simple two-level model (the rising edge of the delay curve excepted), and that the low-temperature phase memory in ruby behaves as predicted by a "direct-process" model; furthermore, $T_2' \simeq 50$ nsec near 0°K. This emphasizes the ability of atomic coherence phenomena to provide useful information on phase-relaxation mechanisms in solid-state systems.

*Work supported in part by the USAF Office of Scientific Research and Kirtland Air Force Base.

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PHYSICAL REVIEW A VOLUME 8, NUMBER 1 JULY 1973

Renormalization Group Equation for Critical Phenomena

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An exact renormalization equation is derived by making an infinitesimal change in the cutoff in momentum space. From this equation the expansion for critical exponents around dimensionality 4 and the limit $n=\infty$ of the n-vector model are calculated. We obtain agreement with the results of Wilson and Fisher, and with the spherical model.

I. INTRODUCTION

Recently Wilson[1] has developed a powerful method of calculating critical exponents of the n-vector model in powers of $\epsilon = 4-d$, where d is the dimensionality of the system. Although he uses some properties of the renormalization group, his procedure is not directly based on this group. Here a renormalization-group equation is derived by eliminating the Fourier components of the order parameter in an infinitesimally small shell in k space (Sec. II). This equation yields the generator of the renormalization group. The change in the Hamiltonian under the infinitesimal change of momentum cutoff can be expressed by a closed equation. Integration of this equation up to a momentum-cutoff factor b will presumably yield the recursion formulae mentioned at the end of the paper by Wilson and Fisher.[2]

To demonstrate the usefulness of our equation, we consider (a) the expansion around dimensionality 4 for the n-vector model and rederive critical exponents to order ϵ and η to order ϵ^2 (Sec. III) and (b) the limit $n=\infty$ of the n-vector model (Sec. IV). In this limit the equation for the fixed point decomposes into several equations which can be solved successively. The analysis yields the spherical-model results as expected from Stanley's proof.[3]

II. BASIC RENORMALIZATION GROUP EQUATIONS

In this section we derive renormalization-group equations by making an infinitesimal change in the momentum cutoff. We start from an effective Hamiltonian $H_0\{S_k\}$ [implicit in H_0 is the factor $\beta = (k_B T)^{-1}$] in which S is a classical field ($-\infty \leq S \leq \infty$) and S_k are its Fourier components (in Sec. III we generalize our discussion to an n-vector model). It is assumed that Fourier components with $k>1$ have already been eliminated; consequently H_0 depends only on the N remaining S_k with $k<1$. Sometimes it is convenient to expand the effective Hamiltonian in powers of S:

$$H_0 = Nv_0 + \frac{1}{2!}\sum_k v_2(k) S_k S_{-k} + \frac{1}{4!N}\sum_{k_1\cdots k_4} v_4(k_1,k_2,k_3,k_4) \times S_{k_1}S_{k_2}S_{k_3}S_{k_4}\delta_{k_1+k_2+k_3+k_4,0} + \cdots . \quad (2.1)$$

Here only even terms have been retained, odd terms are easily included. The Fourier components S_k have been normalized such that $\langle S_k S_{-k}\rangle$ is of order N^0 (above criticality or for $k\neq 0$). The partition function for the system is given by

$$Z = \mathrm{Tr}\, e^{-H_0} = \prod_k \frac{1}{\sqrt{\pi}}\int dS_k\; e^{-H_0}; \quad (2.2)$$

note that S_{-k} is the complex conjugate of S_k,

$$S_k = S_{-k}^* , \quad (2.3)$$

and that the integral over the complex components should be understood as

$$\int dS_k\, dS_{-k} = \int d\,\mathrm{Re}(S_k)\int d\,\mathrm{Im}(S_k) . \quad (2.4)$$

Our aim is to carry out the renormalization procedure with a momentum-cutoff factor e^{-l}. This procedure allows the construction of a Hamiltonian H_l from H_0 which leaves the partition function invariant. To obtain H_l we take the following steps (compare Sec. II of Ref. 4): (a) Eliminate all Fourier components with wave vector $|k|>e^{-l}$; (b) renumber and rescale the Fourier components (the transformation to new variables); (c) extend the system in all linear dimensions by a scale factor e^l. We perform only an infinitesimal transformation, $l=\delta$; in this way we derive the generator for the renormalization procedure.

To eliminate all Fourier components with $k>e^{-l}=1-\delta$, it is convenient to introduce an operator P which sets all S_k with $k>1-\delta$ equal to zero:

$$PH_0\{S_k\} = H_0\{S_k\Theta(1-\delta-|k|)\} . \quad (2.5)$$

Then the expansion of H_0 in powers of the operators S_k within the shell $1-\delta<k<1$ has the simple form

$$H_0 = PH_0 + \sum{}' S_k P\frac{\partial H_0}{\partial S_k} + \frac{1}{2}\sum{}' S_k S_{k'} P\frac{\partial^2 H_0}{\partial S_k\,\partial S_{k'}} + \cdots . \quad (2.6)$$

Here the prime indicates that the summation is only over k in the shell. We split off the two-spin interaction

$$H^{(2)} = \tfrac{1}{2}\sum_k v_2(k) S_k S_{-k} \quad (2.7)$$

from the Hamiltonian

$$H_0 = H^{(2)} + H' , \quad (2.8)$$

and find that the change δH_e in the Hamiltonian, owing to integrating the partition function over the shell in k space, is given by

$$e^{-\delta H_e} = \prod_k{}' \left[(1/\sqrt{\pi}\int dS_k\right] e^{-H^{(2)}-H'} = \prod{}' [v_2(k)]^{-1/2}\langle e^{-\hat{H}}\rangle , \quad (2.9)$$

where

$$\hat{H} = \sum{}' S_k P\frac{\partial H'}{\partial S_k} + \frac{1}{2}\sum{}' S_k S_{k'} P\frac{\partial^2 H'}{\partial S_k\,\partial S_{k'}} + \cdots . \quad (2.10)$$

The expectation value in Eq. (2.9) is taken with respect to $H^{(2)}$. It follows immediately that δH_e is given by the cumulant expansion

$$\delta H_e = \tfrac{1}{2}\sum \ln v_2(k) + \langle\hat{H}\rangle_c - \tfrac{1}{2}\langle\hat{H},\hat{H}\rangle_c + \tfrac{1}{6}\langle\hat{H},\hat{H},\hat{H}\rangle_c + \cdots , \quad (2.11)$$

where the averages are taken with respect to $H^{(2)}$ for k in the shell. We show in Appendix A that, in the limit of infinitesimal δ, the only terms which contribute to δH_e are cumulants involving no more than two derivatives of H'; hence

$$e^{-\delta H_e} = \prod_k{}' \left(\frac{1}{\sqrt{\pi}}\int dS_k\right) \times \exp\left(-\sum{}' S_k P\frac{\partial H}{\partial S_k} - \frac{1}{2}\sum{}' S_k S_{-k} P\frac{\partial^2 H}{\partial S_k\,\partial S_{-k}}\right); \quad (2.12)$$

that is

$$\delta H_e = \frac{1}{2}\sum{}' \ln\left(P\frac{\partial^2 H}{\partial S_k\,\partial S_{-k}}\right) - \frac{1}{2}\sum{}'\left(P\frac{\partial H}{\partial S_k}P\frac{\partial H}{\partial S_{-k}}\Big/ P\frac{\partial H}{\partial S_k\,\partial S_{-k}}\right) . \quad (2.13)$$

There are $Nd\delta$ Fourier components in the shell. If we replace the sum $\sum_k'$ over the components by the angular integration $(Nd\delta/\Omega)\int d\Omega$, we obtain finally

$$\delta H_e = \frac{\delta N d}{2\Omega} \int d\Omega \left[\ln\left(P \frac{\partial^2 H}{\partial S_e \partial S_{-e}}\right) - \left(P\frac{\partial H}{\partial S_e} P \frac{\partial H}{\partial S_{-e}}\right) \Big/ P \frac{\partial^2 H}{\partial S_e \partial S_{-e}} \right]. \tag{2.14}$$

Here the integral sums over the unit vector e and P indicates that only components with $k<1$ are retained. Readers familiar with Feynman graphs will realize that we sum only those graphs with propagators having momentum such that $1-\delta<|k|<1$. In the limit $\delta\to 0$ two types remain which are represented by the two terms in Eq. (2.14). The first term corresponds to the sum of all graphs with one closed loop of arbitrary length, with all propagators around the loop having momentum e (no external momentum enters at any vertex), while the second term is a sum over open lines, again with momentum e on every propagator on the line. Only $\partial H/\partial S$ and $\partial^2 H/\partial S^2$ are involved because two propagators at most emerge from a given vertex.

Now we renumber the Fourier components. After the elimination of the Fourier components in the shell, the wave vector q runs up to $e^{-\delta}$ only. If we make the change of variable $k\to k' = ke^{\delta}$, then q again runs up to 1, but the interaction potentials $v(k\ldots)$ have been replaced by $v(k' e^{-\delta}\ldots)$. Expanding it is easy to see that the change in the interaction potential leads to a change in the Hamiltonian:

$$\delta H_q = -\delta \sum_k k S_k \partial_k' \frac{\partial H}{\partial S_k}. \tag{2.15}$$

Here ∂_k' denotes differentiation with respect to k; the prime indicates that the differentiation should not be applied to the δ function in Eq. (2.1).

After summing over the shell, the number of Fourier components has been reduced to $N_\delta = N_0 e^{-d\delta}$. Therefore, if v_n is replaced by $v_n \exp[d\delta(\frac{1}{2}n-1)]$, H_δ can be written in the form (2.1), where $N=N_\delta$ now denotes the number of Fourier components. This transformation is achieved by a change in H of

$$\delta H_n = d\delta H - \tfrac{1}{2} d\delta \sum_k S_k \frac{\partial H}{\partial S_k}. \tag{2.16}$$

In addition to the change in H arising from the elimination of the shell in momentum space, changes arise from the scale transformation in S, which is made in order to obtain the fixed point; as we will see, this transformation is related to the critical exponent η. If all spin components S_k are multiplied by a factor $\exp[\frac{1}{2}(2-\eta)\delta]$, then H changes by

$$\delta H_s = \delta \tfrac{1}{2}(2-\eta) \sum_k S_k \frac{\partial H}{\partial S_k}; \tag{2.17}$$

this transformation also affects the trace, which leads to a further contribution

$$\delta H_t = -\tfrac{1}{2}\delta(2-\eta)N. \tag{2.18}$$

The operator $\partial H/\partial l$ is now obtained by collecting all contributions:

$$\delta \frac{\partial H}{\partial l} = \delta H_e + \delta H_q + \delta H_n + \delta H_s + \delta H_t, \tag{2.19}$$

$$\frac{\partial H}{\partial l} = \frac{Nd}{2\Omega} \int d\Omega \left[\ln\left(P\frac{\partial^2 H}{\partial S_e \partial S_{-e}}\right) - \left(P\frac{\partial H}{\partial S_e}\right)\left(P\frac{\partial H}{\partial S_{-e}}\right) \Big/ P\frac{\partial^2 H}{\partial S_e \partial S_{-e}} \right] - \sum_k k S_k \partial_k' \frac{\partial H}{\partial S_k} + dH + \tfrac{1}{2}(2-\eta-d)\sum_k S_k \frac{\partial H}{\partial S_k} - \tfrac{1}{2}(2-\eta)N. \tag{2.20}$$

This is our basic equation. It describes the change of H under the renormalization procedure. Since a fixed point H^* does not change under the renormalization procedure, it is determined by $\partial H^*/\partial l = 0$. This leads to a nonlinear eigenvalue equation for H^* with eigenvalue η. We now show that η is the critical exponent (for a definition of the critical exponents see Refs. 5 and 6). If we add a magnetic field h to H,

$$\hat{H} = H + hN^{1/2}S_0, \tag{2.21}$$

then we obtain from Eq. (2.20)

$$\frac{\partial \hat{H}}{\partial l} = \frac{\partial H}{\partial l} + \tfrac{1}{2}(2-\eta+d)hN^{1/2}S_0. \tag{2.22}$$

Therefore $h_l = h_0 e^{2-\eta+d}$; the corresponding eigenvalue as defined in Ref. 4 is $y=\frac{1}{2}(2-\eta+d)$; and we obtain the gap exponent $\Delta=\frac{1}{2}(2-\eta+d)$. Using the scaling law

$$\gamma = 2\Delta - (2-\alpha) = 2\Delta - d\nu, \tag{2.23}$$

we obtain

$$\gamma = \nu(2-\eta); \tag{2.24}$$

that is, the coefficient η introduced in Eqs. (2.17) and (2.18) is identical to the exponent η defined by the critical susceptibility $\chi_c(k) \propto k^{\eta-2}$.

III. EXPANSION IN $\epsilon = 4-d$ FOR THE n-VECTOR MODEL

Equation (2.20) is easily generalized to an equation for an n-vector model. Denoting the n components of the vector S_k by S_k^α we find

$$\frac{\partial H}{\partial l}=\frac{Nd}{2\Omega}\int d\Omega\left\{\sum_\alpha\left(\ln P\frac{\partial^2 H}{\partial S_e\,\partial S_{-e}}\right)_{\alpha\alpha}-\sum_{\alpha\beta}P\frac{\partial H}{\partial S_e^\alpha}P\frac{\partial H}{\partial S_{-e}^\beta}\left[\left(P\frac{\partial^2 H}{\partial S_e\,\partial S_{-e}}\right)^{-1}\right]_{\alpha\beta}\right\}$$

$$-\sum kS_k\partial_k'\frac{\partial H}{\partial S_k}+dH+\tfrac{1}{2}(2-\eta-d)\sum_k\vec{S}_k\frac{\partial H}{\partial\vec{S}_k}-\tfrac{1}{2}(2-\eta)Nn\,. \tag{3.1}$$

Here $\partial^2H/\partial S_e\,\partial S_{-e}$ denotes the tensor with components $\partial^2H/\partial S_e^\alpha\partial S_{-e}^\beta$. If we limit our considerations to Hamiltonians H which are isotropic in S space, we may write

$$H=Nn\bar{H}\{(1/N)n(k,k')\}\,, \tag{3.2}$$

where

$$n(k,k')=(1/n)\vec{S}_k\vec{S}_{k'}\,. \tag{3.3}$$

Then, noting that

$$\frac{\partial H}{\partial S_k^\alpha}=2\sum_{k'}\frac{\partial\bar{H}}{\partial[(1/N)n(k,k')]}S_{k'}^\alpha \tag{3.4}$$

and

$$\frac{\partial^2 H}{\partial S_e^\alpha\partial S_{-e}^\beta}=2\frac{\partial\bar{H}}{\partial[(1/N)n(e,-e)]}\delta_{\alpha\beta}+\frac{4}{Nn}\sum_{kk'}\frac{\partial^2\bar{H}}{\partial[(1/N)n(e,k)]\partial[(1/N)n(-e,k')]}S_k^\alpha S_{k'}^\beta\,, \tag{3.5}$$

we find that Eq. (3.1) reduces to

$$\frac{\partial\bar{H}}{\partial l}=\frac{d}{2\Omega}\int d\Omega\ln\left(2P\frac{\partial\bar{H}}{\partial[(1/N)n(e,-e)]}\right)+\frac{d}{2\Omega n}\int d\Omega\,[\ln(1+h\tilde{n})]_{kk}$$

$$-\frac{d}{\Omega}\int d\Omega\sum_{kk'}P\frac{\partial\bar{H}}{\partial[(1/N)n(k,e)]}P\frac{\partial\bar{H}}{\partial[(1/N)n(k',-e)]}\left(P\frac{\partial\bar{H}}{\partial[(1/N)n(e,-e)]}\right)^{-1}[\tilde{n}(1+h\tilde{n})^{-1}]_{kk'}$$

$$-\sum_{kk'}\frac{1}{N}n(k,k')(k\partial_k'+k'\partial_{k'}')\frac{\partial\bar{H}}{\partial[(1/N)n(k,k')]}-\tfrac{1}{2}(2-\eta)+d\bar{H}+(2-\eta-d)\sum_{kk'}\frac{1}{N}n(k,k')\frac{\partial\bar{H}}{\partial[(1/N)n(k,k')]}\,. \tag{3.6}$$

Here the matrices h and $\tilde{n}$ are given by

$$h(e)_{kk'}=\frac{2}{N}P\frac{\partial^2\bar{H}}{\partial[1/N)n(e,k)]\partial[(1/N)n(-e,k')]}\Big/P\frac{\partial\bar{H}}{\partial[(1/N)n(e,-e)]}$$

and (3.7)

$$\tilde{n}_{kk'}=n(k,k')\,.$$

In Sec. IV we will solve Eq. (3.6) to determine the critical behavior in the limit $n=\infty$. However, before carrying out this calculation we will write down explicitly the equations for the potentials v_0 to v_6 and discuss their behavior as a function of $\epsilon=4-d$.

As in Eq. (2.1), we expand the Hamiltonian

$$\bar{\mathrm{H}}=v_0+(1/2N)\sum_k v_2(k)n(k,-k)$$

$$+(1/8N^2)\sum v_4(k_1k_1';k_2k_2')n(k_1,k_1')n(k_2,k_2')\delta_{k_1+k_1'+k_2+k_2',0}$$

$$+(1/48N^3)\sum v_6(k_1k_1';k_2k_2';k_3k_3')n(k_1,k_1')n(k_2,k_2')n(k_3,k_3')\delta\ldots\,. \tag{3.8}$$

Then substituting in (3.1) and equating powers of $n(k,k')$, we find[7]

$$\frac{\partial v_0}{\partial l} = dv_0 - \tfrac{1}{2}(2-\eta) + \frac{d}{2\Omega}\int d\Omega \ln v_2(e)\,, \tag{3.9}$$

$$\frac{\partial v_2(k)}{\partial l} = (2-\eta)v_2(k) - k\partial_k v_2(k) + \frac{d}{2\Omega}\int d\Omega \frac{v_4(k-k;e-e)}{v_2(e)} + \frac{d}{n\Omega}\int d\Omega \frac{v_4(ke;-k-e)}{v_2(e)}\,, \tag{3.10}$$

$$\begin{aligned}\frac{\partial v_4(k)}{\partial l} &= (4-2\eta-d)v_4(k_1k_1';k_2k_2') - \sum k_i\partial_{k_i} v_4(k_1k_1';k_2k_2') \\ &+ \frac{d}{\Omega}\left(\frac{1}{2}\int \frac{d\Omega}{v_2(e)} v_6(k_1k_1';k_2k_2';e-e) + \frac{1}{n}\int \frac{d\Omega}{v_2(e)} v_6(k_1k_1';k_2e;k_2'-e) + v_6(k_1e;k_1'-e;k_2k_2')\right) \\ &- \frac{d}{\Omega}\delta_{k_1+k_1',0}\left(\frac{1}{2}\int \frac{d\Omega}{v_2^2(e)} v_4(k_1-k_1;e-e)v_4(k_2-k_2;e-e) + \frac{1}{n}\int \frac{d\Omega}{v_2^2(e)} v_4(k_1-k_1;e-e)v_4(k_2e;-k_2-e)\right. \\ &\left. + \frac{1}{n}\int \frac{d\Omega}{v_2^2(e)} v_4(k_2-k_2;e-e)v_4(k_2e;-k_2-e)\right) - \frac{d}{n\Omega}(\delta_{k_1+k_2,0} + \delta_{k_1+k_2',0})\int \frac{d\Omega}{v_2^2(e)} v_4(k_1e;-k_1e)v_4(k_2e;-k_2e)\,,\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}\frac{\partial v_6(k_1k_1';k_2k_2';k_3k_3')}{\partial l} &= (6-3\eta-2d)v_6(k_1k_1';k_2k_2';k_3k_3') - \sum k_i\partial_{k_i} v_6(k_1k_1';k_2k_2';k_3k_3') \\ &- \int \frac{d\Omega}{v_2(e)}[v_4(k_1k_1';k_2e)v_4(k_3k_3';k_2'-e)\delta^d(k_1+k_1'+k_2+e) + 5\text{ permutations}] + \cdots\,.\end{aligned} \tag{3.12}$$

The δ functions appearing in Eq. (3.11) are Kronecker δ's.

We now attempt to solve Eqs. (3.9)–(3.12) for small (positive) $\epsilon = 4-d$. It is easy to see that a fixed point $\partial v_n^*/\partial l = 0$ is obtained for $\eta = 0$, $v_2(k) = k^2$, $v_0 = 1/d$, and $v_4 = v_6 = \cdots = 0$. This is the Gaussian solution mentioned by Wilson.[8] It corresponds to an ideal gas of noninteracting fields S_q. Next we derive the nontrivial solution obtained by Wilson and Fisher[2] and further discussed by Fisher and Pfeuty,[9] and Wegner.[10] Let us assume initially that v_4 is a constant v_{40}, and $v_2(e) = 1$. Then we obtain from Eq. (3.12), to order $(v_{40})^2$,

$$v_4 = v_{40}\,, \tag{3.13}$$

$$\begin{aligned}v_6 = -v_{40}^2\,(&f(k_1+k_1'+k_2) + f(k_1+k_1'+k_2') \\ &+ f(k_1+k_1'+k_3) + f(k_1+k_1'+k_3') \\ &+ f(k_2+k_2'+k_1) + f(k_3+k_3'+k_1))\,,\end{aligned} \tag{3.14}$$

with

$$f(k) = \begin{cases} 0, & |k| \le 1 \\ k^{-2}, & |k| > 1\,. \end{cases} \tag{3.15}$$

Substituting Eqs. (3.13) and (3.14) into the integrals of Eq. (3.11), we find

$$\begin{aligned}&(4-2\eta-d)v_4 - \sum k\partial_k v_4 \\ &= v_{40}^2 d(g(k_1+k_1') + (4/n)g(k_1+k_1') \\ &\quad + (2/n)g(k_1+k_2) + (2/n)g(k_1+k_2'))\,,\end{aligned} \tag{3.16}$$

where

$$g(k) = (1/\Omega)\int d\Omega\, f(k+e) \quad \text{and} \quad g(0) = \tfrac{1}{2}\,. \tag{3.17}$$

It should be pointed out that the projection operator P in Eqs. (3.6) and (3.7) limits the value of the wave vector k in Eqs. (3.10)–(3.12) to being at least infinitesimally less than 1: hence the integrals in these equations do not contribute if any wave vector k is a unit vector. This is especially important for the integral in Eq. (3.12), which contains a δ function. As it follows immediately that, at least for any case needed in the calculation, $f(k) = 0$ for $|k| = 1$ [Eq. (3.15)]. For example, as we only need to calculate interactions of the form $v_6(k_1k_1';k_2k_2';e-e)$ or $v_6(k_1k_1';k_2e;k_2'-e)$, conservation of wave vector gives $f(k_1+k_1'+k_2) = f(k_2')$, and hence $f(k) = 0$ for $|k| = 1$. However, if we now define $g(0) = \frac{1}{2}$, as in Eq. (3.17), then $g(k)$ is a continuous function and Eq. (3.16) follows immediately from Eq. (3.11): the extra terms come from the δ functions in that equation.

Then, expanding $g(k)$ and v_4 in powers of k, we obtain

$$(4-2\eta-d)v_{40}=v_{40}^2\,d\tfrac{1}{2}(1+8/n)\,. \tag{3.18}$$

As we will see, $\eta\alpha\propto\epsilon^2$; hence

$$v_{40}=\frac{2\epsilon}{d(1+8/n)}=\frac{\epsilon}{2(1+8/n)}+O(\epsilon^2)\,. \tag{3.19}$$

Moreover, as the k-dependent contributions to v_4 are proportional to v_{40}^2 and therefore proportional to ϵ^2, we have

$$v_4=\frac{\epsilon}{2(1+8/n)}+O(\epsilon^2)\,. \tag{3.20}$$

If we now substitute this expression for v_4 into Eq. (3.10), we find that, to order ϵ,

$$v_2=k^2-\frac{d}{4}v_{40}-\frac{d}{2n\Omega}v_{40}=k^2-\left(1+\frac{2}{n}\right)v_{40}$$

$$=k^2-\frac{(n+2)}{2(n+8)}\epsilon+O(\epsilon^2)\,. \tag{3.21}$$

We can now calculate η to order ϵ^2. We note that, apart from a k-independent contribution, v_4 to order ϵ^2 is given by Eq. (3.16). The reason is that a change in v_4 at order ϵ^2 affects v_6 only at order ϵ^3. From Eq. (3.16) we find that

$$v_4(k_1k_1';k_2k_2')-v_4(0)$$

$$=\left(\frac{\epsilon}{2(1+8/n)}\right)^2 d\left(h(k_1+k_1')+\frac{4}{n}h(k_1+k_1')\right.$$

$$\left.+\frac{2}{n}h(k_1+k_2)+\frac{2}{n}h(k_1+k_2')\right)+O(\epsilon^3), \tag{3.22}$$

with

$$-k\partial_k h(k)=g(k)-g(0);\; h(0)=0\,, \tag{3.23}$$

and therefore we obtain from Eq. (3.10)

$$(2-\eta)[v_2(k)-v_2(0)]-k\partial_k v_2(k)$$

$$=-\frac{12\epsilon^2(n+2)}{(n+8)^2}\frac{1}{\Omega}\int d\Omega[h(k+e)-h(e)]\,. \tag{3.24}$$

The integral on the right-hand side of Eq. (3.24) is an analytic function of k; expanding for small k we find

$$(2-\eta)[v_2(k)-v_2(0)]-k\partial_k v_2(k)$$

$$=-\frac{3}{2}\frac{\epsilon^2(n+2)}{(n+8)^2}k^2[3h'(1)+h''(1)]\,, \tag{3.25}$$

which, making use of Eq. (3.23), becomes

$$(2-\eta)[v_2(k)-v_2(0)]-k\partial_k v_2(k)$$

$$=-\frac{3}{2}\frac{\epsilon^2(n+2)}{(n+8)^2}k^2[-2g(1)+2g(0)-g'(1)]+O(k^4)\,. \tag{3.26}$$

For $\eta=0$ the inhomogeneous term, proportional to k^2, on the right-hand side of the equation would lead to a nonanalyticity

$$v_2(k)-k^2\propto k^2\ln k\,. \tag{3.27}$$

If, on the other hand, $v_2(k)$, as a function of l, is renormalized according to Eq. (3.10), such a nonanalyticity never arises; but the amplitude of the k^2 term is changed. This corresponds to a rescaling of S_k with $\eta=0$. We choose η so that $v_2(k)$ is analytic. From this condition we find that

$$\eta=\tfrac{3}{2}[-2g(1)+2g(0)-g'(1)]\epsilon^2(n+2)/(n+8)^2\,. \tag{3.28}$$

From Eq. (3.17) we find $g(1)$ and $g'(1)$ are given by

$$g(1)=\tfrac{1}{3}-\sqrt{3}/4\pi,\quad g'(1)=\sqrt{3}/2\pi\,, \tag{3.29}$$

and therefore

$$\eta=\tfrac{1}{2}\epsilon^2(n+2)/(n+8)^2\,, \tag{3.30}$$

in agreement with Wilson and Fisher.

As explained by Wilson[8] and in Refs. 2, 9, and 10, one obtains the other critical exponents by calculating the eigenperturbations in linear approximation. For a perturbation growing like e^{yl}, we obtain from Eq. (3.10)

$$y\delta v_2(k)=(2-\eta)\delta v_2(k)-k\partial_k\delta v_2(k)$$

$$+\left(\frac{1}{2}+\frac{1}{n}\right)\frac{d}{\Omega}\int d\Omega\frac{\delta v_4}{v_2(e)}$$

$$-\left(\frac{1}{2}+\frac{1}{n}\right)\frac{d}{\Omega}\int d\Omega\frac{v_4}{v_2(e)^2}\delta v_2(e)\,. \tag{3.31}$$

One can see from Eqs. (3.10)–(3.12) that for $\epsilon=0$ there is a solution $y=2$, $\delta v_2(k)=(\text{const})$, $\delta v_4=\delta v_6=\cdots=0$, $\delta v_0=-\tfrac{1}{2}\delta v_2$. For small ϵ one finds from Eqs. (3.11) and (3.12) that $\delta v_4\propto\epsilon^2\delta v_2$. Therefore we may neglect δv_4 in Eq. (3.31) to obtain y to order ϵ:

$$y_{1s}=2-(n+2)/(n+8)\epsilon\,. \tag{3.32}$$

According to the classification in Ref. 9 we have called this index y_{1s}. As is well known [2,4,8–10] the critical exponents α, β, γ, and ν can be calculated from η and y_{1s}.

IV. LIMIT $n=\infty$

Stanley has shown that in the limit $n=\infty$ the n-vector model reduces to the spherical model.[3] As this model can be solved exactly,[11,12] it is tempting to look for its solution within the framework of the renormalization group. When $n\to\infty$ the second term on the right-hand side of Eq. (3.6) vanishes because of the factor $1/n$. Let us expand the Hamiltonian into terms which contain products of zero, two, three, etc. off-diagonal factors $n(k, k')$ with $k+k' \neq 0$:

$$\overline{H} = \hat{H} + \hat{H}_2 + \hat{H}_3 + \cdots . \tag{4.1}$$

Then $\hat{H}$ depends only on the diagonal terms

$$n_k = n(k, -k), \tag{4.2}$$

$$H = \overline{H}\{n(k, k') = n_k \delta_{k+k',0}\}, \tag{4.3}$$

in which all the off-diagonal terms $n(k, k')$ have been put equal to zero. For $\hat{H}$ we obtain the closed equation

$$\frac{\partial \hat{H}}{\partial l} = \frac{d}{2\Omega}\int d\Omega \ln\left(2\frac{\partial \hat{H}}{\partial[(1/N)n_e]}\right) - \sum \frac{1}{N} n_k k \partial_k \frac{\partial \hat{H}}{\partial[(1/N)n_k]} + (2-\eta-d)\sum \frac{1}{N} n_k \frac{\partial \hat{H}}{\partial[(1/N)n_k]} - \tfrac{1}{2}(2-\eta) + d\hat{H}, \tag{4.4}$$

since the contributions from H_2, H_3, etc. to $\partial\overline{H}/\partial l$ contain at least two off-diagonal factors. Similarly $\partial\hat{H}_2/\partial l$ depends only on $\hat{H}$, and $\hat{H}_2$, etc. Here we restrict ourselves to the solution for $\hat{H}$.[12] To solve Eq. (4.4) we make the ansatz

$$\hat{H} = (c/2N)\sum k^2 n_k + f(z) + (2-\eta)/2d, \tag{4.5}$$

where

$$z = \tfrac{1}{2}\sum (1/N) n_k . \tag{4.6}$$

Then substituting Eqs. (4.5) and (4.6) into Eq. (4.4) we obtain the nonlinear differential equation

$$\frac{\partial f}{\partial l} = \frac{d}{2}\ln(c+f') + (2-\eta-d)zf' + df - \frac{\eta c}{2N}\sum k^2 n(k), \tag{4.7}$$

which for $\eta = 0$ becomes a differential equation for the function f only,

$$\frac{\partial f}{\partial l} = \frac{d}{2}\ln(c+f') + (2-d)zf' + df . \tag{4.8}$$

In order to obtain the fixed point f^*, when $\eta = 0$, we must solve the equation

$$\tfrac{1}{2}\ln(c+f^{*\prime}) + [(2-d)/d]zf^{*\prime} + f^* = 0 . \tag{4.9}$$

The solution is given in Appendix B; however, as we will see, in order to determine the critical exponents it is only necessary to determine the essential singularities of the equation. First, we consider $f^{*\prime}$ as a funtion of f^* and z. In Fig. 1 we plot

$$g_1 = \tfrac{1}{2}\ln(c+f^{*\prime}) \quad\text{and}\quad g_2 = [(d-2)/d]zf^{*\prime} - f^* \tag{4.10}$$

as a function of $f^{*\prime}$. Depending on the value of f^* the line g_2 cuts g_1 twice, touches g_1, or does not intersect g_1; that is, we obtain two solutions $f^{*\prime}(f^*, z)$ for $f^* > f_{\text{lim}}(z)$, one solution for $f^* = f_{\text{lim}}(z)$, but no solution for $f^* < f_{\text{lim}}(z)$. The limit curve is given by

$$g_1 = g_2 \quad\text{and}\quad \frac{\partial g_1}{\partial f^{*\prime}} = \frac{\partial g_2}{\partial f^{*\prime}}; \tag{4.11}$$

that is,

$$f_{\text{lim}} = \tfrac{1}{2} + \tfrac{1}{2}\ln[2(d-2)z/d] - (d-2)cz/d, \tag{4.12}$$

which has a maximum at

$$z = z_0 = d/2(d-2)c . \tag{4.13}$$

Except at $z = z_0$, the slope $f^{*\prime}$ at the limit curve

$$f^{*\prime}(f_{\text{lim}}, z) = \frac{d}{d-2}\frac{\partial f_{\text{lim}}}{\partial z} \tag{4.14}$$

is steeper than the gradient of the curve, $\partial f_{\text{lim}}/\partial z$. Therefore, solutions which reach $f_{\text{lim}}(z)$ form a cusp with finite slope at the limit curve. These solutions have no (real) value on one side of the cusp and two values on the other side; because of this unphysical behavior, they are rejected. The only physical (single-valued) solutions then, either touch $f_{\text{lim}}(z)$ at $z = z_0$ or they do not touch the limit curve at all.

We now show that there are no solutions which do not touch the curve $f_{\text{lim}}(z)$. From Fig. 1 it is apparent that for $f^* > f_{\text{lim}}(z)$ there is one solution with

$$f_+^{*\prime}(f^*, z) > f^{*\prime}(f_{\text{lim}}, z) = \frac{d}{d-2}\frac{\partial f_{\text{lim}}}{\partial z} \tag{4.15}$$

and one solution with

$$f_-^{*\prime}(f^*, z) < f^{*\prime}(f_{\text{lim}}, z) = \frac{d}{d-2}\frac{\partial f_{\text{lim}}}{\partial z} . \tag{4.16}$$

If a solution does not touch $f_{\text{lim}}(z)$, then it is either of type

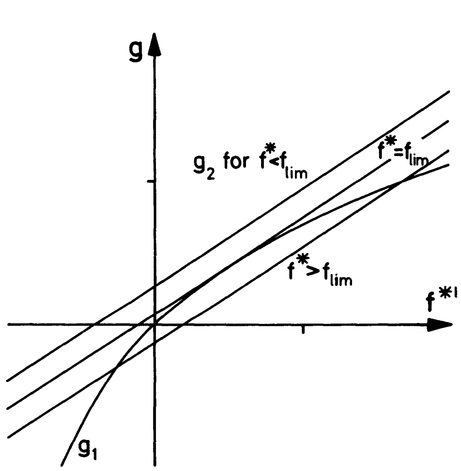

Fig. 1. Determination of $f^{*\prime}(f^*, z)$ from $g_1 = g_2$.

$$f^{*\prime}=f_{+}^{*\prime} \text{ or } f^{*\prime}=f_{-}^{*\prime}. \quad (4.17)$$

Now from Eq. (4.16) we obtain, for $z>z_0$,

$$f^*(z)-f^*(z_0)<[d/(d-2)][f_{\lim}(z)-f_{\lim}(z_0)] \quad (4.18)$$

which with a little rearrangement becomes

$$f^*(z)-f_{\lim}(z)<[2/(d-2)]f_{\lim}(z) - [d/(d-2)]f_{\lim}(z_0)+f^*(z_0). \quad (4.19)$$

It is easy to see that the right-hand side of Eq. (4.19) vanishes for sufficiently large $z=z_c$; therefore, the solution f^* of $f^{*\prime}=f_{-}^{*\prime}(f^*,z)$ comes down to $f_{\lim}(z)$ for some finite $z>z_0$. In a similar way one can show that the solution f^* of $f^{*\prime}=f_{+}^{*\prime}(f^*,z)$ comes down to the limit curve for some finite z, such that $0<z<z_0$. Therefore all solutions f^* touch $f_{\lim}(z)$: moreover only the smooth solutions of Eq. (4.9) go through

$$f^*(z_0)=-\tfrac{1}{2}\ln c. \quad (4.20)$$

To obtain these solutions we expand Eq. (4.9) around $z=z_0$ and obtain

$$\tfrac{1}{2}\ln c-\frac{1}{4c^2}(f^{*\prime})^2+\frac{1}{6c^3}(f^{*\prime})^3+\cdots +\frac{2-d}{d}\delta z f^{*\prime}+f^*=0, \quad (4.21)$$

which has the solutions

$$f^*=-\tfrac{1}{2}\ln c, \quad (4.22)$$

$$f^*=-\tfrac{1}{2}\ln c+\frac{c^2(4-d)}{d}(\delta z)^2+\frac{4c^3(4-d)^3}{3d^2(6-d)}(\delta z)^3+O(\delta z)^4. \quad (4.23)$$

Equation (4.22) corresponds to a system of spins that only interact via a two-spin interaction; this is the Gaussian solution. Equation (4.23) does not exist for $d=6, 8, 10\ldots$. For $d=4$ it is identical to solution (4.22). It corresponds to the nontrivial solution of Wilson and Fisher.[2] To first order in $\epsilon=4-d$ it agrees with the solution Eq. (3.8), (3.20), and (3.21) since for $c=1$

$$f^*=\tfrac{1}{4}\epsilon(z-1)^2 =\tfrac{1}{16}\epsilon\Big(\sum (1/N)n_k\Big)^2-\tfrac{1}{4}\epsilon\sum (1/N)n_k+\tfrac{1}{4}\epsilon. \quad (4.24)$$

Besides the solutions, Eqs. (4.22) and (4.23), considered here, there is the additional set of solutions discussed in the Appendix B.

We consider perturbations to the fixed-point solutions f^*. We start with a perturbation

$$\delta\hat{H}=\delta f(z). \quad (4.25)$$

For this perturbation we obtain from Eq. (4.7) the eigenvalue problem

$$y\delta f=\frac{d}{2(c+f^{*\prime})}\delta f'+(2-d)z\delta f'+d\delta f, \quad (4.26)$$

which leads to

$$\ln\delta f=\tfrac{1}{2}(d-y)\int dz/Q(z), \quad (4.27)$$

with

$$Q(z)=-\frac{d}{4(c+f^{*\prime})}+\frac{(d-2)z}{2} =-\frac{d}{4(c+f^{*\prime})}+\frac{d}{4\{c+[d/(d-2)]f'_{\lim}\}}. \quad (4.28)$$

Since

$$f^{*\prime}=[d/(d-2)]f'_{\lim}$$

only when $f^*=f_{\lim}$, we see that $Q=0$ only at $z=z_0$. For the solution (4.22) we find

$$Q(z)=\tfrac{1}{2}(d-2)\delta z, \quad (4.29)$$

and for (4.23)

$$Q(z)=\delta z-[2c(4-d)^2/d(6-d)](\delta z)^2+O(\delta z)^3; \quad (4.30)$$

therefore for the Gaussian fixed point,

$$\delta f=(\delta z)^{(d-y)/(d-2)}, \quad (4.31)$$

and for the nontrivial fixed point

$$\delta f=[q(z)]^{(d-y)/2}, \quad (4.32)$$

where

$$q(z)=c\delta z+[2c^2(4-d)^2/d(6-d)](\delta z)^2+O(\delta z)^3. \quad (4.33)$$

The perturbations [Eqs. (4.31) and (4.32)] are analytic for positive-integer exponents, $(d-y)/(d-2)$ and $\frac{1}{2}(d-y)$, respectively; hence

$$y_m=d-(d-2)m \quad (4.34)$$

for the Gaussian fixed point and

$$y_m=d-2m \quad (4.35)$$

for the nontrivial solution.

The solution (4.23) and perturbation (4.32) allow the following interpretation. The Hamiltonian $Nn\hat{H}$ describes a system of spins interacting via $\frac{1}{2}c\sum k^2S_k^\alpha S_{-k}^\alpha$ in a potential $Nnf(\delta z)\approx Nnc^2(4-d)\times(\delta z)^2/d$. As long as $d<4$ the potential is attractive and forces the spins to a mean value of $\sum n_k/2N\simeq z_0$, this is no longer true for $d>4$ where the potential is repulsive. The application of a perturbation of type (4.32) with $m=1$ shifts the minimum of the potential; this corresponds to a change in temperature. The critical indices are easily obtained from Eq. (4.35):

$$\nu=1/y_1=1/(d-2), \quad (4.36)$$

$$\alpha = 2 - d\nu = (d-4)/(d-2)\,, \tag{4.37}$$

$$\gamma = \nu(2-\eta) = 2/(d-2)\,, \tag{4.38}$$

and

$$\beta = \nu(\tfrac{1}{2}d - 1 + \eta) = \tfrac{1}{2}\,, \tag{4.39}$$

in agreement with the results obtained previously for the spherical model.

We now consider more-general perturbations. From Eq. (4.4) we find the linear response to a perturbation $\delta\hat{H}$,

$$(y-d)\delta\hat{H} = \frac{d}{(c+f')\Omega}\int d\Omega \frac{\partial\,\delta\hat{H}}{\partial((1/N)n_e)} - \sum \frac{1}{N} n_k k \partial_k \frac{\partial\delta\hat{H}}{\partial((1/N)n_k)} + (2-d)\sum \frac{1}{N} n_k \frac{\partial\delta\hat{H}}{\partial((1/N)n_k)}\,. \tag{4.40}$$

If $\delta\hat{H}_1$ and $\delta\hat{H}_2$ are solutions of Eq. (4.40) with eigenvalues y_1 and y_2, then $\delta\hat{H}_1\delta\hat{H}_2$ is also an eigensolution with

$$y - d = y_1 - d + y_2 - d\,. \tag{4.41}$$

We now calculate a set of eigensolutions

$$\delta\hat{H}_{pl\bar{m}} = \frac{c}{2N}\sum k^{2p+l} Y_l^{\bar{m}}(\Omega) n_k + \delta_{l,0} h_p(q) \tag{4.42}$$

for $l \neq 0$ or $p \neq 0$ and

$$\delta\hat{H}_{00} = q(z)\,. \tag{4.43}$$

Substituting into Eq. (4.41) we find

$$y_{pl} = 2 - l - 2p \quad \text{for } l \neq 0 \text{ or } p \neq 0, \tag{4.44}$$

and

$$(2 - d - 2p)h_p(z) = -2q\frac{\partial h_p}{\partial q} + \frac{dc}{2(c+f^{*\prime})}\,. \tag{4.45}$$

Then using

$$\frac{dc}{2(c+f^{*\prime})} = \frac{d}{2} - (4-d)q + \frac{2(4-d)^2}{d}q^2 + O(q^3)\,, \tag{4.46}$$

we find

$$h_p(q) = -\frac{d}{2(2p+d-2)} + \frac{(4-d)}{(2p+d-4)}q - \frac{2(4-d)^2}{d(2p+d-6)}q^2 + O(q^3), \tag{4.47}$$

from which we can construct the general eigenperturbations

$$\delta\hat{H} = \prod \delta\hat{H}_{pl\bar{m}}\,, \tag{4.48}$$

with

$$y = d + \sum (y_{pl} - d)\,. \tag{4.49}$$

We note that the perturbations $\delta\hat{H}_{10}$ and $\delta\hat{H}_{02}$ have vanishing exponent y. The perturbation δH_{10} corresponds to a scale transformation of S. This transformation can equally well be performed by differentiating with respect to the parameter c. It follows that

$$\delta\hat{H}_{10} = \frac{c\,\partial\hat{H}^*}{\partial c}\,, \tag{4.50}$$

which is easily checked using Eq. (4.5) and differentiating f^* with respect to c in Eq. (4.9). This yields Eq. (4.45) with $h_1(z) = \partial f^*/\partial c$. The perturbations $\delta\hat{H}_{02}$ arise from a change of scale in k space. If we had not eliminated the Fourier components with $k^2e^{2l} > 1$ but, for example, with $k_z{}^2 e^{2l} + k_\perp{}^2 > 1$ (here $k_\perp$ is the component of k perpendicular to k_z), then we would find a fixed point that differed (for infinitesimal l) from H^* by a term proportional to $\delta\hat{H}_{02}$. These are the components of the stress tensor.

Finally, we consider the effect of the perturbations beyond linear order. Following the notation in Ref. 4, we expand

$$\frac{\partial}{\partial l}\Big(H^* + \sum \mu_i O_i\Big) = \sum y_i \mu_i O_i + \frac{1}{2!}\sum a'_{ji_1i_2}\mu_{i_1}\mu_{i_2}O_j + \cdots\,. \tag{4.51}$$

The higher-order terms come from the first term on the right-hand side of Eq. (4.4) and yield the following relation

$$\sum a'_{ji_1\cdots i_n}O_j = \frac{(-1)^{n-1}(n-1)!}{2}\Big(\frac{2}{c+f'}\Big)^n \frac{d}{\Omega} \times \int d\Omega \prod_m \frac{\partial O_{im}}{\partial((1/N)n_e)}\,. \tag{4.52}$$

We find from Eq. (4.48) that

$$\frac{\partial O_i}{\partial((1/N)n_e)} = \sum_{pl}\frac{\partial O_i}{\partial\delta H_{pl}}\frac{\partial\delta H_{pl}}{\partial((1/N)n_e)}, \tag{4.53}$$

and introduce functions ϕ by

$$\frac{\partial\delta H_{pl}}{\partial((1/N)n_e)}\frac{2}{c+f'} = \phi_{pl}(q)Y_l^m(\Omega)\,. \tag{4.54}$$

Then we obtain with

$$K(p_1l_1,\ldots) = \prod_m \phi_{p_ml_m}(q)\cdots\frac{d}{\Omega}\int d\Omega \prod_m Y_l^m(\Omega), \tag{4.55}$$

$$\sum a'_{ji_1\ldots i_n} = \frac{(-1)^{n-1}(n-1)!}{2}\sum_{\{pl\}} K(p_1l_1,\ldots) \times \prod_m \frac{\partial O_i}{\partial\delta H_{p_ml_m}}\,. \tag{4.56}$$

We find, for example,

$$\phi_{00}(q)=\frac{\partial q}{\partial z}\frac{1}{c+f'}=1-\frac{2(4-d)(d-2)}{d(6-d)}q+O(q^2). \tag{4.57}$$

From these equations it is easy to calculate the coefficients a'. For $O_0=1$ and $O_1=q$ we find

$$a'_{011}=-\tfrac{1}{2}d, \tag{4.58}$$

$$a'_{111}=2(4-d)(d-2)/(6-d), \tag{4.59}$$

$$a'_{0111}=d. \tag{4.60}$$

From this we find

$$f_{0111}=a'_{0111}+[3/(d-2)]\,a'_{011}a'_{111}=2d(d-3)/(6-d). \tag{4.61}$$

If f_{0111} did not vanish in three dimensions, then we would find a logarithmic singularity in the specific heat, since $3y_1=y_0$. However, since f_{0111} vanishes for $d=3$, there is no logarithmic singularity in the specific heat, as is well known for the spherical model.

Finally we consider the Hamiltonian

$$\hat{H}=(1/2N)\sum v(k)n_k+f(z)+(2-\eta)/2d \tag{4.62}$$

and try to calculate H_l. Similarly to Eq. (4.7) we obtain

$$\frac{\partial v}{\partial l}=(2-\eta)v-k\frac{\partial v}{\partial k}, \tag{4.63}$$

$$\frac{\partial f}{\partial l}=\frac{d}{2\Omega}\int d\Omega\ln[v(e)+f']+(2-\eta-d)zf'+df. \tag{4.64}$$

Integration of Eq. (4.63) yields

$$v_l(k)=e^{(2-\eta)l}v_0(ke^{-l}). \tag{4.65}$$

Differentiating Eq. (4.64) with respect to z and considering z as a function of f' and l we obtain the linear differential equation (for the l independent solution compare Appendix B)

$$\frac{\partial z}{\partial l}=(d-2)z-2f'\frac{\partial z}{\partial f'}+\frac{d}{2[v_l(e)+f']} \tag{4.66}$$

with the formal solution[14]

$$z_l(f')=e^{(d-2+\eta)l}\left[z_0(f'e^{-(2-\eta)l})-\frac{d}{2\Omega}\int_{e^{-l}}^{1}\frac{d^dk}{v_0(k)+f'e^{-(2-\eta)l}}\right]. \tag{4.67}$$

For $v_0(k)=ck^2$ and $\eta=0$, one obtains from this equation

$$z_l(f')=z^*(f')+e^{(d-2)l}[z_0(f'e^{-2l})-z^*(f'e^{-2l})], \tag{4.68}$$

in which $z^*(f')$ is the fixed-point solution, Eq. (B10). For a potential which behaves like $v_0(k)\propto k^{2-\eta}$ for small k we obtain only a limit $\lim_{l\to\infty}z_l(0)$, if

$$z_0(0)=\frac{d}{2\Omega}\int_0^1\frac{d^dk}{v_0(k)}. \tag{4.69}$$

Since $z_0(0)$ is the minimum of an attractive potential $f(z)$, one finds in the thermodynamic limit using Eq. (4.6):

$$\left\langle\frac{1}{N}\sum n_k\right\rangle=2z_0(0)=\frac{d}{\Omega}\int_0^1\frac{d^dk}{v_0(k)}, \tag{4.70}$$

which is precisely the condition for criticality in the spherical model.[3,11]

APPENDIX A

In this appendix we estimate the order of magnitude in N and δ of the various contributions to the cumulant expansion of Eq. (2.11).

We find

$$\left\langle\frac{\partial^nH'}{\partial S^n}\,\frac{\partial^{n'}H'}{\partial S^{n'}}\cdots\right\rangle\sim N^{1+\Delta-(n+n'+\cdots)/2}; \tag{A1}$$

Δ usually vanishes; indeed the derivative

$$\frac{\partial^nH'}{\partial S^n}=\sum_m\frac{N^{1-m/2}}{(m-n)!}\sum_{k_1\cdots k_{m-n-1}}v_mS_{k_1}\cdots S_{k_{m-n}} \tag{A2}$$

is easily estimated by performing the k summations to be

$$\frac{\partial^nH'}{\partial S^n}\simeq\sum_m N^{m/2-n}S_k^{m-n}, \tag{A3}$$

which, as the expectation value of a product of p operators S_{k_p}, is usually of order

$$\langle S_{k_1}\cdots S_{k_p}\rangle\simeq N^{1-p/2} \tag{A4}$$

gives (A1) with $\Delta=0$. However, strictly speaking (A4) only holds for the cumulant, which is equal to the expectation value (A4) only if all factorizations vanish. A factorization is nonvanishing only if all factors have vanishing total momentum.

Suppose that the product of p spins can be factorized into $\Delta+1$ groups of operators, each with vanishing total momentum; then

$$\langle S_{k_1}\cdots S_{k_p}\rangle=O(N^{1+\Delta-p/2}). \tag{A5}$$

The additional factors N^Δ are not dangerous if only one or two out of N^Δ terms under a summation carry this extra factor. However, if the derivatives in Eq. (A1) can be grouped into $\Delta+1$ groups of operators $\partial^nH'/\partial S^n$, each with vanishing total momentum, then all terms under the summation have the extra factor N^Δ and we obtain (A1). These terms must be considered separately.

Next we consider the summations. The cumulants of Eq. (2.11) are evaluated for products of S_k, k in the shell. Since there are $\frac{1}{2}(n+n'+\cdots)-\Delta$ independent summations [the expectation values with respect to $H^{(2)}$ factorize exactly to products of two-spin correlations $\langle S_k S_{-k}\rangle = 1/v_2(k)$ therefore, we have $\frac{1}{2}(n+n'+\cdots)$ remaining summations restricted by the Δ conditions that the total momentum of each group must vanish] and each summation gives a contribution of the order $N\delta$, we find

$$\sum \langle S_k \cdots, S_{k'} \cdots, \cdots \rangle_c \simeq (N\delta)^{(n+n'+\cdots)/2-\Delta}. \tag{A6}$$

Combining (A6) and (A1) we see that a typical contribution to Eq. (2.11) is of order

$$N\delta^{(n+n'+\cdots)/2-\Delta}.$$

Since we consider infinitesimal δ, we keep only those terms of order $N\delta$; that is, those with

$$2(1+\Delta) = n+n'+\cdots, \tag{A7}$$

the mean number of derivatives per group $(n+n'+\cdots)/(1+\Delta)$ is two.

The derivative $\partial H'/\partial S_k$ cannot form a group of vanishing momentum, as q has to be in the shell; hence, the number of derivatives per group cannot be less than two and, therefore, has to be two. Therefore, as stated in the text, only groups of the form $(\partial H'/\partial S_k)(\partial H'/\partial S_k)$ and $\partial^2 H/\partial S_k \partial S_{-k}$ have to be considered.

APPENDIX B

In this appendix we solve Eqs. (4.9) and (4.45). We differentiate Eq. (4.9) and obtain

$$\frac{f^{*\prime\prime}}{2(c+f^{*\prime})} + \frac{2-d}{d} z f^{*\prime\prime} + \frac{2}{d} f^{*\prime} = 0, \tag{B1}$$

which leads to

$$\left(\frac{d}{2(c+f^{*\prime})} + (2-d)z\right)\frac{\partial f^{*\prime}}{\partial z} + 2f^{*\prime} = 0. \tag{B2}$$

Comparing this equation with Eq. (4.26), we see that $f^{*\prime}$ fulfills the differential equation for q. Therefore $f^{*\prime}$ is proportional to q and comparison shows

$$f^{*\prime} = [2c(4-d)/d]q. \tag{B3}$$

Now let us consider z as a function of $f^{*\prime}$, then we obtain from Eq. (B2) the linear differential equation

$$(2-d)z + 2f^{*\prime}\frac{\partial z}{\partial f^{*\prime}} = -\frac{d}{2(c+f^{*\prime})}. \tag{B4}$$

From this equation we obtain z as a function of $f^{*\prime}$. For simplicity's sake let us introduce the function

$$L_m(x) = \sum_{r=0} \frac{1}{m+2r} x^r, \quad m \neq 0, -2, -4, \ldots. \tag{B5}$$

This function satisfies (a) the differential equation

$$mL_m(x) + 2x\frac{\partial L_m}{\partial x} = \frac{1}{1-x}, \tag{B6}$$

(b) the recursion relation

$$xL_{m+2}(x) = L_m(x) - 1/m, \tag{B7}$$

(c) the integral representation for $m > 0$:

$$L_m(x) = x^{-m/2}\int_0^x \frac{x^{m/2-1}}{2(1-x)}dx. \tag{B8}$$

We note that

$$L_1(x) = (-x)^{-1/2}\arctan(\sqrt{-x}), \quad x \leq 0$$
$$= x^{-1/2}\operatorname{arctanh}(\sqrt{x}), \quad x \geq 0. \tag{B9}$$

Therefore Eq. (B4) has the solution

$$z = -(d/2c)L_{2-d}(-f^{*\prime}/c) + (af^{*\prime})^{(d-2)/2}, \tag{B10}$$

in which a is an arbitrary constant. The function f^* can be obtained from inserting Eq. (B10) into Eq. (4.9). Then both z and f^* are represented as functions of the parameter $f^{*\prime}$. One easily checks that for $a=0$ we obtain the solution (4.23), and for $a=\infty$ we obtain the trivial solution (4.22). We note that for even d the function L_{2-d} is not defined. In these cases the solution of Eq. (B4) contains a nonanalytic term proportional to $(f^{*\prime})^{(d-2)/2}\ln f^{*\prime}$. Because of this term we do not obtain analytic solutions of type (4.23) for $d=6, 8, \ldots$. {For $d=4$ the solution (4.23) reduces to the solution (4.22) because of the factor $(4-d)$ in Eq. (B3). For $d=2$ we find $z = -(2c)^{-1}\ln[f^{*\prime}/(c+f^{*\prime})] + a$.}

Next we consider the solutions, Eq. (B10) with $a \neq 0$ and $a \neq \infty$. For $d<4$ we may iterate

$$f^{*\prime} = \frac{[z - z_0 + (d/2c)L_{2-d}(-f^{*\prime}/c) - (d/2c)L_{2-d}(0)]^{2/(d-2)}}{a}, \tag{B11}$$

which leads to

$$f^* = -\tfrac{1}{2}\ln c + \frac{d-2}{da}(\delta z)^{d/(d-2)} - \frac{d}{4c^2(4-d)a^2}(\delta z)^{4/(d-2)} + O(a^{-3}). \qquad \text{(B12)}$$

These solutions are analytic around $\delta z = 0$, provided $2/(d-2)$ is an integer.

In particular, for $d = 3$ we have

$$z = \frac{3}{2c} + \frac{3}{2c}\left(\frac{f^{*\prime}}{c}\right)^{1/2}\arctan\left[\left(\frac{f^{*\prime}}{c}\right)^{1/2}\right] \pm (af^{*\prime})^{1/2} \quad \text{for } f^{*\prime} \geq 0, \qquad \text{(B13a)}$$

$$z = \frac{3}{2c} - \frac{3}{2c}\left(-\frac{f^{*\prime}}{c}\right)^{1/2}\operatorname{arctanh}\left[\left(-\frac{f^{*\prime}}{c}\right)^{1/2}\right] \pm (af^{*\prime})^{1/2} \quad \text{for } f^{*\prime} \leq 0. \qquad \text{(B13b)}$$

We note that for $a \neq 0$ the analytic functions $f^{*\prime}(z)$ around z_0 are obtained either from (B13a) or (B13b) by using both signs of the square root in the last term. One easily finds that the solution (B13b) has a maximum for some finite z. Therefore it does not cover the whole positive z axis. Solution (B13a), however, behaves asymptotically like $(\frac{3}{4}\pi c^{-3/2} \pm \sqrt{a})/f^{*\prime}$. Therefore choosing $a > (\frac{3}{4}\pi)^2 c^{-3}$, that is, $0 \leq a^{-1} < 16c^3/(9\pi^2)$, we obtain a solution analytic in the whole region[14] $0 \leq z < \infty$:

$$f^* = -\tfrac{1}{2}\ln c + \tfrac{1}{3}a^{-1}(\delta z)^3 - \tfrac{3}{4}c^{-2}a^{-2}(\delta z)^4 + O(a^{-3}). \qquad \text{(B14)}$$

We note that this solution has the same critical exponents as the trivial solution Eq. (4.22).

For $d > 4$ one finds from Eq. (B10)

$$f^* = f^*_{\text{reg}}(z) - (2/d)a^{(d-2)/2}[f^{*\prime}_{\text{reg}}(z)]^{(d-2)/2} + O(a^{d-2}), \qquad \text{(B15)}$$

in which $f^*_{\text{reg}}(z)$ is the solution with $a = 0$. Since $f^*_{\text{reg}}(z)$ is nonanalytic for $d = 4, 6, 8, \ldots$ and the exponent $\frac{1}{2}(d-2)$ is not an integer for $d \neq 4, 6, 8, \ldots$, there is no analytic solution (B15) for $a = 0$.

From Eqs. (4.45) and (B6) we find immediately the solution

$$h_p = \tfrac{1}{2}d\, L_{2-d-2p}(-f^{*\prime}/c) = \tfrac{1}{2}d\, L_{2-d-2p}\{-[2(4-d)/d]q\}. \qquad \text{(B16)}$$

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*Work supported in part by the National Science Foundation under Grant No. GP9445 and by the Brown University Materials Research Program supported by the National Science Foundation under Grant No. GH-33631.

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