Relativistic Effect in the Light-Scattering Spectrum of a θ-Pinch Plasma

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(Received 19 June 1972)

Precise measurements of incoherently scattered light from a θ-pinch plasma revealed a blue shift of the spectrum which could not be accounted for either by the diamagnetic drift velocity of the electrons or by the motion of the plasma as observed stereoscopically by a streak camera. We find that a major part of the blue shift can be attributed to a relativistic effect. We discuss the consequences of ignoring the relativistic corrections and point out the possibility of making large systematic errors in electron-temperature measurement in certain experimental arrangements employing Thomson scattering. Even at temperatures well below 1000 eV the effect is not negligible.

INTRODUCTION

Thomson scattering of electromagnetic waves from a plasma has become almost a routine diagnostic tool for the determination of ion and electron densities and temperatures in laboratory plasmas. This diagnostic technique has been most successfully used to determine the temperatures and densities in dense, moderately hot plasmas in θ-pinch devices. In most cases lack of sufficient precision in the experiment has concealed the small deviations of the experimental spectra from the widely used spectra derived from nonrelativistic theories.[1] We have been able to overcome several sources of error in the usual light-scattering experiments and have considerably improved the accuracy of such measurements.[2] Several deviations from the spectra of equilibrium plasmas have become evident. One of the deviations was a large shift to the blue of the spectrum of light scattered from a plasma in the absence of any plasma drifts. In this paper, we wish to report the first experimental evidence of a relativistic blue shift in Thomson scattering from a laboratory plasma. Formulations of light scattering from plasmas, including various relativistic aspects, have been put forward in several theoretical works.[3-7] Though Theimer and Sollid[6] mention specifically a blue shift of the scattering spectrum, it was not expected that these effects might be important for the proper interpretation of light-scattering experiments on plasmas of only 100-eV electron temperature.

RELATIVISTIC CORRECTIONS

Implicit in the usual theories is the assumption that the motion of the plasma electrons is nonrelativistic, and the theories are therefore adequate for cold plasmas where $v/c \simeq 0$. But even in moderately hot plasmas with $T_e = 100$ eV, $v_{\text{thermal}}/c \sim 2.5 \times 10^{-2}$, and therefore there are a substantial number of electrons in the tail of the electron-velocity distribution function for which the approximation $v/c \simeq 0$ is violated. For even higher-energy plasmas that are to be encountered in the next generation of plasma devices, or when the high-energy tail of the distribution function is of particular interest, relativistic corrections can be significant.

In evaluating the relativistically correct expression for the spectrum of incoherently scattered electromagnetic waves from a hot plasma, consideration has to be given to the following points.

First, the motion of a single electron of mass m and charge e in the field of a linearly polarized electromagnetic wave is usually approximated by

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) = e\vec{E}(t). \tag{1}$$

The complete equation of motion is, however,[8]

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}\,\frac{m\vec{v}}{(1-\beta^2)^{1/2}} = e[\vec{E}(t) + \vec{\beta}\times\vec{H}(t)]. \tag{2}$$

Here, $\vec{p}$ is the momentum of the electron, $\vec{\beta} = \vec{v}/c$, and $\vec{E}$ and $\vec{H}$ are the electric and magnetic fields associated with the electromagnetic wave.

Second, the electric field at a distance R from the electron, owing to its acceleration by the electromagnetic wave, is usually written as

$$E_s(t) = \frac{e}{c}\left(\frac{\hat{k}_s\times(\hat{k}_s\times\dot{\vec{\beta}})}{R}\right)_{t'}. \tag{3}$$

This again has to be replaced by a more accurate expression,[8]

$$E_s(t) = \frac{e}{c}\left(\frac{\hat{k}_s\times[(\hat{k}_s-\vec{\beta})\times\dot{\vec{\beta}}]}{R(1-\hat{k}_s\cdot\vec{\beta})^3}\right)_{t'}, \tag{4}$$

where $\hat{k}_s$ is a unit vector in the direction of scattering. The scattered field $E_s(t)$ above is calculated at the retarded time t'.

Equations (1) and (3) give the usual Thomson cross section σ_T for the scattering cross section of a stationary electron. Using the more complete Eqs. (2) and (4) gives the more complicated differential scattering cross section of a moving electron as given, for instance, in the work of Pechacek and Trivelpiece,[5] Theimer and Sollid,[6] and Sheffield.[7]

Third, in the usual Thomson scattering theory, the approximation is made that

$$|\vec{k}_s| = |\vec{k}_i| \quad , \tag{5}$$

and therefore

$$|\vec{k}| = |\vec{k}_s - \vec{k}_i| = 2|\vec{k}_i|\sin\tfrac{1}{2}\theta.$$

Here $\vec{k}_s$ and $\vec{k}_i$ are the wave vectors of the scattered and the incident wave, respectively, and θ is the scattering angle. Equation (5) implies that the change in momentum of the scattered photon is a function only of the scattering angle. This is true in elastic scattering from a stationary electron. Such an approximation will not, however, be valid when the frequency of the scattered wave is very different from the incident frequency. For instance, Thomson scattering has been observed from a plasma of electron temperature 1 keV, using a ruby laser.[9] The full half-width of the scattered spectrum is about 1000 Å, and the variation of $\vec{k}$ about its mean value, i.e., $2|\vec{k}_i|\sin\frac{1}{2}\theta$, is about 8%, and cannot be ignored. One then has to write

$$\vec{k} = (1/c)(\hat{k}_s\omega_s - \hat{k}_i\omega_i), \tag{6}$$

where ω_s and ω_i are the scattered and incident frequencies, and $\hat{k}_s$ and $\hat{k}_i$ are unit vectors in these respective directions. Williamson and Clark[10] have shown how to avoid this difficulty by either doing a backscattering experiment or interposing a diffraction grating in the scattered beam. Otherwise, precise expressions for the wave vector of the scattered wave have to be carried into the calculations.

When these effects are included, there are still two situations to consider.

(a) The spectrum of waves scattered from an incident beam of infinite extent by an infinite plasma, so that during the scattering event, all electrons remain within the region of observation, which is also infinite. The usual Thomson scattering theory is further specialized to this situation.

(b) The spectrum of scattered waves from an incident beam such that the beam, or the region of observation, or any combination of these is finite. This is the situation most commonly encountered in laboratory scattering experiments. In such an experiment with a laser, for example, the laser beam is focused to a spot a few millimeters wide. With $T_e = 100$ eV, we have $v_{\text{thermal}} \sim 6\times10^8$ cm/sec, and therefore a typical electron moves out of the scattering region in a little over 1 nsec. This is much shorter than the duration of a normal scattering experiment using a Q-switched ruby laser, which typically lasts for 20–50 nsec.

The combined result of Eqs. (2), (4), and (6) to order v/c is that the spectrum of waves incoherently scattered from a plasma with an isotropic Maxwellian velocity distribution function is no longer a Gaussian spectrum symmetrical about the frequency of the incident wave, as expected from the usual theories. The peak of the spectrum is now shifted to higher frequencies, and the spectrum is asymmetric about this new peak. This is demonstrated in the numerical calculations of Pechacek and Trivelpiece.[5] These authors have further shown that scattering from a finite volume introduces a broadening of the spectrum of scattered waves in a way analogous to collisional broadening. This broadening amounts, however, to only 6.5×10^{-2} Å, and is thus much smaller than the thermal broadening of the spectrum due to any high-temperature plasma. Moreover, the total scattering cross section of the plasma is reduced by a factor $1-\hat{k}_s\cdot\vec{\beta}$ due to the finite transit time of the electron through the scattering volume. Neglecting the influence of a finite scattering volume, Theimer and Sollid[6] were able to derive simple expressions which are valid also for collective light scattering and show consequences similar to those of the spectrum of scattered waves as shown by Pechacek and Trivelpiece.

If these relativistic effects are ignored, the experimentally observed asymmetry of the spectrum may be falsely interpreted as a drift of the electron plasma, and an anisotropy of the electron-velocity distribution function. Theimer and Sollid[6] have shown that the peak of the spectrum of incoherently scattered waves from a plasma with an isotropic Maxwellian electron-velocity distribution function would be shifted to higher frequency by an amount that would give an apparent drift velocity v_d of the electron plasma parallel to the scattering vector $\vec{k}$ such that

$$v_d = \tfrac{5}{3}(v^2_{\text{thermal}}/c)\sin\tfrac{1}{2}\theta ,$$

where

$$v_{\text{thermal}} = (3kT/m)^{1/2}$$

is the rms velocity of the electrons, T (°K) is the temperature, and θ is the scattering angle. In terms of wavelength shift, the peak would be shifted to the blue by

$$\Delta\lambda \cong -1.69\times10^{-9}\lambda_0 T\sin^2\tfrac{1}{2}\theta .$$

A 30% higher value of this blue shift was obtained by Sheffield,[7] who restricted his treatment to the scattering by single electrons and accounted for finite-transit-time effects.

From the expressions derived by Theimer and Sollid,[6] it can also be shown that, when $\alpha_e \ll 1$ for scattering from a plasma with an isotropic Maxwellian electron-velocity distribution function, the

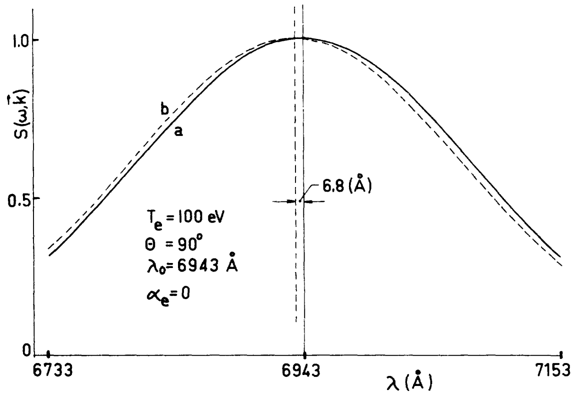

FIG. 1. Theoretical light-scattering spectra for the indicated parameters: (a) neglecting relativistic corrections and (b) including relativistic corrections according to the treatment of Theimer and Sollid (Ref. 6).

spectrum of scattered waves has the form

$$F(\delta\lambda) \simeq \left(1 - \frac{5}{2}\frac{\delta\lambda}{\lambda_0}\right) \exp\left(-\frac{3}{8}\frac{c^2}{v_{\text{thermal}}^2}\frac{\delta\lambda^2}{\lambda_0^2}\frac{1}{\sin^2\frac{1}{2}\theta}\right),$$

where $\delta\lambda$ is the wavelength shift from the incident wavelength λ_0. Curve (b) in Fig. 1 shows this function for the case $\lambda_0 = 6943$ Å, $\theta = 90°$, $T_e = 100$ eV, $\alpha_e = 0$. Curve (a) in this same diagram shows a Gaussian scattered spectrum, also for the above conditions, which would arise if we ignored the relativistic corrections.

As an illustration of what erroneous conclusions one might draw if one ignored the relativistic effect, we observe that in scattering from a deuterium plasma with $T_e = T_i = 100$ eV ($= 1.16 \times 10^6$ °K) and $\theta = 90°$, we would see an apparent drift of the electron plasma of $\simeq 2 \times 10^7$ cm/sec.

The rms thermal velocity of the ions in this plasma v_{thermal} (ions) $\simeq 1.2 \times 10^7$ cm/sec. Therefore $v_d > v_{\text{thermal}}$ (ions). For certain plasma conditions, this result could be misinterpreted as the cause for enhanced ion-acoustic oscillations.

Another important consideration is the possibility of making systematic errors in electron-temperature measurement in certain experiments if these relativistic corrections are ignored. Thomson scattering has been used to determine the electron temperature in low-density hot plasma.[9] High-power ruby lasers are most commonly employed for such experiments. In order to obtain optimum detection of the low-intensity scattered light, it is customary to detect the scattered spectrum on the blue side of the ruby-laser line. One would detect the spectrum, say, at ten points between the laser line and a point at half the peak amplitude. These points would lie on a curve similar to curve (b) in Fig. 1, with certain standard deviations associated with each point. Ignoring the relativistic corrections, one would mistakenly fit a Gaussian curve to these points (since $\alpha_e \ll 1$ in these experiments), centered on the incident wavelength, and deduce a temperature from the half half-width of the fitted curve. We have computed the errors one would make if a Thomson scattering experiment was performed in the above manner, using a ruby laser for a light source, and scattering at $\theta = 90°$. Some of the results are shown in Table I. Clearly, substantial errors can arise in the determination of electron temperature even when the temperature is well below 1000 eV.

EXPERIMENT

We next give results of a Thomson scattering experiment which, we believe, gives the first experimental evidence of this relativistic blue shift. A preliminary account of this work has been given earlier.[11]

The plasma under study is produced in a 100-kJ θ pinch. Details of this plasma device and the ruby-laser scattering experiment are given in an earlier publication.[12] Light scattered at 83° to the direction of the incident beam was collected and analyzed using an eight-channel polychromator. The scattering measurements were made at several phases of the evolution of the plasma. Scattering was observed from plasmas with and without initially trapped axial magnetic field. Thus, a range of plasma conditions with varying electron density, temperature, and also azimuthal electron drift velocity were examined. The θ pinch has been adjusted to give highly reproducible plasma conditions. Several refinements of technique in the scattering experiment allow great precision to be achieved, such that the normalized scattered intensity of one of the center channels, averaged over 10–15 discharges, gives a standard deviation of less than 3%. Errors in the relative sensitivities of the channels give rise to additional errors in the final spectrum. Typical scattering spectra are shown in Fig. 2. This is for a plasma with no initially trapped axial magnetic field. In Fig. 2(a) the scattering was from the center of the plasma column. We have fitted a theoretical curve to this

TABLE I. The correct electron temperature T_e by comparison with values of T_e' derived from measurements of only the blue side of the light-scattering spectra ignoring relativistic corrections.

Real electron temperature T_e (eV)	Measured electron temperature T_e' (eV)	Error (%)
100	108	8
500	596	19
1000	1256	25.6
2000	2790	39

experimental spectrum with the help of a nonlinear regression computer program which is based on the theories of Fejer[12] and Salpeter.[13] Both these treatments are nonrelativistic. The computer program assumes a Maxwellian velocity distribution for the electrons and incorporates an option of a small drift in this distribution according to the treatment by Theimer.[14] The justification for the use of these nonrelativistic formulations is derived from the remark in Theimer and Sollid's paper[6] that, for the conditions of our plasma (low temperature of 100 eV, and drift velocities, if any, negligible compared to thermal velocities), "the Doppler-shift corrections and relativistic corrections ... have almost the same effect as replacing the true drift w_e by an apparent drift w_e' in the uncorrected Salpeter formulas." The best-fitting curve had a value of $\alpha_e \sim 0.2$ and $T_e = 117$ eV, and the peak of the best-fitting curve was shifted to the blue by 9 ± 3.5 Å. Since we are looking at the center of the plasma column where the electron drifts are low and, moreover, both directions of this azimuthal drift can be seen in the scattering spectrum, it is unlikely that this 9-Å shift arises from azimuthal electron drift. To confirm this point, we have also obtained scattering spectra from the edge of the plasma column, shown in Fig. 2(b). This again shows a shift of the spectrum to the blue of 7.5 ± 3.5 Å. The diamagnetic drift of the electrons in the sheath of the θ-pinch plasma is expected to give a shift of the spectrum of below 1 Å, and, according to the scattering geometry, this extra blue shift should add to the shift shown in Fig. 2(a). Instead, an opposite tendency is observed.

In this situation it would be desirable to extend such measurements also to the opposite edge of the plasma, where scattering wave vector and diamagnetic drift are such that a red shift would result. Unfortunately, our setup did not permit us to follow the plasma that far. However, in experiments preceding those reported here, a then unintended confirmation that the blue shift is independent of plasma drift was obtained. When scattering was observed from the same scattering volume but with the plasma current reversed by reversing all external discharge voltages, a blue shift was still observed. It was this startling observation which necessitated an improvement of the wavelength and intensity calibration of the light-scattering setup and which led to the detailed investigation presented in this paper.

Measurements were also made on a plasma with an antiparallel trapped magnetic field. A much larger azimuthal diamagnetic drift velocity of the electrons obtains in such a plasma. The blue shift due to this drift, if any, should be more pronounced, and it should increase with the azimuthal current. Again, the observed blue shift shows an opposite tendency. Figure 3(a) shows the variation of the electron temperature and the azimuthal current with the time at which the scattering experiment was performed. Figure 3(b) shows the corresponding blue shift in the spectra obtained. We see that the blue shift is closely correlated to the variation

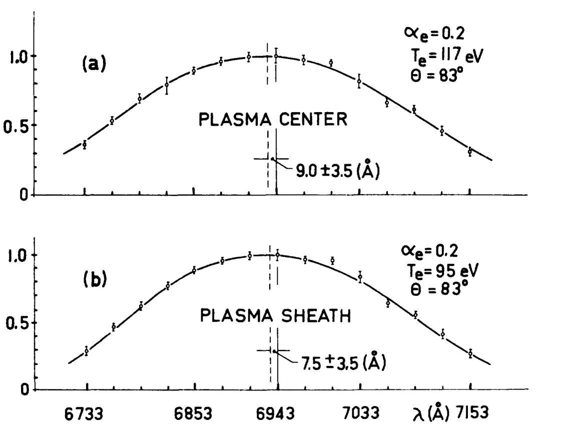

FIG. 2. Experimental light-scattering spectra obtained on a 100-kJ θ-pinch plasma: (a) from the plasma center and (b) from the current-carrying plasma sheath. The indicated errors represent combined standard deviations of many measurements and possible systematic errors arising from the relative calibration of the wavelength channels. Best-fitted curves are calculated according to the theoretical treatment of Fejer (Ref. 12) and Salpeter (Ref. 13), and include the convolution due to the finite instrumental width of the polychromator.

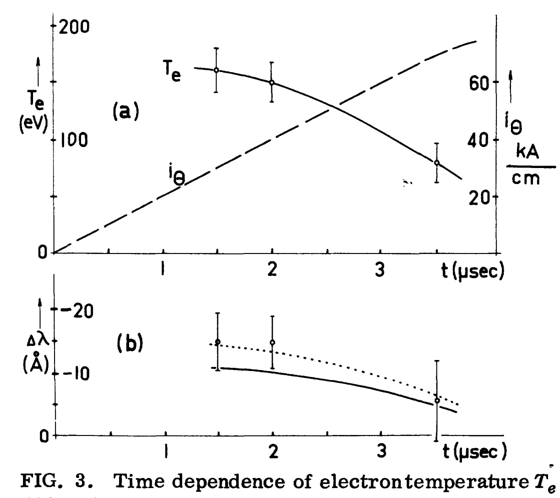

FIG. 3. Time dependence of electron temperature T_e and blue shift $\Delta\lambda$ as evaluated from light-scattering spectra which were obtained on a θ-pinch plasma with antiparallel trapped magnetic field. Also indicated is the time dependence of the azimuthal line current i_θ. The curves plotted in (b) show the blue shift in the spectra expected from the relativistic corrections according to the treatment of Theimer and Sollid (Ref. 6) (solid curve) and Sheffield (Ref. 7) (dotted curve).

of the electron temperature and not to the azimuthal current. Knowing the magnetic field gradient and the density in this annular shaped plasma cross section, we can calculate the diamagnetic drift velocity of the electrons, and the corresponding shift of the peak of the spectrum due to this drift. Such a shift of the peak is below 1 Å. The solid curve in Fig. 3(b) shows the blue shift in the spectra expected from the relativistic corrections in Theimer and Sollid's approach,[6] whereas the dotted curve represents that expected from Sheffield's treatment.[7] The errors associated with the measured blue shifts do not allow us to discriminate between either one of the above approaches. We may, however, conclude that a large part of the observed blue shift in our spectra is not due to electron drifts in the plasma, but is a straightforward consequence of a more complete treatment of incoherent scattering of an electromagnetic wave by a moving electron.

CONCLUSIONS

We find that the major part of the blue shift of the scattered spectrum that we observe is independent of any plasma drifts. The blue shift is also much larger than what would arise from any possible plasma drift. Lastly, the blue shift is temperature dependent and can be attributed to a relativistic effect in Thomson scattering. However, the accuracy of our measurements does not allow us to discriminate between two different treatments of this effect, one due to Theimer and Sollid[6] and the other due to Sheffield.[7]

We note that Thomson scattering experiments, and the most frequently invoked theories used to explain these experiments, ignore these relativistic corrections, which can be important even at temperatures as low as 100 eV. For instance, as pointed out by Theimer and Sollid,[6] if the relativistic corrections are ignored, then the experimentally observed asymmetry of the spectrum may be falsely interpreted as an anisotropy of the electron-velocity distribution function. This can lead to very misleading conclusions about the plasma. In certain experimental arrangements frequently employed to measure the electron temperature in low-density high-temperature plasmas where only one-half of the scattered spectrum is measured, this can lead to large systematic errors in the determination of the temperature, and these errors get larger at higher plasma temperatures, and therefore have particular significance for the next generation of fusion plasma experiments.

ACKNOWLEDGMENTS

We are grateful to Dr. J. Sheffield and Dr. J. H. Williamson for many valuable comments and suggestions, and we want to acknowledge the valuable technical contributions of R. Benesch.

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