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Repeated-Cascade Theory of Turbulence in an Inhomogeneous Plasma*

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Turbulent-plasma phenomena in laboratories and in the ionosphere are often associated with plasma inhomogeneities. As a plasma inhomogeneity, we consider a partially ionized gas with a mean density gradient in a nonuniform mean electric field and a strong magnetic field. Because of drift instabilities, the plasma becomes turbulent in the plane transverse to the magnetic field. We propose a method of a repeated cascade, and derive the spectral distributions for the turbulent density and field fluctuations in the production, inertia, and dissipation subranges of the universal spectrum. The results predict k^{-3}, k^{-1}, and k^{-5} laws in the respective subranges for the density spectrum, and a single k^{-3} law for the field spectrum. As the plasma inhomogeneity may be embedded in an atmospheric turbulence, we also study the effect of the atmospheric turbulence which drives the plasma inhomogeneity. The theoretical predictions are found to agree with data from laboratory plasma experiments.

I. INTRODUCTION

For plasma turbulence we can distinguish between a homogeneous plasma and a plasma inhomogeneity. The latter possesses a mean density gradient and a nonuniform electric field which serve as sources of energy input into the flow of energy across the spectrum, and therefore has the full sequence of production, inertia, and dissipation subranges in its spectrum. Theories of plasma turbulence have treated weak turbulence[1] and strong turbulence[2] in a homogeneous medium. The turbulent motion in plasma inhomogeneities contains too many physical parameters to be treated by dimensional analyses, which become too ambiguous. On the other hand, most analytical theories in hydrodynamic turbulence cannot even satisfactorily predict a Kolmogoroff law with one parameter. Therefore, we shall resort to the method of a repeated cascade, which extends the single cascade[3] describing the transfer of energy across a spectrum to include further cascades which describe a memory chain and which determine the eddy viscosity.[4] Mathematically speaking, a hydrodynamic turbulence is described by one equation of motion, since the Navier–Stokes equation of momentum and the equation of continuity can be combined to determine the velocity after eliminating the pressure, while a plasma turbulence is described by a system of two equations governing two variables: density and field fluctuations. Therefore, the plasma turbulence will be treated by means of the above repeated-cascade method extended to two equations. Since the mathematical details of the technique of the repeated cascade have been fully developed in Ref. 4, we shall restrict ourselves to a discussion of the physical features and emphasize the application of the method to a plasma and the derivation of the subsequent results.

II. DYNAMICS OF PLASMA INHOMOGENEITY

A. General Considerations

Theories of the dynamics of plasma inhomogeneities are based upon a fundamental system of equations governing the evolution of density and of induced electric field. In the past such a fundamental system of equations[5-11] had been derived for a laminar plasma in a background of a constant, external electric field. They are therefore, valid for weak plasma inhomogeneities. However, in strong plasma inhomogeneities, turbulent motions can be generated, and they require a system of dynamic equations for turbulent motions in a nonuniform background, i.e., in the presence of an inhomogeneous density and an inhomogeneous electric field.

B. Transport Properties and Dynamical Equations

We analyze the transport properties of a plasma by means of the following equations of continuity and momenta for individual species:

$$\frac{\partial n_a}{\partial t} + \nabla \cdot n_a \vec{v}_a = 0, \tag{2.1}$$

$$m_a n_a \left(\frac{\partial \vec{v}_a}{\partial t} + \vec{v}_a \cdot \nabla \vec{v}_a\right) = m_a n_a \vec{g} - k(T_a \nabla n_a + n_a \nabla T_a) + n_a e_a(\vec{E} + \vec{v}_a \times \vec{B}/c) - m_a n_a \nu_a(\vec{v}_a - \vec{V}) \ , \tag{2.2}$$

$$\nabla \cdot \vec{E} = 4\pi \sum_a n_a e_a \ , \tag{2.3a}$$

$$\nabla \times \vec{E} = 0 \ , \tag{2.3b}$$

where $\vec{v}_a$ is the velocity, n_a is the number density, T_a is the temperature, m_a is the mass, and e_a is the electric charge. The subscript a refers to the species electron or ion. Further, k is the Boltzmann constant, $\vec{g}$ is the gravitational acceleration, $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field, $\vec{V}$ is the wind velocity, c is the speed of light, and ν_a is the collision frequency between the ionized and neutral particles. Collisions between the ionized particles are neglected.

In a plasma dominated by collisions, the relaxation time for the approach to equilibrium of the transport properties is short, so that any flux should be proportional to ∇n_a or ∇T_a only, and not to their higher powers or derivatives. Consequently an adiabatic approximation can be adopted to simplify the determination of the particle velocities and their elimination between (2.1) and (2.2). By omitting the details of the calculation, we find as a result of such elimination

$$\frac{\partial n_a}{\partial t} + \nabla \cdot [n_a(\vec{Q}_a + \vec{q}_a)] = \nabla \cdot (\overleftrightarrow{D}_a \nabla n_a) \ , \tag{2.4}$$

where

$$\vec{q}_a = -(\overleftrightarrow{\alpha}_a / e_a) \nabla \varphi$$

is an induced drift, and

$$\vec{Q}_a = \vec{V} + \vec{Q}_a^* + \vec{Q}_a^{(g)}$$

is the sum of a wind drift $\vec{V}$, an external drift from electric and magnetic fields

$$\vec{Q}_a^* = \frac{\overleftrightarrow{\alpha}_a}{e_a} (\vec{E}_0 + \vec{V} \times \vec{B}/c) \ ,$$

and an external drift from gravitational and temperature fields

$$\vec{Q}_a^{(g)} = \frac{\overleftrightarrow{\alpha}_a}{e_a} \frac{m_a}{e_a} \vec{g}^* \ ,$$

with

$$\vec{g}^* = \vec{g} - \left(\frac{\partial}{\partial t} + \vec{V} \cdot \nabla\right) \vec{V} - \frac{kT_a}{m_a} \nabla \ln T_a \ . \tag{2.5}$$

Here $n_a \overleftrightarrow{\alpha}_a$ is a conductivity tensor and $\overleftrightarrow{D}_a$ is a diffusivity tensor.

III. SIMPLIFIED EQUATIONS OF MOTION

The system of Eqs. (2.3) and (2.4) is too complicated for the study of turbulence. It is necessary to simplify them by introducing the following approximations:

(i) The turbulent plasma motions will be studied in the plane transverse to the magnetic field.

(ii) The magnetic field is strong, so that the ratio κ_i of the gyrofrequencies to the collision frequencies of ions and electrons is large.

(iii) The time scale for the realization of the quasineutrality condition $(n_i = n_e = n)$ is smaller than the time scale for the rate of change of density in the low-frequency plasma turbulence with which we are concerned.

(iv) The mean temperature gradients, as well as temperature fluctuations, will not be considered. As a result, the present study will not apply to very large scales of turbulence, comparable to those of gravitational instability, or to plasma turbulence induced by heating, e.g., by radar and laser.

(v) The diffusion coefficients are assumed constant.

(vi) The dynamical states of the atmospheric environment, including mean motions and fluctuations, are assumed given, or not to be modified by the introduction of plasma turbulence. In this manner, it is not necessary to write down the equation of motion of the atmospheric environment.

Under those conditions, we can simplify drastically the dynamical equation (2.4). We shall rewrite it for ions and electrons separately and take the sum and the difference of the two equations. By omitting the details, we find for two dimensions

$$\frac{\partial n}{\partial t} + \nabla \cdot n(\vec{U} + \vec{u}) \times \hat{e}_B = D\nabla^2 n \ , \tag{3.1}$$

$$\nabla \cdot n[\vec{U} + \vec{u} + (\vec{V} + \vec{U}_g) \times \hat{e}_B] = \lambda \nabla^2 n \ , \tag{3.2}$$

where

$$\vec{u} = -\frac{c}{B}\nabla\varphi \ , \quad \vec{U} = \frac{c\vec{E}_0}{B} \ , \quad \vec{U}_g = \frac{\vec{g}^*}{\nu_i} \ , \qquad D \equiv D_e \ , \quad \lambda = \kappa_i (D_i - D_e) \ , \tag{3.3a}$$

and $\vec{g}^*$ is defined by (2.5), which is approximated by

$$\vec{g}^* = \vec{g} - \left(\frac{\partial}{\partial t} + \vec{V} \cdot \nabla\right)\vec{V} \ . \tag{3.3b}$$

The diffusion coefficients D and λ originate only from the two-dimensional diagonal components which are isotropic. The off-diagonal components, which have opposite signs, will not contribute to the right-hand side of Eq. (2.4), upon which Eqs. (3.1) and (3.2) are based. We note that

$$\lambda > D \ . \tag{3.3c}$$

IV. DYNAMICAL EQUATIONS FOR A TURBULENT PLASMA

For the sake of simplification of writing, we introduce the following new variables:

$$\vec{Y} = \vec{W} + \vec{w}, \quad \vec{Z} = \vec{U} + \vec{u} + (\vec{V} + \vec{U}_g) \times \hat{e}_B \ , \qquad \vec{W} \equiv \vec{U} \times \hat{e}_B \ , \quad \vec{w} = \vec{u} \times \hat{e}_B \ , \tag{4.1}$$

reducing the dynamical equations (3.1) and (3.2) for low-frequency plasma to the form

$$\frac{\partial n}{\partial t} + \nabla \cdot n\vec{Y} = D\nabla^2 n \ , \quad \nabla \cdot n\vec{Z} = \lambda \nabla^2 n \ . \tag{4.2}$$

The variables n, Y, and Z can be decomposed into a mean and a fluctuation:

$$n = \overline{N} + N' + n' \ , \quad \vec{Y} = \langle\vec{Y}\rangle + \vec{Y}' \ , \quad \vec{Z} = \langle\vec{Z}\rangle + \vec{Z}' \ ,$$

with

$$\langle\vec{Y}\rangle = \langle\vec{U}\rangle \times \hat{e}_B \ , \quad \langle\vec{Z}\rangle = \langle\vec{U}\rangle + (\langle\vec{V}\rangle + \langle\vec{U}_g\rangle) \times \hat{e}_B \ ,$$

$$\vec{Y}' = (\vec{U}' + \vec{u}) \times \hat{e}_B \ , \quad \vec{Z}' = \vec{U}' + \vec{u} + (\vec{V}' + \vec{U}_g') \times \hat{e}_B \ ,$$

where $\overline{N}$, $\langle\vec{U}\rangle$, $\langle\vec{V}\rangle$, and $\langle\vec{U}_g\rangle$ are permanent, or mean, environmental quantities, while N', $\vec{U}'$, $\vec{V}'$, and $\vec{U}_g'$ are environmental fluctuations representative of the atmospheric turbulence. Finally n' and $\vec{u} \equiv -(c/B)\nabla\varphi$ are fluctuations representative of the plasma turbulence. We note that $\overline{N}$, $\langle\vec{U}\rangle$, N', $\vec{U}'$ refer to an ionized environment, and $\langle\vec{V}\rangle$, $\langle\vec{U}_g\rangle$, $\vec{V}'$, $\vec{U}_g'$ refer to a neutral atmosphere.

By means of the above decompositions into a mean and fluctuations, we can transform Eqs. (4.2) for the total motion into equations for the mean quantities and for the fluctuations. The latter lumps together the turbulent motions from the plasma and the atmosphere, and it is necessary to separate them. Omitting the details of the calculation, separation gives the dynamical equations for the fluctuations of plasma turbulence in $\vec{x}$ space as follows:

$$\frac{\partial n'}{\partial t} + \vec{Y} \cdot \nabla n' = -\vec{w} \cdot \nabla(N + N') + D\nabla^2 n' \ , \tag{4.3a}$$

$$\vec{Z} \cdot \nabla n' = -\vec{u} \cdot \nabla(N + N') + \lambda \nabla^2 n' \ . \tag{4.3b}$$

In the derivation of the system (4.3) for turbulent plasma motions, we have made use of the properties of a solenoidal field (2.3b), and of the Poisson relation (2.3a), which are degenerated to the quasineutral form

$$\nabla \cdot \vec{u} = 0 \ ,$$

entailing

$$\nabla \cdot \vec{Y} = 0 \ , \quad \nabla \cdot \vec{Z} = 0 \ .$$

Also we have omitted the terms involving global averages appearing in (4.3), i.e.,

$$\langle\vec{Y}' \cdot \nabla n'\rangle, \quad \langle\vec{Z}' \cdot \nabla n'\rangle \ , \quad \langle\vec{Y}' \cdot \nabla N'\rangle \ ,$$

$$\langle\vec{Z}' \cdot \nabla N'\rangle \ , \quad \langle\vec{w} \cdot \nabla N'\rangle \ , \quad \langle\vec{u} \cdot \nabla N'\rangle \ ,$$

since they will not contribute to the spectral balance. In a locally homogeneous turbulence, i.e., where the homogeneity extends to a limited region, the global average, as denoted by brackets, can be obtained by a spatial average over that limited region.

In Eqs. (4.3a) and (4.3b) for turbulent plasmas, the following transport processes appear:

(a) The convections are represented by

$$\langle\vec{Y}\rangle \cdot \nabla n', \quad \langle\vec{Z}\rangle \cdot \nabla n' \ .$$

(b) The productions are

$$\vec{w} \cdot \nabla\overline{N} \ , \quad \vec{u} \cdot \nabla\overline{N} \ .$$

(c) The atmospheric coupling between plasma turbulence and atmospheric turbulence are

$\vec{w}\cdot\nabla N', \quad \vec{u}\cdot\nabla N'$.

(d) The mode couplings are

$\vec{w}\cdot\nabla n', \quad \vec{u}\cdot\nabla n'$.

(e) The molecular dissipations take the form

$D\nabla^2 n', \quad \lambda\nabla^2 n'$.

In order to analyze the wave interactions more adequately, we introduce a Fourier transform

$$n(t,\vec{x}) = \int_{-\infty}^{\infty} d\vec{k}\, e^{i\vec{k}\cdot\vec{x}} n(t,\vec{k}) \quad ,$$

and transform the system (4.3) into the following dynamical equations in $\vec{k}$ space:

$$\frac{\partial n'(\vec{k})}{\partial t} + \int_{-\infty}^{\infty} d\vec{k}'\, i\vec{k}'\cdot\vec{Y}'(\vec{k}-\vec{k}')n(\vec{k}') = -\int_{-\infty}^{\infty} d\vec{k}'\, i\vec{k}'\cdot\vec{w}(\vec{k}-\vec{k}')N(\vec{k}') - Dk^2 n'(\vec{k}), \quad (4.4a)$$

$$\int_{-\infty}^{\infty} d\vec{k}'\, i\vec{k}'\cdot\vec{Z}(\vec{k}-\vec{k}')n'(\vec{k}') = -\int_{-\infty}^{\infty} d\vec{k}'\, i\vec{k}'\cdot\vec{u}(\vec{k}-\vec{k}')N(\vec{k}') - \lambda k^2 n'(\vec{k}) \quad . \quad (4.4b)$$

For the sake of brevity, we have omitted the time argument t in the equations, and shall restore it when the need arises.

Although the writing of the mean background in $\vec{k}$ space could permit going to strong inhomogeneities, we neglect the presence of a curvature in the mean profile; thus

$$\nabla\nabla\bar{N} \cong 0 \quad . \quad (4.4c)$$

Also we shall neglect

$$(\nabla\bar{N})^2 \cong 0, \quad (4.4d)$$

since any turbulent flux should be linearly proportional to $\nabla\bar{N}$ as a driving force, in compatibility with the Onsager general linear relation between fluxes and forces in transport processes.

V. CASCADE SYSTEM

We shall apply the repeated cascade method, introduced by Tchen[4] for hydrodynamic turbulence, to the dynamic system (4.4) for plasma turbulence. For this purpose we decompose n' into a series of ranks

$$n' = n^0 + n^{(1)} + \cdots \quad ,$$

and, similarly, for the other fluctuations. We take the wave numbers $k^0, k^{(1)}, \ldots,$ which are the boundaries between adjacent ranks as independent variables which will enter later in differential or integral equations. "Cascade ensemble averages," i.e., ensemble averages of different ranks, will be used to discriminate among the ranks. They give relative macroscopic and random components, and are denoted by $\langle\cdots\rangle^0, \langle\cdots\rangle^{(1)}, \ldots$. They are determined[4] by cascade distribution functions, which govern the velocities of many ranks, analogous to the distribution functions governing the velocities of many particles in the statistical mechanics of many bodies.

Upon applying the method of cascade decomposition[4] and its rules of screening, we can transform the dynamical system (4.4) into a cascade system of zeroth rank,

$$\frac{\partial n^0(\vec{k})}{\partial t} + \int_{-\infty}^{\infty} d\vec{k}'\, i\vec{k}'\cdot[\langle\vec{Y}(\vec{k}-\vec{k}')\rangle + \vec{Y}^0(\vec{k}-\vec{k}')]n^0(\vec{k}') = -\int_{-\infty}^{\infty} d\vec{k}'\, i\vec{k}'\cdot[\vec{w}^0(\vec{k}-\vec{k}')N(\vec{k}') + \langle Y^{(1)}(\vec{k}-\vec{k}')n^{(1)}(\vec{k}')\rangle^{(1)}] - Dk^2 n^0(\vec{k}) \quad , \quad (5.1a)$$

$$\int_{-\infty}^{\infty} d\vec{k}'\, i\vec{k}'\cdot[\langle\vec{Z}(\vec{k}-\vec{k}')\rangle + \vec{Z}^0(\vec{k}-\vec{k}')]n^0(\vec{k}') = -\int_{-\infty}^{\infty} d\vec{k}'\, i\vec{k}'\cdot[\vec{u}^0(\vec{k}-\vec{k}')N(\vec{k}') + \langle\vec{Z}^{(1)}(\vec{k}-\vec{k}')n^{(1)}(\vec{k}')\rangle^{(1)}] - \lambda k^2 n^0(\vec{k}) \quad , \quad (5.1b)$$

and a cascade equation of first rank,

$$d_t n^{(1)}(t,\vec{k}) + Dk^2 n^{(1)}(t,\vec{k}) = h^{(1)}(t,\vec{k}) \quad , \quad (5.2a)$$

called the "Langevin equation of turbulence," where

$$h^{(1)}(t,\vec{k}) \equiv -\int_{-\infty}^{\infty} d\vec{k}'\, i\vec{k}'\cdot\{\vec{w}^{(1)}(\vec{k}-\vec{k}')[N(\vec{k}') + n^0(\vec{k}')] + \langle\vec{Y}^{(2)}(\vec{k}-\vec{k}')n^{(2)}(\vec{k}')\rangle^{(2)}\} \quad (5.2b)$$

is a driving force, and

$$d_t n^{(1)}(\vec{k}) \equiv \frac{\partial n^{(1)}(\vec{k})}{\partial t} + \int_{-\infty}^{\infty} d\vec{k}'\, i\vec{k}'\cdot[\langle\vec{Y}(\vec{k}-\vec{k}')\rangle + \vec{Y}^0(\vec{k}-\vec{k}') + Y^{(1)}(\vec{k}-\vec{k}')]n^{(1)}(\vec{k}') \quad (5.2c)$$

is a substantive, or Lagrangian, time derivative. Terms from the first-rank fluctuations in (5.1) will generate transport properties.

In analogy with the Brownian motion of molecules, we have called (5.2a) a Langevin equation of turbulence, with the wave number as a parameter, in the sense that $d_t n^{(1)}(t,\vec{k})$ represents the time rate of change of $n^{(1)}(t,\vec{k})$ carried by a fluid element which has a total streaming velocity $\langle\vec{Y}\rangle + \vec{Y}^0 + \vec{Y}^{(1)}$ composed of ranks equal to and lower than the first. Since the calculations of all transport properties, molecular or turbulent alike, are based upon such a Langevin equation, it will be used here also as a basis for calculating eddy viscosities. For that purpose, we make a formal integration, giving

$$n^{(1)}(t,\vec{k}) = \int_0^t dt' e^{-Dk^2(t-t')} h^{(1)}(t',\vec{k}) + n^{(1)}(0,\vec{k}) e^{-Dk^2 t} \quad . \tag{5.3a}$$

Since a correlation from variables of first rank contributes to a transport property, and is attached to a background ∇n^0 of rank zero, the upper limit t will belong to the time scale of the rank 0, which is much larger than the duration of that correlation. Therefore, that upper limit can be replaced by ∞. By the same token, the inital value will not be correlated with any fluctuation at time t, thus simplifying (5.3a) to

$$n^{(1)}(t,\vec{k}) = \int_0^{t\to\infty} dt' e^{-Dk^2(t-t')} h^{(1)}(t',\vec{k}) \tag{5.3b}$$

$$\cong \int_0^{t\to\infty} dt' h^{(1)}(t',\vec{k}) \tag{5.4a}$$

for the first rank, or to

$$n^{(\alpha)}(t,\vec{k}) \cong \int_0^{t\to\infty} dt' h^{(\alpha)}(t',\vec{k}) \tag{5.4b}$$

for a more general rank α. The viscous damping in formulas (5.4) has been neglected, being small compared to the eddy mixing process. If such is not the case, formulas (5.3) should be employed.

VI. TRANSPORT PHENOMENA IN TURBULENT PLASMA

A. General Considerations

The density n^0 and the field $\vec{u}^0$ are governed by the cascade system (5.1), from which we can derive the equations for the development of the spectral distributions of the density and the field, called the "equations of spectral balance," determining $\langle n^0(t,\vec{k})n^0(t,-\vec{k})\rangle^0$ and $\langle \vec{u}^0(t,\vec{k})\cdot\vec{u}^0(t,-\vec{k}\rangle^0$. Since several transport processes are involved in the dynamical system (4.3), we must expect that similar transport processes will appear in the equations of spectral balance, which will be categorized as transport functions of the density and the field in Secs. VIB and VIC, respectively. Simplifications obtained by assuming local homogeneity will be discussed in Sec. VID.

The turbulent transport processes are controlled by eddy mixing, and therefore will call for "eddy diffusivities of density and field fluxes," to be studied in Secs. VIE and VIF.

B. Transport of Density

The time evolution of the density spectrum

$$\langle n^0(\vec{k})n^0(-\vec{k})\rangle^0 \tag{6.1}$$

is obtained by multiplying (5.1a) by $n^0(-\vec{k})$, giving

$$\frac{\partial}{\partial t}\langle n^0(\vec{k})n^0(-\vec{k})\rangle^0 + K^0(\vec{k}) + K^0(-\vec{k})$$

$$= -\int_{-\infty}^{\infty} d\vec{k}' i\vec{k}'_j [\langle w_j^0(\vec{k}-\vec{k}')n^0(-\vec{k})\rangle^0 \overline{N}(\vec{k}') + \langle w_j^0(\vec{k}-\vec{k}')n^0(-\vec{k})N^0(\vec{k}')\rangle^0]$$

$$-\int_{-\infty}^{\infty} d\vec{k}' ik'_j \langle\langle Y_j^{(1)}(\vec{k}-\vec{k}')n^{(1)}(\vec{k}')\rangle^{(1)} n^0(-\vec{k})\rangle^0$$

$$- Dk^2\langle n^0(\vec{k})n^0(-\vec{k})\rangle^0 + (\vec{k}\to -\vec{k}) \quad , \tag{6.2}$$

where

$$K^0(\vec{k}) = \int_{-\infty}^{\infty} d\vec{k}' i\vec{k}'\cdot\langle[\langle\vec{Y}(\vec{k}-\vec{k}')\rangle + \vec{Y}^0(\vec{k}-\vec{k}')]n^0(\vec{k}')n^0(-\vec{k})\rangle^0 \tag{6.3}$$

is a convection of an inhomogeneous spectrum (6.1) by a streaming velocity $\langle\vec{Y}\rangle + \vec{Y}^0$. The notation

$$(\vec{k}\to -\vec{k}) \tag{6.4}$$

represents the complex-conjugate part, obtained by replacing $\vec{k}$ by $-\vec{k}$. We shall sum up all the wave numbers contributing to the zeroth rank, i.e., covering the spectrum in the range of wave numbers between 0 and k^0,

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} d\vec{k}\,\chi^0\langle n^0(\vec{k})n^0(-\vec{k})\rangle^0 + K_n^0 = S_n^0 - C_n^0 - T_n^0 - D_n^0 \quad , \tag{6.5}$$

where K_n^0, S_n^0, T_n^0, C_n^0, and D_n^0 are called the convection, production, transfer, atmospheric coupling, and dissipation functions, respectively, and are defined by

$$S_n^0 = -\int\int_{-\infty}^{\infty} d\vec{k}\,d\vec{k}'\, ik'_j\chi^0\langle w_j^0(\vec{k}-\vec{k}')n^0(-\vec{k})\rangle^0\overline{N}(\vec{k}') \quad ,$$

$$C_n^0 = \int\int_{-\infty}^{\infty} d\vec{k}\,d\vec{k}'\, ik'_j\chi^0\langle w_j^0(\vec{k}-\vec{k}')n^0(-\vec{k})N^0(\vec{k}')\rangle^0 \quad ,$$

$$T_n^0 = \int\int_{-\infty}^{\infty} d\vec{k}\,d\vec{k}'\, ik'_j\chi^0\langle\langle Y_j^{(1)}(\vec{k}-\vec{k}')n^{(1)}(\vec{k}')\rangle^{(1)}n^0(-\vec{k})\rangle^0,$$

$$D_n^0 = DJ^0, \quad J^0 \equiv \int_{-\infty}^{\infty} d\vec{k}\, k^2\chi^0\langle n^0(\vec{k})n^0(-\vec{k})\rangle^0 \quad ,$$

$$K_n^0 = \int\int_{-\infty}^{\infty} d\vec{k}\,\chi^0 K^0(\vec{k}) \ . \tag{6.6}$$

$K^0(\vec{k})$ is defined by (6.3). The inclusion of a complex-conjugate part, obtained by the property of reversibility (6.4), is understood, and therefore is not written explicitly.

C. Transports of Field

Upon multiplying (5.1b) by $n^0(-\vec{k})$, we obtain an equation for the field energy in the form

$$K_\varphi^0 = S_\varphi^0 - C_\varphi^0 - T_\varphi^0 - D_\varphi^0 \tag{6.7}$$

similar to (6.5), where K_φ^0, S_φ^0, C_φ^0, T_φ^0, and D_φ^0 are the convection, production, atmospheric coupling, transfer and dissipation functions, respectively, governing the field spectrum. They are defined by the expressions

$$K_\varphi^0 = \int \int_{-\infty}^{\infty} d\vec{k}\, d\vec{k}'\, ik_j' \chi^0 \langle [\langle \vec{Z}(\vec{k}-\vec{k}')\rangle + Z^0(\vec{k}-\vec{k}')] \times n^0(\vec{k}') n^0(-\vec{k})\rangle^0 ,$$

$$S_\varphi^0 = -\int \int_{-\infty}^{\infty} d\vec{k}\, d\vec{k}'\, ik_j' \chi^0 \langle u_j^0(\vec{k}-\vec{k}') n^0(-\vec{k})\rangle^0 \overline{N}(\vec{k}') ,$$

$$C_\varphi^0 = \int \int_{-\infty}^{\infty} d\vec{k}\, d\vec{k}'\, ik_j' \chi^0 \langle u_j^0(\vec{k}-\vec{k}') n^0(-\vec{k}) N^0(\vec{k}')\rangle^0 ,$$

$$T_\varphi^0 = \int \int_{-\infty}^{\infty} d\vec{k}\, d\vec{k}'\, ik_j' \chi^0 \langle \langle Z_j^{(1)}(\vec{k}-\vec{k}') n^{(1)}(\vec{k}')\rangle^{(1)} \times n^0(-\vec{k})\rangle^0 ,$$

$$D_\varphi^0 = \lambda J^0 , \qquad (6.8)$$

similar to (6.6).

D. Locally Homogeneous Turbulence

We shall assume a locally homogeneous turbulence, in which we have

$$K_n^0 \cong 0 , \quad K_\varphi^0 \cong 0 .$$

For the same reason, we also have

$$\langle Y_j^{(1)}(\vec{k}) Y_s^{(1)}(\vec{k}')\rangle^{(1)} = \chi^{(1)} \langle Y_j^{(1)}(\vec{k}) Y_s^{(1)}(-\vec{k})\rangle^{(1)} \delta(\vec{k}+\vec{k}') , \qquad (6.9)$$

where

$$\chi^{(\alpha)} = (\pi / X^{(\alpha)})^2$$

and $X^{(\alpha)}$ is a length, characteristic of the extent of the local homogeneity of $\langle Y_j^{(\alpha)}(\vec{x}) Y_s^{(\alpha)}(\vec{x})\rangle^{(\alpha)}$, when a space average is taken. Similarly, we have

$$\langle n^0(\vec{k}) n^0(-\vec{k}')\rangle^0 = \chi^0 \langle n^0(\vec{k}) n^0(-\vec{k})\rangle^0 \delta(\vec{k}+\vec{k}') .$$

Therefore X^0 is the size of the region within which $\langle [n^0(\vec{x})]^2\rangle^0$ is locally homogeneous.

E. Basic Coupling and Transport Processes

The basic variables of the plasma turbulence are $n^{(\alpha)}$ and $\vec{u}^{(\alpha)}$. The zeroth ranks of these variables are governed by the system of Eqs. (5.1a) and (5.1b), and the transport functions are derived in (6.6) and (6.8). The coupling of the plasma motion, as represented by the variables $n^{(\alpha)}$ and $\vec{u}^{(\alpha)}$, with the driving mean density gradient $\nabla\overline{N}$, is responsible for the generation of drift plasma turbulence, and is represented by the production functions S_n^0 and S_φ^0. A corresponding coupling with the mean field drift $\vec{U}$ is represented by the convection functions K_n^0 and K_φ^0, which we assumed to vanish in a locally homogeneous turbulence. The coupling between the plasma motion and the density fluctuation $N^{(\alpha)}$ of the ionized atmosphere resides in the atmospheric coupling functions C_n^0 and C_φ^0. Finally the plasma motions may couple among themselves, or with the field fluctuation $\vec{U}^{(\alpha)}$ of the ionized atmosphere and the velocity fluctuations $\vec{V}^{(\alpha)}$, $\vec{U}_\xi^{(\alpha)}$ of the neutral atmosphere, all entering into the transfer functions T_n^0 and T_φ^0.

The transfer functions are the most difficult to calculate, as they are in the form of triple correlations. As will be shown in Sec. VII, a repeated cascade method will upgrade them to quadruple correlations in the form of a product of two pair correlations. The two pair correlations may be two autocorrelations, one from the plasma motions and the other from the atmospheric motions, or, alternatively, the pair correlations may be in the form of a cross-correlation between the plasma and atmospheric motions, such as

$$\langle \nabla N^0 \nabla n^0\rangle^0 \qquad (6.10a)$$

and

$$\langle \vec{U}^{(1)} \vec{u}^{(1)}\rangle^{(1)}, \quad \langle (\vec{V}^{(1)} + \vec{U}_\xi^{(1)})(\vec{U}^{(1)} + \vec{u}^{(1)})\rangle^{(1)} . \qquad (6.10b)$$

For cross correlations of the types (6.10a) and (6.10b) to exist, a mean gradient driving a turbulent fluctuation with a mixing length $\vec{l}^{(\alpha)}$ is necessary. For example, in connection with (6.10a), we have

$$n^0 = -\vec{l}^0 \cdot \nabla\overline{N} .$$

Consequently

$$N^0 n^0 = -N^0 \vec{l}^0 \cdot \nabla\overline{N}$$

or

$$\langle \nabla N^0 \nabla n^0\rangle^0 = -\langle \vec{l}^0 \cdot \nabla N^0\rangle^0 \nabla\nabla\overline{N} \cong 0 , \qquad (6.11a)$$

following the assumption (4.4c).

In an analogous way, we can show that the cross correlations (6.10b) will also involve a gradient of a lower-rank quantity. As they contribute to an eddy diffusivity, such correlations of the gradient type should be left out of consideration in the present transport processes, according to the assumption (4.4d). Hence we shall write the following:

$$\langle \vec{U}^{(1)} \vec{u}^{(1)}\rangle^{(1)} \cong 0 , \quad \langle (\vec{V}^{(1)} + \vec{U}_\xi^{(1)})(\vec{U}^{(1)} + \vec{u}^{(1)})\rangle^{(1)} \cong 0 . \qquad (6.11b)$$

In conclusion, the transfer functions will be the product of two autocorrelations, either both arising from the plasma motions, or one each from the plasma and atmospheric motions.

F. Eddy Diffusivity of Density Flux

The time correlation of $\vec{Y}^{(\alpha)}$ gives, by an integration, the expression

$$\Pi_{sj}^{(\alpha)}(k) = \int_0^{t-\infty} dt' \chi^{(\alpha)} \langle Y_j^{(\alpha)}(t, \vec{k}) Y_s^{(\alpha)}(t', -\vec{k})\rangle^{(\alpha)} , \qquad (6.12a)$$

called the "eddy diffusivity of density flux."

By inspecting (5.1a), it is recognized that the average $\langle \vec{Y}^{(\alpha)} n^{(\alpha)} \rangle^{(\alpha)}$ has a rank value $\alpha - 1$, and that the time integration (6.12a), which has been subjected to a further smoothing process, will yield a lower-rank value, i.e.,

$$\Pi_{sj}^{(\alpha)}(k) \text{ has a rank value less than } \alpha - 1. \quad (6.12\text{b})$$

In view of the property (6.12b), we can easily show that

$$\Pi_{sj}^{(\alpha)}(k) = \sum_{\beta=\alpha}^{\infty} \Pi_{sj}^{(\beta)}(k) \equiv \tilde{\Pi}_{sj}^{(\alpha)}(k) \quad . \qquad (6.13\text{a})$$

An eddy diffusivity in $\vec{x}$ space is obtained by an integration with respect to $d\vec{k}$:

$$\Pi_{sj}^{(\alpha)} \equiv \Pi_{sj}^{(\alpha)}(\vec{x}/k^{\alpha-1}) = \int_{-\infty}^{\infty} d\vec{k}\, \Pi_{sj}^{(\alpha)}(k) \qquad (6.13\text{b})$$

have $k^{\alpha-1}$ as a lower-bound wave number.

A reduction to isotropic form is obtained by writing

$$\Pi_{sj}^{(\alpha)} = \Pi^{(\alpha)} \delta_{sj} \quad . \qquad (6.13\text{c})$$

G. Eddy Diffusivity of Field Flux

We introduce an "eddy diffusivity of field flux," similar to (6.12a) and (6.13b),

$$\Lambda_{js}^{(\alpha)}(k) = \int_0^{t\to\infty} dt' \chi^{(\alpha)} \langle Z_j^{(\alpha)}(t, \vec{k}) Y_s^{(\alpha)}(t', -\vec{k}) \rangle^{(\alpha)} \qquad (6.14\text{a})$$

in $\vec{k}$ space, or

$$\Lambda_{js}^{(\alpha)} \equiv \Lambda_{js}^{(\alpha)}(\vec{x}/k^{(\alpha-1)}) \equiv \int_{-\infty}^{\infty} d\vec{k}\, \Lambda_{js}^{(\alpha)}(k) \qquad (6.14\text{b})$$

in $\vec{x}$ space. The symmetry in the indices j and s may be noted.

If the external fluctuations, represented by $\vec{U}'$, $\vec{V}'$, and $\vec{U}_\varepsilon'$, are not correlated to the internal-field fluctuation $\vec{u}$, according to (6.11b), we can write the correlation

$$\langle \vec{Z}^{(\alpha)} \vec{Y}^{(\alpha)} \rangle^{(\alpha)} = \langle \vec{u}^{(\alpha)} \vec{w}^{(\alpha)} \rangle^{(\alpha)} = \langle \vec{u}^{(\alpha)} (\vec{u}^{(\alpha)} \times \hat{e}_B) \rangle^{(\alpha)}$$

using the defintions of (4.1). This indicates the absence of a trace. Hence $\Lambda_{js}^{(\alpha)}$ has the shear components

$$\Lambda_{12}^{(\alpha)}, \ \Lambda_{21}^{(\alpha)} \neq 0 \qquad (6.15\text{a})$$

and the diagonal components

$$\Lambda_{11}^{(\alpha)} = -\Lambda_{22}^{(\alpha)} \neq 0 \ , \quad \Lambda_{ii}^{(\alpha)} = 0 \quad . \qquad (6.15\text{b})$$

The system of Eqs. (6.5) and (6.7) represents the fundamental equations of spectral balance for nonequilibrium turbulence. The transport functions (6.6) and (6.8) will be calculated in Secs. VII and VIII. The eddy diffusivities $\Pi_{js}^{(\alpha)}$ and $\Lambda_{js}^{(\alpha)}$ for density and field fluxes, as introduced in (6.13) and (6.14), will be convenient quantities in those calculations.

VII. TURBULENT TRANSPORT FUNCTIONS

A. Transport Functions Covering Density Spectrum

From the repeated-cascade theory of hydrodynamic turbulence,[4] the transfer of modes across a spectrum occurs in the direction of increasing wave numbers. Therefore, we shall calculate the mode coupling

$$\langle\langle Y_j^{(1)}(t, \vec{k}-\vec{k}') n^{(1)}(t, \vec{k}') \rangle^{(1)} n^0(t, -\vec{k}) \rangle^0 \ , \qquad (7.1)$$

as appearing in the function T_n^0 of (6.6), by taking this unidirectional property into account. Consequently, we neglect the term involving $\vec{Y}^{(2)}$ in the Langevin equation (5.2), and write the solution (5.3b) in the following form:

$$\begin{aligned} n^{(1)}(t, \vec{k}') &= -\int_{-\infty}^{\infty} d\vec{k}''\, ik_s'' \int_0^t dt'\, Y_s^{(1)}(t', \vec{k}'-\vec{k}'') \\ &\qquad \times [N(t', \vec{k}'') + n^0(t', \vec{k}'')] \\ &\cong -\int_{-\infty}^{\infty} d\vec{k}''\, ik_s'' \int_0^t dt'\, Y_s^{(1)}(t', \vec{k}'-\vec{k}'') \\ &\qquad \times [N(t, \vec{k}'') + n^0(t, \vec{k}'')] \ , \end{aligned} \qquad (7.2)$$

where we have replaced $n^0(t', \vec{k}'')$ by $n^0(t, \vec{k}'')$ in view of the stationarity of n^0 as compared to $\vec{Y}^{(1)}$. Upon substituting (7.2) into (7.1), we have

$$\begin{aligned} &\langle\langle Y_j^{(1)}(t, \vec{k}-\vec{k}') n^{(1)}(t, \vec{k}') \rangle^{(1)} n^0(t, -\vec{k}) \rangle^0 \\ &\quad \cong \int_{-\infty}^{\infty} d\vec{k}''\, ik_s'' \int_0^t dt' \langle Y_j^{(1)}(t, \vec{k}-\vec{k}') Y_s^{(1)}(t', \vec{k}'-\vec{k}'') \rangle^{(1)} \\ &\qquad \times \langle n^0(t, \vec{k}'') n^0(t, -\vec{k}) \rangle^0 \\ &\quad = -ik_s \Pi_{js}^{(1)}(|\vec{k}'-\vec{k}|) \langle n^0(t, \vec{k}) n^0(t, -\vec{k}) \rangle^0 \ , \end{aligned} \qquad (7.3)$$

with the aid of (6.9), (6.11a), and (6.12a).

The mode coupling (7.3) gives a transfer function, as defined by (6.6),

$$\begin{aligned} T_n^0 &\cong \int_{-\infty}^{\infty} d\vec{k}\, \chi^0 \langle n^0(t, \vec{k}) n^0(t, -\vec{k}) \rangle^0 \\ &\qquad \times \int_{-\infty}^{\infty} d\vec{k}'\, k_j' k_s \Pi_{js}^{(1)}(|\vec{k}'-\vec{k}|) \\ &\cong c_1 \Pi_{js}^{(1)} \int_{-\infty}^{\infty} d\vec{k}\, k_j k_s \chi^0 \langle n^0(t, \vec{k}) n^0(t, -\vec{k}) \rangle^0 \\ &\equiv c_1 \Pi_{js}^{(1)} J_{js}^0 \ , \end{aligned} \qquad (7.4)$$

with

$$J_{js}^0 = \int_{-\infty}^{\infty} d\vec{k}\, k_j k_s \chi^0 \langle n^0(t, \vec{k}) n^0(t, -\vec{k}) \rangle^0 \quad . \qquad (7.5)$$

Equation (7.4) involves a normal eddy diffusivity, defined by (6.12) and (6.13c). In arriving at (7.4), we have made the additional assumption that $\Pi_{js}^{(1)}(|\vec{k}'-\vec{k}|)$ decreases rapidly with its argument, in view of its stationary lower rank value (<0); see (6.12b).

The coefficient c_1 should not differ much from unity:

$$c_1 \cong 1 \quad . \tag{7.6}$$

A calculation of the coefficient c_1 without the above approximation of the rapid decrease of $\Pi_{js}^{(1)}(|\vec{k}'-\vec{k}|)$, but with the hypothesis of isotropy, gives

$$c_1 = 2/\pi \quad . \tag{7.7}$$

Since isotropy does not hold for all contributions to T_n^0, and a precise determination of the coefficient requires a model of anisotropy, we shall not pursue a more precise value of c_1, which will lie between (7.6) and (7.7), but simply use the value (7.6) on account of its simplicity.

By a calculation similar to that leading to (7.3) and (7.4), and with the use of (5.4b) for $\alpha = 0$, we find

$$\begin{aligned} -ik_j'\chi^0\langle w_j^0(t,\vec{k}-\vec{k}')n^0(t,-\vec{k})N(\vec{k}')\rangle^0 &= -\int_{-\infty}^{\infty} d\vec{k}''k_j'k_s''\int_0^{t\to\infty} dt'\langle w_j^0(t,\vec{k}-\vec{k}') \\ &\quad \times Y_s^0(t',-\vec{k}-\vec{k}'')\rangle^0\chi^0\langle N(t,\vec{k}'')N(t,\vec{k}')\rangle^0 \\ &= -k_j'k_s'[\Pi_{js}^0(|\vec{k}-\vec{k}'|)]_F\chi^0\langle N(t,\vec{k}')N(t,-\vec{k}')\rangle^0 \quad , \end{aligned}$$

which transforms the functions S_n^0 and C_n^0 of (6.6) to the forms

$$\begin{aligned} S_n^0 &= -\int_{-\infty}^{\infty} d\vec{k}'k_j'k_s'\chi^0\bar{N}(\vec{k}')\bar{N}(-\vec{k}') \\ &\quad \times\int_{-\infty}^{\infty} d\vec{k}\,[\Pi_{js}^0(|\vec{k}-\vec{k}'|)]_F \\ &\cong c_0(\Pi_{js}^0)_F\frac{\partial\bar{N}}{\partial x_j}\frac{\partial\bar{N}}{\partial x_s} \quad , \quad c_0 = 1 \end{aligned} \tag{7.8}$$

and

$$C_n^0 = c_0(\Pi_{js}^0)_F\left\langle\frac{\partial N^0}{\partial x_j}\frac{\partial N^0}{\partial x_s}\right\rangle^0 \quad , \tag{7.9}$$

where the subscript F refers to the contribution from the field $\vec{u}$ alone, as a consequence of neglecting the contributions from $\langle\nabla\vec{w}^0\nabla\vec{W}^0\rangle^0$, on the basis of (6.11b).

B. Transport Function Governing Field Spectrum

Omitting the calculations, which are similar to those leading to (7.4b), (7.8), and (7.9), we find the transfer function, the production function, and the coupling functions governing the field spectrum in (6.8), to be as follows:

$$\begin{aligned} T_\varphi^0 &= c_1\Lambda_{js}^{(1)}J_{js}^0 \quad , \\ S_\varphi^0 &= c_0(\Lambda_{js}^0)_F\frac{\partial\bar{N}}{\partial x_j}\frac{\partial\bar{N}}{\partial x_s} \quad , \\ C_\varphi^0 &= c_0(\Lambda_{js}^0)_F\left\langle\frac{\partial N^0}{\partial x_j}\frac{\partial N^0}{\partial x_s}\right\rangle^0 \quad . \end{aligned} \tag{7.10}$$

It is to be noted that these transport functions involve the shear tensor $\Lambda_{js}^{(\alpha)}$, defined by (6.14), with the properties (6.15), and the shear tensor $(\Lambda_{js}^{(\alpha)})_F$ obtained from $\Lambda_{js}^{(\alpha)}$ by retaining the contribution from the field $\vec{u}$ alone.

C. Eddy Viscosity Tensors

Parallel to the eddy viscosity of the plasma field fluctuation

$$\eta_{js}^{(\alpha)} = \int_{-\infty}^{\infty} d\vec{k}\int_0^t dt'\,\chi^{(\alpha)}\langle u_j^{(\alpha)}(t,\vec{k})u_s^{(\alpha)}(t',-\vec{k})\rangle^{(\alpha)}, \tag{7.11a}$$

we introduce the eddy viscosity of the field fluctuation in the atmospheric turbulence:

$$\xi_{js}^{(\alpha)} = \int_{-\infty}^{\infty} d\vec{k}\int_0^t dt'\,\chi^{(\alpha)}\langle U_j^{(\alpha)}(t,\vec{k})U_s^{(\alpha)}(t',-\vec{k})\rangle^{(\alpha)} \quad . \tag{7.11b}$$

We shall calculate the diffusivities $\Pi_{js}^{(\alpha)}$ and $\Lambda_{js}^{(\alpha)}$, as defined by (6.12a) and (6.14a), in terms of $\xi_{js}^{(\alpha)}$ and $\eta_{js}^{(\alpha)}$. For this purpose, we write

$$\begin{aligned} \langle\vec{Y}^{(\alpha)}\vec{Y}^{(\alpha)}\rangle^{(\alpha)} &= \langle\vec{W}^{(\alpha)}\vec{W}^{(\alpha)}\rangle^{(\alpha)} + \langle\vec{w}^{(\alpha)}\vec{w}^{(\alpha)}\rangle^{(\alpha)} \quad , \\ \langle\vec{Z}^{(\alpha)}\vec{Y}^{(\alpha)}\rangle^{(\alpha)} &= \langle\vec{U}^{(\alpha)}\vec{W}^{(\alpha)}\rangle^{(\alpha)} + \langle\vec{u}^{(\alpha)}\vec{w}^{(\alpha)}\rangle^{(\alpha)} \quad . \end{aligned} \tag{7.12}$$

When we substitute (7.12) into (6.12a) and (6.14a), and make use of (7.11), we find the components of $\Pi_{js}^{(\alpha)}$ and $\Lambda_{js}^{(\alpha)}$ as follows:

$$\begin{aligned} &\Pi_{11}^{(\alpha)} = \xi_{22}^{(\alpha)} + \eta_{22}^{(\alpha)}, \quad \Pi_{22}^{\alpha} = \xi_{11}^{(\alpha)} + \eta_{11}^{(\alpha)}, \\ &\Pi_{12}^{(\alpha)} = \Pi_{21}^{(\alpha)} = -(\xi_{12}^{(\alpha)} + \eta_{12}^{(\alpha)}) \ , \\ &\Lambda_{11}^{(\alpha)} = -\Lambda_{22}^{(\alpha)} = \xi_{12}^{(\alpha)} + \eta_{12}^{(\alpha)} \quad , \\ &\Lambda_{12}^{(\alpha)} = -(\xi_{11}^{(\alpha)} + \eta_{11}^{(\alpha)}) \ , \quad \Lambda_{21}^{(\alpha)} = \xi_{22}^{(\alpha)} + \eta_{22}^{(\alpha)} \quad . \end{aligned} \tag{7.13}$$

D. Transport Functions in Explicit and Simplified Forms

Upon substituting (7.13) into (7.4b), (7.8), (7.9), and (7.10), we transform the transport functions into the following explicit forms:

$$\begin{aligned} S_n^0 &= c_0\eta^0\bar{J}, \quad \bar{J} = (\Delta\bar{N})^2, \quad \xi^{(\alpha)} \equiv \tfrac{1}{2}\xi_{ii}^{(\alpha)}, \quad \eta^{(\alpha)} \equiv \tfrac{1}{2}\eta_{ii}^{(\alpha)}, \\ C_n^0 &= c_0\eta^0\langle(\nabla N^0)^2\rangle^0 - 2c_0\eta_{12}^0\left\langle\frac{\partial N^0}{\partial x_1}\frac{\partial N^0}{\partial x_2}\right\rangle^0 \quad , \\ T_n^0 &= c_1(\xi^{(1)} + \eta^{(1)})J^0 - 2c_1(\xi_{12}^{(1)} + \eta_{12}^{(1)})J_{12}^0 \quad ; \\ S_\varphi^0 &= c_0\eta_{12}^0\left[\left(\frac{\partial\bar{N}}{\partial x_1}\right)^2 - \left(\frac{\partial\bar{N}}{\partial x_2}\right)^2\right] \quad , \\ C_\varphi^0 &= c_0\eta_{12}^0\left\langle\left(\frac{\partial N^0}{\partial x_1}\right)^2 - \left(\frac{\partial N^0}{\partial x_2}\right)^2\right\rangle^0 \\ &\quad + c_0(\eta_{22}^0 - \eta_{11}^0)\left\langle\frac{\partial N^0}{\partial x_1}\frac{\partial N^0}{\partial x_2}\right\rangle^0 \quad , \\ T_\varphi^0 &= c_1(\xi_{12}^{(1)} + \eta_{12}^{(1)})(J_{11}^0 - J_{22}^0) \\ &\quad + c_1[(\xi_{22}^{(1)} - \xi_{11}^{(1)}) + (\eta_{22}^{(1)} - \eta_{11}^{(1)})]J_{12}^0 \quad . \end{aligned} \tag{7.14}$$

From (7.14), we note that the production of field turbulence requires a preferential density gradient

$\partial\bar{N}/\partial x_1$, giving rise to a preferred second moment

$$J_{11}^0 \gg J_{22}^0 \gg J_{12}^0$$

which has negligible shear component J_{12}^0. For the atmospheric turbulence, we assume similarly that

$$\left\langle\left(\frac{\partial N^0}{\partial x_1}\right)^2\right\rangle^0 \gg \left\langle\left(\frac{\partial N^0}{\partial x_2}\right)^2\right\rangle^0 \gg \left\langle\frac{\partial N^0}{\partial x_1}\,\frac{\partial N^0}{\partial x_2}\right\rangle^0 .$$

Under these circumstances, we can simplify (7.14) to read for the density spectrum

$$S_n^0 = c_0\eta^0\bar{J} ,$$
$$C_n^0 = c_0\eta^0 J_a^0 , \quad J_a^0 = \langle(\nabla N^0)^2\rangle^0 , \qquad (7.15a)$$
$$T_n^0 = c_1(\xi^{(1)} + \eta^{(1)}) J^0 ,$$

and for the field spectrum

$$S_\varphi^0 = c_0\eta_{12}^0\bar{J} ,$$
$$C_\varphi^0 = c_0\eta_{12}^0 J_a^0 , \qquad (7.15b)$$
$$T_\varphi^0 = c_1(\xi_{12}^{(1)} + \eta_{12}^{(1)}) J^0 .$$

We note that the above two kinds of transport functions (7.15a) and (7.15b) are similar: those determining the density spectrum are isotropic, while those determining the field spectrum require shear viscosities $\xi_{12}^{(1)}$ and $\eta_{12}^{(1)}$, consistent with the properties of the tensors $\Pi_{js}^{(\alpha)}$ and $\Lambda_{js}^{(\alpha)}$, as expected. The spectral structure of the isotropic and shear viscosities will be determined in Sec. VIII.

VIII. STRUCTURE OF EDDY VISCOSITIES

A. Normal Eddy Viscosity for Plasma Turbulence

An eddy viscosity tensor has both normal components and shear components, which will be treated here and in Sec. VIII B, respectively, for plasma turbulence and in Sec. VIII C for atmospheric turbulence.

For plasma turbulence, we introduce F as the spectral distribution function of $\vec{u}$, and attempt to derive the relationship between $\eta_{ij}^{(\alpha)}$ and F. The normal components are assumed isotropic, and in the following will be denoted by $\eta^{(\alpha)}$:

$$\eta_{ij}^{(\alpha)} = \eta^{(\alpha)}\delta_{ij} .$$

This relationship has been derived in Ref. 4, the results of which are

$$\eta^0 = c_2\int_0^{k^0} dk'\,\frac{F(k')}{k'^2\bar{\eta}^{(1)}(\vec{x}/k')}, \quad c_2 = c_1^{-1} , \qquad (8.1a)$$

$$\eta^{(1)} = \bar{\eta}^{(1)}(\vec{x}/k^0) \qquad (8.1b)$$

$$= c_2\int_{k^0}^{\infty} dk'\,\frac{F(k')}{k'^2\bar{\eta}^{(2)}(\vec{x}/k')} \qquad (8.1c)$$

$\vdots$

$$\bar{\eta}^{(\alpha)} = c_2\int_{k^{(\alpha-1)}}^{\infty} dk^{(\alpha)}\,\frac{F(k^{(\alpha)})}{k^{(\alpha)2}\bar{\eta}^{(\alpha+1)}(\vec{x}/k^{(\alpha)})} . \qquad (8.1d)$$

The identity (8.1b) is based upon the property (6.13a). The hierarchy (8.1) represents a memory chain, when the eddy viscosity $\bar{\eta}^{(\alpha)}(x/k^{(\alpha-1)})$ of rank α is governed by a relaxation frequency $k^{(\alpha)2}\bar{\eta}^{(\alpha+1)}(x/k^{(\alpha)})$ of a higher rank. In Ref. 4 the closure of the hierarchy was achieved by a viscous cutoff

$$\bar{\eta}^{(\alpha+1)}(\vec{x}/k^{(\alpha)}) = \bar{\eta}^{(\alpha)}(\vec{x}/k^{(\alpha)})e^{-k^{(\alpha)}/k_c^{(\alpha)}} ,$$

where $k_c^{(\alpha)}$ is a viscous cutoff wave number, the value of which has been determined.

The simplest closure of the memory chain (8.1) is an inviscid one, reducing (8.1) to

$$\eta^0 = c_2\int_0^{k^0} dk\,\frac{F(k)}{\omega_F^{(1)}(k)} , \qquad (8.2a)$$

with

$$\omega_F^{(1)}(k) = k^2\bar{\eta}^{(1)}(\vec{x}/k) \qquad (8.2b)$$

and

$$\bar{\eta}^{(1)} \cong c_2\int_{k^0}^{\infty} dk'\,\frac{F(k')}{\omega_F^{(1)}(k')} .$$

The solution of the integral equation gives

$$\bar{\eta}^{(1)}(\vec{x}/k^0) = [2c_2 \textstyle\int_{k^0}^{\infty} dk'k'^{-2}F(k')]^{1/2} . \qquad (8.2c)$$

Formulas (8.2a) and (8.2c) for the eddy viscosities will be used in Secs. IX and X.

It is to be remarked that the isotropic eddy viscosity (8.2a) can also be written in the following form:

$$\eta_{ij}^0 \equiv \eta^0\delta_{ij}$$
$$= \tfrac{1}{2}c_2 \int_{-\infty}^{\infty} d\vec{k}\,\chi^0\langle\vec{u}^0(\vec{k})\cdot\vec{u}^0(-\vec{k})\rangle^0[\omega_F^{(1)}(k)]^{-1}\delta_{ij} . \qquad (8.3)$$

B. Shear Eddy Viscosity for Plasma Turbulence

The isotropic viscosity is given by (8.3). Now we shall calculate the corresponding shear components, by first rewriting them in the following form:

$$\eta_{ij}^0 = c_2^* \int_{-\infty}^{\infty} d\vec{k}\,\chi^0\langle u_i^0(\vec{k})u_j^0(-\vec{k})\rangle^0[\omega_F^{(1)}(k)]^{-1} \qquad (8.4a)$$

for $i\neq j$, with a shear coefficient c_2^* which is different from the isotropic coefficient c_2. The shear coefficient is

$$c_2^* = c_0^{-1} \quad . \tag{8.4b}$$

From the calculation of shear stress made in Ref. 4, we find

$$\chi^0\langle u_i^0(\vec{k})u_j^0(-\vec{k})\rangle^0 = -c_0\eta^0(k)\frac{\partial U_i}{\partial x_j}$$

$$= -c_0c_2\frac{\partial U_i}{\partial x_j}\frac{1}{2}\chi^0\langle\vec{u}^0(\vec{k})\cdot\vec{u}^0(-\vec{k})\rangle^0 \times[\omega_F^{(1)}(k)]^{-1} \quad , \tag{8.5}$$

where use of (8.2) has been made.

When we substitute (8.5) into (8.4a), we obtain

$$\eta_{ij}^0 = -c_2^* c_0 c_2 \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right) \times\int_{-\infty}^{\infty} d\vec{k}\,\tfrac{1}{2}\chi^0\langle\vec{u}^0(\vec{k})\cdot\vec{u}^0(-\vec{k})\rangle^0[\omega_F^{(1)}(k)]^{-2} \quad ,$$

which may be rewritten

$$\eta_{ij}^0 = -\Gamma_{ij}\nu_F^0 \quad , \tag{8.6a}$$

using the notations

$$\Gamma_{ij} = \frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i} \quad , \tag{8.6b}$$

$$\nu_F^0 = c_3\int_0^{k^0} dk\,F(k)[\omega_F^{(1)}(k)]^{-2} \quad , \tag{8.6c}$$

$$c_3 = \tfrac{1}{2}c_2^* c_0 c_2 \quad . \tag{8.6d}$$

Similarly for the next rank, we have

$$\eta_{ij}^{(1)} = -\Gamma_{ij}\nu_F^{(1)} \quad , \tag{8.7a}$$

with

$$\nu_F^{(1)} \equiv \tilde{\nu}_F^{(1)} = c_3\int_{k^0}^{\infty} dk\,F(k)[\omega_F^{(1)}(k)]^{-2} \,, \tag{8.7b}$$

where we can write

$$\nu_F^{(1)} = \tilde{\nu}_F^{(1)}$$

for the same reason as stated in (6.12b).

The formulas (8.6a) and (8.7a) give the shear components of the eddy viscosity.

When we differentiate (8.7b), we find

$$\frac{d\tilde{\nu}_F^{(1)}}{dk^0} = -c_3F(k^0)[\omega_F^{(1)}(k^0)]^{-2}$$

$$= -c_3(k_0)^{-4}F(k^0)[\tilde{\eta}^{(1)}]^{-2} \quad , \tag{8.8}$$

with the use of notation (8.2b). When we further make use of (8.2c), we can transform (8.8) into

$$\frac{d\tilde{\nu}_F^{(1)}}{dk^0} = c_2^{-1}c_3(k^0)^{-2}\frac{d\ln\tilde{\eta}^{(1)}}{dk^0} \tag{8.9}$$

a useful relation between $\tilde{\nu}_F^{(1)}$ and $\tilde{\eta}^{(1)}$.

C. Normal and Shear Components of Eddy Viscosity for Atmospheric Turbulence

If we introduce a spectral function F_a for the velocity $\vec{U}'$ of the atmospheric turbulence, and assume a cascade decomposition of $\vec{U}'$, we can define F_a by

$$\int_0^{k^0} dk\,F_a(k) = \tfrac{1}{2}\int_{-\infty}^{\infty} d\vec{k}\,\chi^0\langle\vec{U}^0(\vec{k})\cdot\vec{U}^0(-\vec{k})\rangle^0 \quad ,$$

and find the eddy viscosities $\tilde{\xi}^0$, $\tilde{\xi}^{(1)}$, ξ_{ij}^0, and $\tilde{\xi}_{ij}^{(1)}$ in the atmosphere as follows:

$$\begin{aligned}
\xi^0 &= c_2\int_0^{k^0} dk\,\frac{F_a(k)}{\omega_a^{(1)}(k)} \quad ,\\
\tilde{\xi}^{(1)} &= [2c_2\int_{k^0}^{\infty} dk'\,k'^{-2}F_a(k')]^{1/2} \quad ,\\
\nu_a^0 &= c_3\int_0^{k^0} dk\,\frac{F_a(k)}{[\omega_a^{(1)}(k)]^2} \quad ,\\
\nu_a^{(1)} &= c_3\int_{k^0}^{\infty} dk\,\frac{F_a(k)}{[\omega_a^{(1)}(k)]^2} \quad ,\\
\xi_{ij}^0 &= -\nu_a^0\Gamma_{ij} \quad ,\\
\tilde{\xi}_{ij}^{(1)} &= -\tilde{\nu}_a^{(1)}\Gamma_{ij} \quad ,
\end{aligned} \tag{8.10a}$$

where

$$\omega_a^{(1)}(k) = k^2\tilde{\xi}^{(1)}(k)$$

is the relaxation frequency of the eddy viscosity $\tilde{\xi}^{(1)}$. The relationship between $\tilde{\nu}_a^{(1)}$ and $\tilde{\xi}^{(1)}$ is

$$\frac{d\tilde{\nu}_a^{(1)}}{dk^0} = c_2^{-1}c_3k^{0-2}\frac{d\ln\tilde{\xi}^{(1)}}{dk^0} \quad . \tag{8.10b}$$

The derivation of Eqs. (8.10a) and (8.10b) for atmospheric turbulence is analogous to that of Eqs. (8.2)–(8.8) for plasma turbulence, and is therefore omitted.

We conclude that Eqs. (8.6a), (8.7a), and (8.10a) enable the eddy viscosity tensors $\overleftrightarrow{\xi}^{(\alpha)}$ and $\overleftrightarrow{\eta}^{(\alpha)}$ to be transformed into the scalar quantities $\tilde{\nu}_F^{(\alpha)}$ and $\tilde{\nu}_a^{(\alpha)}$, respectively. These scalar quantities, having the dimensions of the square of a mixing length, will be called the "eddy dispersion" of plasma and atmospheric turbulence, respectively.

IX. EQUATIONS OF SPECTRAL BALANCE IN UNIVERSAL RANGE

We can divide a spectrum into a nonuniversal range, dependent on the special turbulence-generating agents, and a universal range governed exclusively by the transport functions (7.8)–(7.10) at large wave numbers. In the universal range, we can write the transport equations (6.5) and (6.7) in the form

$$\begin{aligned}
S_n^0 + C_n^0 - T_n^0 - D_n^0 &= S_n + C_n - D_n \,, \qquad T_n = 0 \quad ,\\
S_\varphi^0 + C_\varphi^0 - T_\varphi^0 - D_\varphi^0 &= S_\varphi + C_\varphi - D_\varphi \,, \qquad T_\varphi = 0 \quad ,
\end{aligned} \tag{9.1}$$

called the "equations of spectral balance." The functions without superscript 0 are obtained by putting $k^0 = \infty$ in the same functions with a superscript. Thus, we have

$$S_n = S_n^0(k^0=\infty), \quad S_\varphi = S_\varphi^0(k^0=\infty),$$
$$C_n = C_n^0(k^0=\infty), \quad C_\varphi = C_\varphi^0(k^0=\infty),$$
$$D_n = D_n^0(k^0=\infty), \quad D_\varphi = D_\varphi^0(k^0=\infty) \ .$$

With the simplified forms (7.15a) and (7.15b) of the transport functions, we can rewrite the equations of energy balance (9.1) in an explicit form:

$$-c_0\eta^0(\bar{J}+J_a^0)+c_1(\tilde{\xi}^{(1)}+\tilde{\eta}^{(1)})J^0+DJ^0 = -S_n - C_n + D_n \ , \tag{9.2a}$$

$$-c_0\Gamma_{12}[\nu_F^0(\bar{J}+J_a^0)-(\tilde{\nu}_a^{(1)}+\tilde{\nu}_F^{(1)})J^0]+\lambda J^0 = -S_\varphi - C_\varphi + D_\varphi \ . \tag{9.2b}$$

Here the formulas (8.6a), (8.7a), and (8.10a) for expressing the tensorial eddy viscosities, as they occur in (7.15), in terms of the scalar eddy diffusivities $\tilde{\nu}_F^{(1)}$ and $\tilde{\nu}_a^{(1)}$ have been applied.

The system (9.2) governs the density spectrum G, as carried by the function J^0, and the field spectrum F, as carried by the functions $\tilde{\eta}^{(1)}$ and $\tilde{\nu}_F^{(1)}$. The two spectra are coupled in general. Because the plasma consists of electrons and ions, two diffusivities D and λ determine the dissipations D_n and D_φ. The plasma inhomogeneity is represented by a mean density gradient $\bar{J}^{1/2}$ and the gradient Γ_{12} of the mean electric field. The eddy viscosity $\tilde{\eta}^{(1)}$ controls the transfer of density from small to large wave numbers across the spectrum in a homogeneous field, while the eddy dispersion $\tilde{\nu}_F^{(1)}$ controls a similar transfer in the field gradient. In addition, an eddy viscosity $\tilde{\xi}^{(1)}$ and an eddy dispersion $\tilde{\nu}_a^{(1)}$ of the atmospheric turbulence will couple the plasma turbulence with the atmospheric turbulence.

We distinguish two cases. *Case a*: *Self-generated plasma turbulence in the absence of atmospheric turbulence.* The governing equations of spectral balance are obtained from (9.2), by neglecting

$$\tilde{\xi}^{(1)} \ll \tilde{\eta}^{(1)}, \quad \tilde{\nu}_a^{(1)} \ll \tilde{\nu}_F^{(1)} \ ,$$

and thus transforming (9.2) into

$$\tilde{\eta}^{(1)}(\bar{J}+J^0)+DJ^0 = D_n \ , \tag{9.3a}$$

$$\Gamma_{12}\tilde{\nu}_F^{(1)}(\bar{J}+J^0)+\lambda J^0 = D_\varphi \ . \tag{9.3b}$$

Case b: *Passive plasma driven by atmospheric turbulence.* The governing equations of spectral balance are obtained from (9.2), by making the approximations

$$\tilde{\eta}^{(1)} \ll \tilde{\xi}^{(1)}, \quad \tilde{\nu}_F^{(1)} \ll \tilde{\nu}_a^{(1)} \ ,$$

and are therefore reduced to

$$c_0\tilde{\eta}^{(1)}(\bar{J}+J_a^0)+c_1\xi^{(1)}J^0+DJ^0 = D_n \ , \tag{9.4a}$$

$$c_0\Gamma_{12}[\tilde{\nu}_F^{(1)}(\bar{J}+J_a^0)+\tilde{\nu}_a^{(1)}J^0]+\lambda J^0 = D_\varphi \ . \tag{9.4b}$$

The system (9.4) indicates that the production and coupling functions require the presence of $\tilde{\eta}^{(1)}$ and $\tilde{\nu}_F^{(1)}$, while the transfer and dissipation functions do not.

X. SPECTRAL LAWS IN SELF-GENERATED PLASMA TURBULENCE

A. Inertia Subrange

We distinguish the following subranges: inertia, production, and dissipation subranges. The inertia subrange is the simplest, as it is governed by a transfer function alone. All other subranges require the flow from one process to another, along each spectrum or between the two spectra. We shall discuss the inertia subrange first.

The inertia subrange is characterized by the mode coupling alone, with a constant transfer across each spectrum. Thus, by neglecting the production and dissipation functions, the equations of spectral balance (9.3) are degenerated to

$$c_0\tilde{\eta}^{(1)}J^0 = D_n \ , \tag{10.1a}$$

$$c_0\Gamma_{12}\tilde{\nu}^{(1)}J^0 = D_\varphi \ . \tag{10.1b}$$

By eliminating J^0, we find the relation

$$\tilde{\eta}^{(1)} = \Gamma_{12}(D_n/D_\varphi)\tilde{\nu}_F^{(1)} \ ,$$

which gives, after a differentiation,

$$\frac{d\tilde{\eta}^{(1)}}{dk} = \Gamma_{12}\frac{D_n}{D_\varphi}\frac{d\tilde{\nu}^{(1)}}{dk} = \frac{c_3}{c_2}\Gamma_{12}\frac{D_N}{D_\varphi}\frac{1}{k^2}\frac{d\ln\tilde{\eta}^{(1)}}{dk} \ ,$$

on account of (8.9), and yields the solution

$$\tilde{\eta}^{(1)} = \frac{c_3}{c_2}\frac{\Gamma_{12}D_N}{D_\varphi}k^{-2} \ . \tag{10.2}$$

Here and in the following, we have omitted the superscript in k^0 by replacing k^0 by k as an independent variable.

With the use of the relation (8.2c) between $\tilde{\eta}^{(1)}$ and F, we find, from (10.2),

$$F = A(\Gamma_{12}D/\lambda)^2k^{-3}, \quad A = 2c_2^{-3}c_3^2 \cong \tfrac{1}{2}. \tag{10.3}$$

A substitution of (10.3) into (10.1a) further gives

$$G = B_i(J\lambda/\Gamma_{12})k^{-1}, \quad B_i = c_2/c_0c_3 \cong 2 \ . \tag{10.4}$$

B. Production Subrange

In the production subrange, molecular dissipations are not effective, simplifying the equations of spectral balance (9.3) to read

$$c_0\tilde{\eta}^{(1)}(\bar{J}+J^0)=D_n \;, \tag{10.5a}$$

$$c_0\Gamma_{12}\tilde{\nu}_F^{(1)}(\bar{J}+J^0)=D_\varphi \;. \tag{10.5b}$$

The equation governing the F spectrum is found by eliminating $\bar{J}+J^0$ between (10.5a) and (10.5b), yielding, as a solution,

$$F=A(\Gamma_{12}D/\lambda)^2k^{-3} \;, \tag{10.6}$$

which is identical to (10.3).

On the other hand, the G spectrum should be governed by the flow process across the spectrum, as a result of the interaction between the production and the inertia transfer, which process can be best described by a differential form of Eq. (10.5a) for the spectral balance of density; we have

$$\frac{d\tilde{\eta}^{(1)}}{dk}\bar{J}+\tilde{\eta}^{(1)}2k^2G=0 \;, \tag{10.7}$$

where we have neglected J^0, since

$$J^0 \ll J$$

in the range of small wave numbers involved. When we substitute the value of F from (10.6), or equivalently the value of $\tilde{\eta}^{(1)}$ from (10.2), we obtain the density spectrum from (10.7):

$$G=\bar{J}k^{-3} \;. \tag{10.8}$$

C. Dissipation Subrange

In the dissipation subrange, the nonlinear mode transfers are dissipated by molecular diffusion. In order to prescribe such a dissipation, we differentiate the system (9.3) of spectral balance and neglect the production functions, giving

$$\begin{aligned} &c_0J^0\frac{d\tilde{\eta}^{(1)}}{dk}+(D+c_0\tilde{\eta}^{(1)})2k^2G=0 \;, \\ &c_0J^0\frac{d\tilde{\nu}^{(1)}}{dk}+(\lambda\Gamma_{12}^{-1}+c_0\tilde{\nu}_F^{(1)})2k^2G=0 \;. \end{aligned} \tag{10.9}$$

The dissipation subrange is characterized by large-enough wave numbers to justify the approximation

$$J^0 \cong J \;,$$

where $J\equiv J^0$ $(k=\infty)$. In view of $\lambda \gg D$, according to (3.3c), and of the large wave numbers considered, we can also make the approximations that

$$\lambda+c_0\Gamma_{12}\tilde{\nu}_F^{(1)}\cong\lambda \;, \qquad D+c_0\tilde{\eta}^{(1)}\cong D \;,$$

reducing the system (10.9) to

$$-c_0JD^{-1}\frac{d\tilde{\eta}^{(1)}}{dk}=2k^2G \;, \tag{10.10a}$$

$$-c_0J\Gamma_{12}\lambda^{-1}\frac{d\tilde{\nu}_F^{(1)}}{dk}=2k^2G \;. \tag{10.10b}$$

Equation (10.10b) may be replaced by

$$-c_0c_2^{-1}c_3J\Gamma_{12}\lambda^{-1}k^{-2}\frac{d\ln\tilde{\eta}^{(1)}}{dk}=2k^2G \;. \tag{10.10c}$$

The system of Eqs. (10.10a) and (10.10b) or (10.10c) yield the solutions

$$F=A(\Gamma_{12}D/\lambda)^2k^{-3}, \quad A=2c_2^{-3}c_3^2\cong\tfrac{1}{2} \;, \tag{10.11}$$

$$G=B_D(J\Gamma_{12}/\lambda)k^{-5}, \quad B_D=2c_0c_2^{-1}c_3\cong 1. \tag{10.12}$$

It is to be noted that the field spectrum $F(k)$ takes the identical law in all subranges (10.3), (10.6), and (10.11).

D. Critical Wave Numbers

The critical wave number k_s separating the production and inertia subranges is calculated from the ratio

$$(J^0/\bar{J})_{k=k_s}=1 \;,$$

with J^0 determined by an inertial spectrum (10.4). We find

$$k_s=B_i^{-1/2}(\Gamma_{12}\bar{J}/\lambda J)^{1/2}, \quad B_i^{-1/2}\cong 1/\sqrt{2} \;. \tag{10.13}$$

The critical wave number k_λ separating the inertia and dissipation subranges is calculated from the equalization at this wave number between the transfer and dissipation functions, i.e.,

$$c_1(\tilde{\eta}^{(1)})_{k=k_\lambda}=D \;,$$

giving

$$k_\lambda=(\tfrac{1}{2}c_1^2c_2A)^{1/4}(\Gamma_{12}/\lambda)^{1/2} \;, \tag{10.14a}$$

with

$$(\tfrac{1}{2}c_1^2c_2A)^{1/4}\cong 1/\sqrt{2} \;. \tag{10.14b}$$

XI. INHOMOGENEOUS PLASMA DRIVEN BY ATMOSPHERIC TURBULENCE

A. General Considerations

We consider a plasma inhomogeneity embedded in an ionospheric background which is highly turbulent, so that the field fluctuations generated by the turbulent motions of the plasma are negligibly small, as compared with the background turbulence in the ionosphere. Under such a circumstance, the plasma can be considered as being a simple passive species driven by an external turbulence. The plasma itself may possess a mean density gradient.

B. Inertia and Dissipation Subrange

In the study of the spectrum of the plasma turbulence in the inertia and dissipation subranges, we neglect the production function, reducing the equation of spectral balance (9.4a) to the form

$$J^0 = DJ/(c_1\bar{\xi}^{(1)} + D) . \tag{11.1}$$

where $\bar{\xi}^{(1)}$ is given by (8.10a).

A simple differentiation of (11.1) yields the density spectrum G in terms of the background field turbulence:

$$2k^2G = \frac{d}{dk}\,\frac{DJ}{c_1\bar{\xi}^{(1)} + D} . \tag{11.2}$$

If the background field turbulence, as represented by the fluctuation U', has a spectrum F_a following the power law

$$F_a = A_a k^{-m} , \tag{11.3a}$$

then the density spectrum G in the plasma inhomogeneity is derived as

$$G = \frac{m+1}{4}\left(\frac{2c_2}{m+1}\right)^{1/2} A_a^{1/2} DJ \times\left[D + c_0\left(\frac{2c_2}{m+1}\right)^{1/2} A_0^{1/2} k^{-(m+1)/2}\right]^{-2} k^{-(m+7)/2} , \tag{11.3b}$$

which degenerates to the following asymptotic forms for the inertia and viscous subranges.

a. Inertia subrange ($k < k_D$). We find

$$k_D = \left[\left(\frac{c_0}{D}\right)^2 \frac{2c_2A_a}{m+1}\right]^{1/(m+1)} , \tag{11.4}$$

as the critical wave number separating the inertia and viscous subranges.

For $k < k_D$, Eq. (11.3b) degenerates to

$$G = \frac{m+1}{4}\, c_0^{-2}\left(\frac{2c_2}{m+1}\right)^{-1/2} A_a^{-1/2} DJk^{(m-5)/2} . \tag{11.5}$$

b. Viscous subrange ($k > k_D$). For $k > k_D$, Eq. (11.5) degenerates to

$$G = \frac{m+1}{4}\left(\frac{2c_2}{m+1}\right)^{1/2} A_a^{1/2}\,\frac{J}{D}\, k^{-(m+7)/2} . \tag{11.6}$$

It is to be remarked that if we put

$$m = 3 \quad\text{and}\quad A_a = A(\Gamma_{12}D/\lambda)^2 , \tag{11.7}$$

we would reduce (11.3a), (11.5), and (11.6) to (10.3), (10.4), and (10.12), respectively.

C. Production and Coupling Subranges

The equations of spectral balance in the production and coupling subranges are given by (9.4) without D and λ, and are rewritten in the form

$$\begin{aligned} &c_0\bar{\eta}^{(1)}(\bar{J} + J_a^0) + c_1\bar{\xi}^{(1)}J^0 = D_n , \\ &c_0\Gamma_{12}[\bar{\nu}_F(\bar{J} + J_a^0) + \bar{\nu}_a^{(1)}J^0] = D_\varphi . \end{aligned} \tag{11.8a}$$

In order to discuss the flow process across the spectra, we differentiate (11.8a), giving

$$\begin{aligned} &c_0\frac{d\bar{\eta}^{(1)}}{dk}(\bar{J} + J_a^0) + c_0\bar{\eta}^{(1)}\frac{dJ_a^0}{dk} + c_1\frac{d\bar{\xi}^{(1)}}{dk}J^0 + c_1\bar{\xi}^{(1)}\frac{dJ^0}{dk} = 0 , \\ &\frac{d\bar{\nu}^{(1)}}{dk}(\bar{J} + J_a^0) + \bar{\nu}_F^{(1)}\frac{dJ_a^0}{dk} + \frac{d\bar{\nu}_a^{(1)}}{dk}J^0 + \bar{\nu}_a^{(1)}\frac{dJ^0}{dk} = 0 . \end{aligned}$$

We shall neglect J_a^0 and J^0, because the production subrange is restricted to small wave numbers, reducing the differential system to

$$\begin{aligned} &\frac{d\bar{\eta}^{(1)}}{dk}\bar{J} + \bar{\eta}^{(1)}2k^2G_a + \bar{\xi}^{(1)}2k^2G = 0 , \\ &\frac{d\bar{\nu}^{(1)}}{dk}\bar{J} + \bar{\nu}_F^{(1)}2k^2G_a + \bar{\nu}_a^{(1)}2k^2G = 0 . \end{aligned} \tag{11.8b}$$

a. Production subrange. If the field spectrum in atmospheric turbulence has a power law

$$F_a = A_a k^{-m},$$

with $m > 3$, the production by the gradient $\bar{J}$ is more effective than the coupling function in (11.8b), and we find

$$F = \left(\frac{4}{m+1}\right)^2 A_a k^{-m}, \tag{11.9a}$$

$$G = \bar{J}k^{-3} . \tag{11.9b}$$

b. Combined production and coupling subranges. Let the field and density spectra in the atmospheric turbulence be

$$F_a = A_a k^{-m} \tag{11.10a}$$

and

$$G_a = B_a k^{-r} . \tag{11.10b}$$

If

$$m = r = 3 ,$$

the production and coupling functions become equally important in (11.8b), and we find the plasma turbulence spectra

$$F = A_a k^{-3} , \qquad G = (\bar{J} - B_a)k^{-3} , \tag{11.11}$$

in equilibrium with the atmospheric turbulence having spectra (11.10).

XII. CONCLUSIONS

In the present paper, we have considered a plasma inhomogeneity which possesses, on the average, a density gradient and a nonuniform electric field. Since the plasma is unstable owing to drift instability, it will become turbulent. We have proposed a

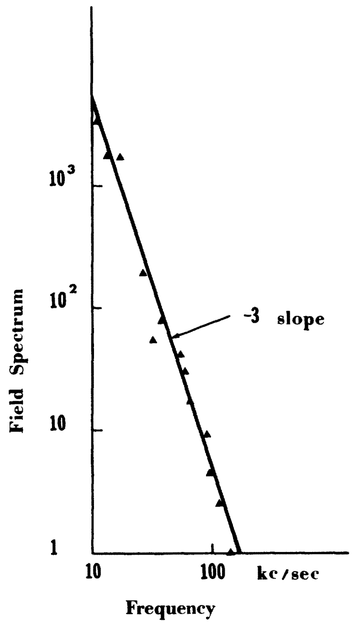

FIG. 1. Power spectrum of electric field in zeta. The k^{-3} spectrum is observed by Robinson and Rusbridge in their experiments in zeta (Ref. 12). The same power is found in the theoretical prediction by Eqs. (10.3), (10.4), and (10.12) for an inhomogeneous plasma in this paper, and is also found by Tchen for a homogeneous plasma with (Refs. 14 and 15) or without (Ref. 16) an external magnetic field.

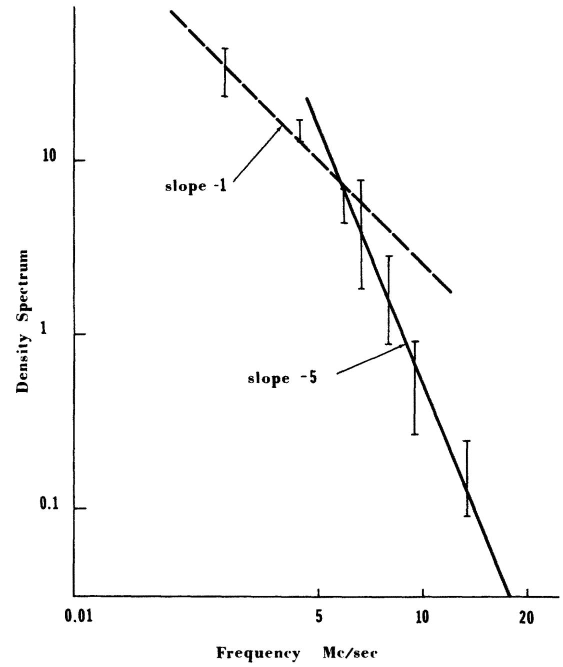

FIG. 2. Density spectrum in zeta. The experiments are reported by Wort and Heald using microwave scattering (Ref. 13). The ordinate has an arbitrary unit. The solid line giving the power law k^{-5} and the dotted line giving the power law k^{-1} are the theoretical predictions from Eqs. (10.12) and (10.4) for the dissipation and inertia subranges, respectively.

repeated cascade theory for calculating the spectral structure of the self-generated turbulence in the plasma inhomogeneity.

When a plasma inhomogeneity is embedded in an ionosphere which is already turbulent, the atmospheric turbulence may drive the plasma into a turbulent state. The theory also includes the analysis of the spectrum of the passive plasma fluctuations which are driven by the atmospheric turbulence.

For a self-generated plasma turbulence, the density spectral laws are found to be k^{-3}, k^{-1}, and k^{-5}, in the production, inertia, and dissipation subranges, respectively [see Eqs. (10.8), (10.4), and (10.12)], while the field spectrum is predicted to be k^{-3} over all subranges (see Sec. X). The effects of atmospheric turbulence driving the plasma fluctuations are analyzed in Sec. XI.

The plasma drift turbulence seems to prevail in plasma laboratory experiments and in plasma inhomogeneities in the ionosphere. The field spectra of laboratory plasmas have been measured by means of Langmuir probes,[12] and their density spectra by microwave scattering.[13] These measurements show a k^{-3} law for the field spectrum, and the k^{-1} and k^{-5} laws for the density spectrum (see Figs. 1 and 2) as predicted by our Eqs. (10.3), (10.4), and (10.12). Although Figs. 1 and 2 are plotted in frequencies, the spectra are in wave numbers, as the plasma inhomogeneities are moving with a large constant drift. The magnetic field fluctuations were measured to be of a lower order of magnitude. The k^{-3} spectral law of electric field energy has also been found by Tchen for a homogeneous plasma with[14,15] or without[16] an external magnetic field.

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Pressure Dependence of Charge Carrier Mobilities in Superfluid Helium*†

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Accurate measurements were made of the mobility of positive and negative charge carriers in He^4, at pressures up to the melting pressure and temperatures in the range $0.27 < T < 1.0$ °K. The data are analyzed in terms of the scattering of phonons and rotons by the charge carriers. The phonon-dominated mobility of the negatives agrees well with existing theories, and analysis of these results provides a new determination of the electron-bubble radius as a function of pressure. The phonon-scattering term for positives is calculated from the electrostriction model, and is in good agreement with the experimental results. Roton-limited mobilities are obtained by subtracting the known phonon contributions. One may express the results in the form $(e/\mu)_\pm = f_\pm(P,T)e^{-\Delta(P,T)/kT}$, where $\Delta(P,T)$ is the pressure- and temperature-dependent energy gap derived from neutron-scattering experiments. The measured pressure and temperature dependences of the prefactors $f_\pm(P,T)$ are discussed in light of recent theories for the scattering of rotons by the charge carriers.

I. INTRODUCTION

Free charges, when injected into liquid helium, form stable microscopically large structures. The negative carrier, for example, can be characterized as a bare electron localized in a bubble with a radius of about 17 Å. The positive carrier, on the other hand, consists of a solid central core having a radius of about 5 Å along with substantial radial density and pressure gradients in the surrounding liquid.

If an electric field is applied to a charge carrier in superfluid helium, the carrier quickly attains an equilibrium drift velocity limited by its interactions with the elementary excitations of the liquid. At low fields this drift velocity is proportional to field: $v_D = \mu E$, where the proportionality constant μ is the mobility. It is of interest to study how a given charge carrier interacts with a particular type of elementary excitation, and how these various types of interactions give rise to the observed behavior of the mobility. Under circumstances where either rotons, phonons, or He^3 impurities are dominant, it is then possible to directly compare the measurements of $\mu_\pm$ with various microscopic models for the scattering of these excitations by the charge carriers.

Meyer and Reif,[1] in their pioneering study of charge carriers in liquid helium, were the first to measure mobility as a function of both temperature and pressure. More recently Brody[2] has made a thorough study of $\mu_\pm(P, T)$ for $1.3\ °K < T < T_\lambda$ and for pressures up to the melting pressure. Our work is essentially an extension of that of Schwarz and Stark,[3,4] whose vapor-pressure data represent the first accurate mobility values at low temperatures.

In this paper we present extensive and accurate measurements of the mobility of positive and negative carriers for various pressures up to the melting pressure and temperatures in the range $0.27\ °K < T < 1.0\ °K$. The observed mobility of the negative carriers in the phonon-limited regime is adequately explained in terms of resonance scattering of sound waves by the electron bubble, in agreement with earlier treatments of the vapor-pressure data.[5] Our analysis yields a new determination of the bubble radius as a function of pressure. In the case of the positives, the observed phonon contribution to the scattering is in good agreement with calculations based on the electrostriction model for the