solids where the scattering processes between phonons of different polarization branches are not concentrated at small angles. To reduce the mesh dependence of the results, we propose starting with the method described here and extrapolating the resulting solution ν to a ν_0 for a finer mesh. ν_0 can then be used as an initial value for a solution of (15) by iteration. Such an investigation for liquid ^{4}He, however, should first wait for a more precise knowledge of the phonon dispersion.[12]

Finally, we remark that conclusions from our results about the sign of γ have to be taken with caution. Due to the finite linewidths of the thermal phonons, three-phonon processes are, in principle, also allowed if $\gamma > 0$. The corresponding generalized Boltzmann equation,[8] however, then has to be solved self-consistently, which is beyond the scope of the present possibilities.

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Application of the Noniterative-Integral-Equation Method to Electron-Hydrogen Scattering*

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A noniterative-integral-equation method (NIEM) is successfully applied to S-wave scattering of electrons by atomic hydrogen. Reactance-matrix elements and partial-wave cross sections are computed for energies above and below the $n = 2$ threshold. The results are stable despite the strong degenerate dipole coupling between the $2s$ and $2p$ channels. In addition, it is demonstrated, by locating the first ^1S resonance, that the NIEM works in the resonance region below threshold.

In a previous paper,[1] we presented a formalism that describes electron-atom or electron-ion scattering. The procedure was applied to a two-channel model that resembled a $1s$-$2p$ approximation for e-H scattering. In the current paper we demonstrate the applicability of the noniterative-integral-equation method (NIEM) to a physical scattering problem.

Owing to the simplicity of the e-H scattering equations, untested techniques are often applied to them. However, the strong degenerate dipole coupling between the $2s$ and $2p$ channels presents a formidable test[2] of the reliability of a technique. In addition, there exists a series of resonances below the $n = 2$ threshold that can be used to judge the ability of a method to locate resonances. Therefore, we have chosen the e-H scattering problem as a physical test of the applicability of the NIEM.

The e-H scattering equations[3] in the $1s$-$2s$-$2p$

TABLE I. Reactance-matrix elements at energies above the $n=2$ threshold. Row (a) present results; (b) Ref. 5 also in the $1s$-$2s$-$2p$ approximation; (c) Numerov's results.

		R_{11}	R_{12}	R_{13}	R_{22}	R_{23}	R_{33}
				$k^2=0.76$			
	(a)	−0.5438	3.047	−5.891	−7.274	13.17	−23.19
Singlet	(b)	...	...	...	...	...	...
	(c)	−0.5480	3.055	−5.909	−7.286	13.19	−23.22
	(a)	36.61	0.5491	−1.053	−1.276	1.510	−0.5590
Triplet	(b)	...	...	...	...	...	...
	(c)	37.23	0.5560	−1.076	−1.276	1.510	−0.5573
				$k^2=0.78$			
	(a)	0.9330	−0.2851	0.5295	−0.6427	0.1377	1.923
Singlet	(b)	...	...	...	...	...	...
	(c)	0.9339	−0.2853	0.5303	−0.6426	0.1376	1.923
	(a)	25.23	−0.7765	1.107	0.3852	−1.426	3.955
Triplet	(b)	...	...	...	...	...	...
	(c)	25.50	−0.7880	1.122	0.3864	−1.427	3.957
				$k^2=0.81$			
Singlet	(a)	0.8232	−0.2759	0.5897	−1.101	1.021	0.5887
	(b)	0.8189	−0.2763	0.5863	−1.099	1.021	0.5841
	(a)	16.90	−0.3859	0.3943	−0.6846	−0.0108	1.403
Triplet	(b)	16.66	−0.3650	0.3940	−0.6858	−0.0094	1.402
	(c)	17.01	−0.3876	0.3978	−0.6845	−0.0106	1.403
				$k^2=1.00$			
	(a)	0.2926	−0.0043	1.698	0.0198	1.475	−5.509
Singlet	(b)	0.2891	−0.0059	1.701	0.0184	1.478	−5.524
	(c)	0.2930	−0.0043	1.700	0.0198	1.475	−5.511
	(a)	7.196	4.325	−2.400	15.57	−9.491	5.658
Triplet	(b)	7.205	4.553	−2.561	16.78	−10.32	6.219
	(c)	7.216	4.327	−2.401	15.51	−9.456	5.637
				$k^2=1.21$			
	(a)	0.1491	0.5052	1.621	0.1560	−0.5206	−5.868
Singlet	(b)	0.1443	0.5051	1.632	0.1545	−0.5242	−5.914
	(c)	0.1500	0.5057	1.623	0.1558	−0.5215	−5.871
Triplet	(a)	3.897	0.5419	−0.1948	1.058	−1.493	0.7014
	(b)	3.889	0.5389	−0.1942	1.056	−1.492	0.7018
				$k^2=1.44$			
	(a)	0.2659	0.7380	0.8255	−0.4220	−1.522	−2.974
Singlet	(b)	0.2624	0.7389	0.8292	−0.4261	−1.529	−2.989
	(c)	0.2676	0.7383	0.8256	−0.4231	−1.524	−2.975
	(a)	2.864	0.4289	−0.0880	0.0763	−1.327	0.3798
Triplet	(b)	2.859	0.4270	−0.0879	0.0746	−1.327	0.3798
	(c)	2.863	0.4290	−0.0879	0.0754	−1.326	0.3798
				$k^2=2.25$			
	(a)	0.1244	1.401	0.7454	−3.720	−3.040	−1.517
Singlet	(b)	0.1199	1.408	0.7505	−3.744	−3.055	−1.527
	(c)	0.1234	1.404	0.7472	−3.733	−3.047	−1.522
Triplet	(a)	1.630	0.6000	0.1038	−2.448	−1.990	−0.1196
	(b)	1.628	0.5991	0.1035	−2.457	−1.994	−0.1209
				$k^2=4.00$			
	(a)	0.8502	−1.187	−0.5931	5.620	2.008	1.489
Singlet	(b)	0.8383	−1.167	−0.5836	5.533	1.973	1.473
	(c)	0.8503	−1.186	−0.5927	5.621	2.009	1.489
	(a)	1.231	−0.9270	−0.4796	7.337	2.455	1.720
Triplet	(b)	1.220	−0.9043	−0.4699	7.212	2.406	1.700
	(c)	1.231	−0.9270	−0.4798	7.341	2.457	1.720

close-coupling approximation for $L=0$ consist of three coupled integrodifferential equations that can be cast into three coupled integral equations of the Volterra type. Application of the NIEM is a simple and straightforward procedure.[1]

The three coupled equations are solved using integral-equation theory from zero out to some transformation point r_t where the reactance matrix is projected out to its asymptotic value by using a matricant technique. This value of r_t is the point at which the integrals involving the exchange and orthogonality terms have converged. Thus, any further contribution to the solution comes only from the direct potential. We use the trapezoidal rule, with a variable step size, to integrate the coupled equations. An important advantage of using the method comes from the fact that, as pointed out by Sams and Kouri,[4] the solution at a given value of r depends only on previously calculated values of r.

A comparison of our results with those of Burke *et al.*[5] indicated that there was not agreement at all the published energies. Thus, we solved the three coupled integrodifferential equations by another numerical procedure to obtain solutions as a standard against which results from the NIEM may be judged. The numerical solutions are obtained by integrating the equations outwards and inwards by Numerov's method, with subsequent matching to obtain a final continuous solution. The asymptotic expansion method of Burke and Schey[6] is used to determine the reactance matrix. A combination of these methods has been outlined by Smith *et al.*[7]

Table I presents reactance-matrix elements for energies above the $n=2$ threshold. When applying the NIEM, the results quoted are computed using a transformation value of $500a_0$ (where a_0 is the Bohr radius) with exchange and orthogonality terms omitted for values of $r>30a_0$. Since the degenerate dipole coupling between the $2s$ and $2p$ channels is large for energies just above threshold,[2] in this region a larger transformation point r_t should be used. For example, at an energy of 0.76 Ry, an asymmetry of 1% exists in the reactance matrix calculated with $r_t=500a_0$. However, the partial-wave cross sections are more reliable than the reactance-matrix elements at this energy due to their more rapid convergence.[2,8] Row (a) presents the NIEM results, while row (b) contains the results of Burke *et al.*[5] Whenever any elements between the two rows disagree by more than 1%, we insert row (c), which presents results of the alternate numerical method. We note that some of the reactance-matrix elements of Burke and co-workers contained in Table I are in error by as much as 10%. Table II contains partial-wave cross sections only for the cases where row (c) is needed. Although there is up to 10% disagreement in the reactance-matrix elements, the cross sections agree to within 2%.

Knirk *et al.*[9] have also recently extended the work of Sams and Kouri[4] and applied their version of an NIEM to open-channel scattering of electrons by atomic hydrogen. However, they were unable to obtain reliable cross sections for singlet and triplet S-wave cross sections at an energy of 2.25 Ry. We have not encountered any difficulties in our application of the NIEM, as is seen in Tables I and II.

The phase shifts for energies below threshold agree to within 0.2% with the results of Burke *et al.*[10] The results are computed by integrating the scattering equations out to $30a_0$ with exchange and orthogonality terms being retained to this point. We further calculated the energy and width of the first scattering resonance below the $n=2$ threshold. The energy and width are $E_r=9.573$ and $\Gamma=0.0545$ eV from the NIEM, and $E_r=9.575$ and $\Gamma=0.0543$ eV from Burke *et al.*[11] Just below threshold, the closed-channel components of the

TABLE II. Partial-wave cross sections at energies above the $n=2$ threshold. Row (a) present results; (b) Refs. 5 and 10.

		Q_{11}	Q_{12}	Q_{13}	Q_{22}
		$k^2=0.81$			
Triplet	(a)	3.687	0.0013	0.0007	16.22
	(b)	3.687	0.0012	0.0007	16.24
		$k^2=1.00$			
Singlet	(a)	0.2647	0.0768	0.0360	0.2883
	(b)	0.2635	0.0766	0.0360	0.2876
Triplet	(a)	2.907	0.0037	0.0033	6.310
	(b)	2.906	0.0036	0.0033	6.251
		$k^2=1.21$			
Singlet	(a)	0.1731	0.0589	0.0359	0.0659
	(b)	0.1722	0.0588	0.0359	0.0654
		$k^2=1.44$			
Singlet	(a)	0.1278	0.0381	0.0343	0.1670
	(b)	0.1269	0.0380	0.0343	0.1665
Triplet	(a)	1.827	0.0055	0.0096	1.718
	(b)	1.827	0.0055	0.0095	1.719
		$k^2=2.25$			
Singlet	(a)	0.0842	0.0123	0.0172	0.3737
	(b)	0.0836	0.0123	0.0171	0.3737
		$k^2=4.00$			
Singlet	(a)	0.0589	0.0049	0.0035	0.2415
	(b)	0.0579	0.0049	0.0035	0.2412
Triplet	(a)	0.4002	0.0030	0.0052	0.7894
	(b)	0.3979	0.0030	0.0052	0.7894

scattering wave function are extended; thus, in order to obtain accurate phase shifts, it is necessary to integrate the equations out to $100a_0$, while exchange and orthogonality terms are still dropped at $30a_0$.

Note that there are two basic advantages in using the NIEM. First, due to the simplicity of the method, the programming required to solve a set of scattering equations is neither extensive nor involved. Second, the method offers a computational procedure with reliability, accuracy, and speed. The NIEM was at least a factor of 4 faster than the alternative numerical method, and we consider its accuracy to be superior.

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Errata

Magnetic Interactions of One-Electron Atoms and of Positronium, H. Grotch and R. Kashuba [Phys. Rev. A 7, 78 (1973)]. (a) In the discussion following Eq. (20) a factor of 2 was inadvertently lost and therefore the factor $\Delta\nu/\nu = 22\times10^{-6}$ or 22 ppm and the fractional error in $\Delta\nu/\nu$ should be about 14 ppm. (b) Equation (24a) should read

$$\langle 2|\mathcal{H}|0\rangle = -\sqrt{\tfrac{2}{3}}(e/m)H[1 + a_e - \tfrac{1}{20}\alpha^2 - \tfrac{1}{80}(a_e\alpha^2)]\,.$$

The authors wish to thank M. L. Lewis for bringing these errors to our attention.

Temperature Fluctuations Associated with Gravity Waves at a Vapor-Superfluid Interface, J. Woods Halley [Phys. Rev. A 5, 1807 (1972)]. The following typographical errors should be noted:

(i) The right-hand side of Eq. (11) should read

$$-(\Delta\rho/\rho_g)[\dot{T}_g]_0 P'\,.$$

(ii) The subheading preceding Eq. (26) should read *Step (b): Velocity Potential for the Superfluid.*

(iii) In the sixth line of the section following Eq. (39) the expression in parentheses should read $\nu = 2.5$ Hz.

(iv) The second to the last equation of the same section should read

$$T_l \simeq (-2T\omega A/c_1^2)\exp(-\omega z/\sqrt{2}Dk) \times \sin(kx - \omega t + \omega z/\sqrt{2}Dk)\,.$$

(v) In the next section headed RELATION TO THIRD SOUND..., the fourth line of the third paragraph should read "locking of the normal fluid..."

I apologize for failing to mention the work of David Bergman.[1] This work complements the work of this paper in that it is restricted to films (though not very thin ones) but includes effects of thermal conductivity of the film and of the substrate as our work did not. I wish to thank Professor I. Rudnick for pointing out this work to me.

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