 51	0.649 04	1.0042
40	0.808 64	0.806 08	1.0057
44	0.950 40	0.944 21	1.0091
48	1.0658	1.0556	1.0166
52	1.1520	1.1394	1.0310
56	1.2105	1.2059	1.0478
60	1.2498	1.2469	1.0759
64	1.2732	1.2710	1.1709
68	1.2868	1.2850	1.1363
72	1.2946	1.2932	1.1567

The initial condition is chosen for the sake of convenience to be

$$\rho_{n,n}(0)=\rho_0 e^{-n^2/2n^*}, \quad \sum_n \rho_{nn}(0)=1 .$$

We take $\Delta n=10^3$, and apply a fourth-order Runge-Kutta method (with a relatively large time increment $\Delta t=0.02$ μsec) to Eq. (37). The resulting function $S(n,t)$ is plotted in Fig. 4. The diagonal elements of the density matrix $\rho_{nn}(t)$ can be found by taking the exponential of $S(n,t)$. The density matrix we found is satisfactory in the following sense. (i) To a good approximation the total probability is conserved, i.e., $\sum_n \rho_{nn}(t)\simeq 1$ (see Table II). A further correction for $\rho_{nn}(t)$ can be achieved by multiplying in a normalization factor so that $\sum_n \rho_{nn}(t)=1$. (ii) The moments calculated from the normalized $\rho_{nn}(t)$ are consistent with the result of Sec. III. In Table II, we compare the moments calculated by the two methods.

We plot the normalized $\rho_{nn}(t)$ for the case in Fig. 5. We mention that the transient solution for the case $n^*=1600$ has been considered by Scully, Sargent, and Lamb,[3] using a different numerical integration procedure. Also, Gordon and Aslaksen[4] considered the dynamics of the turn-on of a Q-switched laser using different analytic approximate solutions for different stages of the time development. Their results have some similarity to ours.

V. DISCUSSION

So far, we have considered the transient behavior of the density matrix in the $|n\rangle$ representation. Equivalently, we can use the "coherent representation."[5] For a laser not too far from threshold, the equation of motion in this representation turns out to be a Fokker-Planck equation of the approximate form[6]

$$\frac{2p(\alpha,\alpha^*,t)}{\partial t}=\tfrac{1}{2}(C-A)\left(\frac{\partial}{\partial\alpha}(\alpha p)+\frac{\partial}{\partial\alpha^*}(\alpha^* p)\right)+A\frac{\partial^2 p}{\partial\alpha\partial\alpha^*}+\tfrac{1}{2}B\left(\frac{\partial}{\partial\alpha}(\alpha\alpha^*\alpha p)+\frac{\partial}{\partial\alpha^*}(\alpha^*\alpha\alpha^* p)\right). \quad (39)$$

Hempstead and Lax[7] found that a rotating-wave van der Pol oscillator can be described by a Fokker-Planck equation of the above form, and obtained solutions of the eigenvalue problem. Haken and Risken[8] have also calculated sets of eigenvalues for the approximate Fokker-Planck equation (39). The steady-state solution for $p(\alpha,\alpha^*,t)$ can be found as

$$\rho(\alpha,\alpha^*,\infty)=N\exp\{-[(A-C)/A]\alpha\alpha^*-\tfrac{1}{2}(B/A)\alpha^2\alpha^{*2}\} \quad (39a)$$

but the general solution of (39) is certainly no simpler than the one we found in Sec. IV.

We may apply (39) to find the eigenvalues of Sec. II by a method adopted by Haken and Risken.[8] Let $\alpha=Re^{i\phi}$ and $\alpha^*=Re^{-i\phi}$ and write $p(\alpha,\alpha^*,t)=Q(R,\phi,t)$; then Eq. (39) can be written

$$\frac{\partial Q}{\partial t}=\tfrac{1}{2}(C-A)\left(R\frac{\partial Q}{\partial R}+2Q\right)+\tfrac{1}{2}B\left(4R^2Q+R^3\frac{\partial Q}{\partial R}\right)+A\left[\frac{1}{4}\left(\frac{\partial^2 Q}{\partial R^2}\right)+\frac{1}{4R^2}\left(\frac{\partial^2 Q}{\partial\phi^2}\right)+\frac{1}{4R}\frac{\partial Q}{\partial R}\right], \quad (40)$$

which can be solved by the method of separation of variables. Let $Q(R,\phi,t)=T(t)\Phi(\phi)u(R)$, and substitute it into (40). We have

$$\frac{dT(t)}{dt}=-\lambda T(t),$$

$$\frac{d^2\Phi}{d\phi^2}=-m^2\Phi, \quad (41)$$

$$\tfrac{1}{2}(C-A)\left(R\frac{du}{dR}+2u\right)+\tfrac{1}{2}B\left(4R^2u+R^3\frac{du}{dR}\right)+\tfrac{1}{4}A\left(\frac{d^2u}{dR^2}+\frac{1}{R}\frac{du}{dR}-\frac{m^2}{R^2}u\right)=-\lambda u,$$

where λ and m^2 are the separation constants. The general solution for (41) is of the form

$$Q(R,\phi,t)=\sum_{n,m} A_{nm}e^{-\lambda_{n,m}t+im\phi}U_{nm}(R) \tag{42}$$

where n and m are integers, and $U_{n,m}(R)$ is the eigenfunction corresponding to $\lambda_{n,m}$. From (42), we can identify the separation constants $\lambda_{m,0}$ and $\lambda_{0,1}$ with our eigenvalues of Sec. II.

*Supported in part by the National Aeronautics and Space Administration.

†Paper based on a thesis submitted by Y. K. Wang to Yale University in partial fulfillment of the requirement for the Ph.D. degree. Present address: Institute for Fundamental Studies, Department of Physics and Astronomy, The University of Rochester, Rochester, N. Y. 14627.

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PHYSICAL REVIEW A VOLUME 8, NUMBER 2 AUGUST 1973

Theory of Some Laser Noise Effects*†

Y. K. Wang† and W. E. Lamb, Jr.

Department of Physics, Yale University, New Haven, Connecticut 06520

(Received 11 August 1972)

Using semiclassical theory, we investigate the shot- and thermal-noise effects on the behavior of a laser. The Fokker-Planck equations for the probability distribution of the laser field are derived. These equations are approximately solved, using a Gaussian function, from which we calculate the spectral profile of the laser field. The width constant for the thermal noise is related to the temperature of the cavity.

I. INTRODUCTION

The basic paper on laser theory[1] was semiclassical, in that quantum-mechanical atoms were coupled to a classical electromagnetic field. It gave a satisfactory discussion of phenomena such as the threshold condition, power output, frequency pulling and pushing, mode competition, frequency locking, etc., but omitted any consideration of fluctuation phenomena of the laser. Later, one of us[2] extended the semiclassical method to consider the phase diffusion caused by thermal fluctuations and found the corresponding width of the Lorentzian spectral profile of the laser radiation. The development of a fully quantum-mechanical laser theory by Scully and Lamb[3] made possible[4] a calculation of both thermal and spontaneous emission contributions to the spectral profile. With this as a guide, a simple change in the noise polarization of Ref. 2 leads to the correct linewidth.

Many other papers have been written on laser noise phenomena. Very complete bibliographies have been given by Lax[5] and by Haken.[6] With few exceptions, the emphasis of these papers has been on noise phenomena, and the underlying laser theory has been rather schematic and not as well adapted for a discussion of the actual operating characteristics of a laser, somewhat above threshold, as the semiclassical theory of Ref. 1. The present work applies a simple version of the semiclassical theory to shot effect, and also extends the previous consideration[2] of thermal noise to allow for amplitude fluctuations.

As in Ref. 1, the laser is considered to be a lossy cavity of the Fabry-Perot type in single-mode operation with circular frequency ν driven by an inverted population of active atoms. The electric field is taken to be transverse to the cavity axis:

$$E(z,t)=E(t)\cos[\nu t+\varphi(t)]\sin Kz, \tag{1}$$

where z is the distance measured along the cavity axis and K is the wave number $K=n\pi/L$, with L being the length of the cavity and the mode number