Energy Loss of Charged Particles in a Plasma

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Formulas for the stopping power of a nonrelativistic ionized gas for electrons, protons, and other structureless charged particles are derived. The primary velocities considered extend from values comparable to those of the plasma electrons up to values in the relativistic range. New results are obtained for the lower primary velocities, in that full account is taken of the effect of the initial velocity distribution of the plasma electrons on binary collisions as well as on collective excitations. Formulas for the case of non-negligible damping of plasma oscillations are given and comparisons are made with the impact-parameter method and with particle kinetic theories.

I. INTRODUCTION

The energy loss of charged particles is of interest in several laboratory[1] and stellar[2] plasmas. Among the various possible beam-plasma interactions only the linear ones are encountered when the primary beam has a sufficiently low intensity. We shall confine ourselves to one aspect of this linear interaction, the stopping power of the plasma.

The theory of stopping by inelastic collisions with atoms has been established already for four decades.[3,4] Its generalization to condensed materials was reviewed by Fano,[5] chiefly experimental reviews[6] are due to Raether and Powell, and a more recent extensive tabulation of the stopping power of various elements and chemical compounds is due to Pages *et al.*[7] Many theoretical papers[8-24] have also appeared on the energy loss in ionized gas and on the energy loss due to the "free" electron plasma in a metal. The latter two subjects are closely related to one another in many respects. A subdivision can be made between two different approaches of the problem, the high-energy theories[8-13] in which the test particle is taken much faster than the target electrons, and the particle kinetic theories[18-24] in which the test particle may have about the same energy and/or velocity as the target electrons. The high-energy case is almost fully understood and the final formulas obtained with various techniques[8-13] either agree with each other or differ from one another only in minor aspects. In the particle kinetic theories one or more approximations are usually made. Pines and Bohm[14,15] have studied the collective excitations in dense plasmas for sufficiently large primary velocities (see Sec. V). We shall study this intermediate region using the microscopic high-energy theory, described in detail by Fano[5] for atoms and condensed material, as a starting point for the analysis. The application of this theory to ionized gases is briefly summarized in Sec. II. In Secs. III and IV, stopping-power formulas are derived for smaller primary velocities. The effect of thermal motion of the plasma electrons on binary as well as on the collective excitations is fully accounted for, which is not done in the high-energy theories. The generalization to lower primary velocities is of interest, for example, to describe the interaction between plasma electrons, in particular between those of the tail of the velocity distribution and all other electrons. In Sec. V, some further comparisons are made between the various methods used so far and the results obtained for the stopping power.

II. HIGH-VELOCITY FORMULAS

In this section we first deal with incident particles of charge ze and mass M, where $M \geqslant m$, m being the electron rest mass. Next the formulas for incident electrons are given, which differ from those for unlike primary particles with $M = m$. It is assumed that free electrons, atoms (molecules) and ions contribute separately to the stopping. The conditions for which this assumption is justified are discussed at the end of this section. Unless otherwise specified, cgs units are used.

A. Free Electrons

Energy losses to "free" electrons can be characterized by the spectrum of discrete and continuum energy transfers E_n and momentum transfers q. Alternatively, one can make use of the variable Q instead of q, where

$$Q(1+Q/2mc^2)=q^2/2m\,, \tag{1}$$

and c is the velocity of light in vacuum. As in Ref. 5, it is convenient to make a subdivision between (i) high-Q excitations, (ii) intermediate-Q excitations, and (iii) low-Q excitations. In each of the three Q ranges, different well-established theoretical approximations can be used to obtain the contributions to the stopping power.

In the *high*-Q range, $Q \geq Q_2$ where Q_2 is taken much larger than the mutual interaction energies between the electrons in the nonrelativistic plasma; more precisely, $Q_2 \gg I_e$, where I_e is the average excitation energy of the plasma electrons, as defined by Eq. (6). Consequently, the contribution of high-Q excitations to the stopping power, the energy loss per unit path length, is determined by binary collisions[25] of the incident particle with individual electrons and is given by[5]

$$-\frac{dE}{ds}(Q \geq Q_2) = \frac{2\pi z^2 e^4}{mv^2} n_e \left[\ln \frac{2m^* v^2}{Q_2(1-\beta^2)} - \beta^2\right], \quad (2)$$

where $m^* = mM/(m+M)$, the reduced mass of the two interacting particles, may be replaced by m for $M \gg m$, the case considered in Ref. 5, Eq. (37). In Eq. (2), v is the velocity of the primary particle, n_e is the free-electron density, and $\beta = v/c$. Equation (2) accounts for the longitudinal as well as the transverse excitations. Spin effects, which are important only when the energy of the primary particle becomes comparable to $(M/m)Mc^2$, are not included.

In the *intermediate*-Q range, $Q_1 \leq Q \leq Q_2$. In addition to $Q_2 \gg I_e$, the upper limit is chosen such that simultaneously $Q_2 \ll 2mc^2$ and $Q_2 \ll \frac{1}{2}Mv^2$. The latter limitation justifies the use of the first Born approximation. The lower limit is chosen such that (i) $Q_1 \ll I_e$ and (ii) $Q_1 \gg I_e^2/2mv^2$. Conditions (i) and (ii) can be fulfilled simultaneously, because $2mv^2 \gg I_e$. Condition (ii) implies first that all significant excitations can take place for $Q \geq Q_1$, as follows from energy and momentum conservation, and second that the interactions of primary particle with the plasma electrons can be described by an unshielded Coulomb potential. It is only for $Q \leq I_e^2/2mv^2$, that the shielding becomes effective. For the latter two reasons, and because the first Born approximation is valid, one can directly use Eq. (36) of Ref. 5 for an electron gas; hence

$$-\frac{dE}{ds}(Q_1 \leq Q \leq Q_2) = \frac{2\pi z^2 e^4}{mv^2} n_e \ln \frac{Q_2}{Q_1}. \quad (3)$$

In the *low*-Q range, $Q \leq Q_1$ and the electron gas has to be regarded as a continuous medium. In this case one has to use Eqs. (45) and (47) of Ref. 5, which represent the longitudinal and transverse low-Q excitations, respectively. When the dielectric function of the electron gas is given by[12,17,26]

$$\epsilon_e(\omega) = 1 - \omega_p^2/(\omega^2 + i\Gamma\omega), \quad (4)$$

where Γ is the damping constant, Eq. (45) of Ref. 5 can be solved exactly. The general formulas are

$$-\frac{dE}{ds}(Q \leq Q_1) = \frac{2\pi z^2 e^4}{mv^2} n_e \ln \frac{Q_1 2m^* v^2}{I_e^2} \quad (5)$$

and

$$\ln I_e = \frac{2}{\pi\omega_p^2} \int_0^\infty \omega \,\mathrm{Im}\left[-\frac{1}{\epsilon_e(\omega)}\right] \ln \hbar\omega \, d\omega, \quad (6)$$

where $\omega_p = (4\pi n_e e^2/m)^{1/2}$ is the plasma frequency. When Eq. (4) is substituted in (6) we find

$$\ln I_e = (\Gamma/2\omega_p)[\ln\hbar\omega_p/\sin\theta + (\tfrac{1}{2}\pi - \theta)/\cos\theta], \quad (7)$$

where

$$\theta = \tfrac{1}{2}\arccos(1 - \Gamma^2/2\omega_p^2). \quad (8)$$

In the limit of small damping, $\Gamma \to 0$, $\sin\theta \to \Gamma/2\omega_p$, and

$$\ln I_e = \ln \hbar\omega_p. \quad (9)$$

When Eq. (4) is used again, the contribution of the transverse low-Q excitations to $-dE/ds$ is equal to zero in both limiting cases $\Gamma \to 0$ and $\beta \to 0$.

The total stopping power is then given by the sum of Eqs. (2), (3), and (5):

$$-\frac{dE}{ds} = \frac{4\pi z^2 e^4}{mv^2} n_e \left[\ln \frac{2m^* v^2}{I_e} - \tfrac{1}{2}\ln(1-\beta^2) - \tfrac{1}{2}\beta^2\right]. \quad (10)$$

The theory of stopping of *electrons* differs only in minor aspects from the theory of stopping of unlike particles. The two interacting electrons are indistinguishable from each other, which neccessitates the inclusion of exchange and interference terms in the cross-section formulas. Also, the upper integration limit of the high-Q range changes. These aspects are treated in more detail in Sec. III. By analogy with the known theories[4,27] for atoms, we can directly give $-dE/ds$ for an electron gas. The result is

$$-\frac{dE}{ds} = \frac{2\pi e^4}{mv^2} n_e \left\{\ln \frac{mv^2 T}{I_e^2} + 1 - [2(1-\beta^2)^{1/2} + \beta^2]\ln 2 + \tfrac{1}{8}[1 - (1-\beta^2)^{1/2}]^2\right\}, \quad (11)$$

$T = mc^2[(1-\beta^2)^{-1/2} - 1]$ being the kinetic energy of the primary electron.

B. Atoms and Ions

The contribution of atoms and ions in the plasma to the (electronic) stopping is given by[5]

$$-\frac{dE}{ds} = \frac{4\pi z^2 e^4}{mv^2} \sum_j n_j \left[\ln \frac{2m^* v^2}{I_j(1-\beta^2)} - \beta^2\right] \quad (12)$$

for unlike particles with charge ze, and by[27]

$$-\frac{dE}{ds}=\frac{2\pi e^4}{mv^2}\sum_j n_j\left\{\ln\frac{mv^2T}{I_j^2(1-\beta^2)}+1-\beta^2-[2(1-\beta^2)^{1/2}+\beta^2]\ln 2+\tfrac{1}{8}[1-(1-\beta^2)^{1/2}]^2\right\} \quad (13)$$

for incident electrons, where n_j is the product of the number density of atoms or ions of type j times the number of electrons in this atom or ion. The summation extends over all different, including excited, atoms or ions, I_j is defined by

$$\ln I_j=\sum_n f_{nj}\ln E_n\,, \quad (14)$$

where f_{nj} gives the optical oscillator strength distribution and the summation sign stands for a summation over all discrete and an integration over the continuum energy transfers. Equation (9) is obtained also by substituting $f_{nj}=\delta(E_n-\hbar\omega_p)$ in Eq. (14), as corresponds to free electrons with one possible (plasma) oscillation frequency ω_p and negligible damping.

The total stopping power is obtained by adding Eqs. (10) and (12), or (11) and (13).

C. Separation of Electrons, Atoms, and Ions

So far we have dealt separately with the electrons, atoms, and ions. From the derivation given here and in Ref. 5 it follows that the general formulas for the high- and intermediate-Q contributions to $-dE/ds$ are obtained by simply replacing n_e in Eqs. (2) and (3) by the total electron density $n=n_e+\sum_j n_j$. In the low-Q range, n_e and I_e in Eq. (5) have to be replaced by n and I, where

$$\ln I=\frac{2}{\pi\omega_0^2}\int_0^\infty\omega\,\mathrm{Im}\left[-\frac{1}{\epsilon(\omega)}\right]\ln\hbar\omega\,d\omega\,, \quad (15)$$

$\omega_0=(4\pi ne^2/m)^{1/2}$ and $\epsilon(\omega)$ is the total dielectric function of the plasma. Similarly, the $\epsilon(\omega)$ in Eq. (47) of Ref. 5 then represents the total dielectric function. When the density and temperature of the plasma are not so large as to appreciably affect the spectrum of energy levels of the atoms and ions, $\hbar\omega_p\ll 1$ eV, and $\epsilon(\omega)$ may be separated into contributions from the different constituents of the plasma,

$$\epsilon(\omega)=\epsilon_e(\omega)+\sum_j\epsilon_j(\omega)\,. \quad (16)$$

Here $\epsilon_e(\omega)$ is given by (4) and

$$\epsilon_j(\omega)=\frac{4\pi e^2}{m}n_j\sum_n\frac{f_{nj}}{\omega_n^2-\omega^2-i\gamma\omega}\,, \quad (17)$$

where $\hbar\omega_n=E_n$ and γ is a very small damping constant. Using $\mathrm{Im}[-1/x]=|x|^{-2}\,\mathrm{Im}\,x$, one finds

$$\mathrm{Im}[-1/\epsilon(\omega)]=\mathrm{Im}[\epsilon_e(\omega)]/|\epsilon(\omega)|^2+\sum_j\mathrm{Im}[\epsilon_j(\omega)]/|\epsilon(\omega)|^2\,. \quad (18)$$

For all tenuous plasmas and also for most high-pressure plasmas, e.g., discharges at atmospheric pressure, $\hbar\omega_p\ll 1$ eV. For ground-state atoms or ions and also for many lower excited atoms and ions, all excitation energies $E_n=\hbar\omega_n$ lie above 1 eV and $\omega_n\gg\omega_p$. From the structure of Eqs. (4) and (16)–(18) it then follows that the first term on the right-hand side of Eq. (18) only contributes to $\ln I$ for $\hbar\omega\ll 1$ eV, and that the second term only contributes to $\ln I$ for $\hbar\omega>1$ eV. For $\hbar\omega<1$ eV, the denominator $|\epsilon(\omega)|^2$ can be replaced by $|\epsilon_e(\omega)|^2$, because $\epsilon_j(\omega)\ll 1$ for these small values of ω. For $\hbar\omega>1$ eV, $\epsilon_e(\omega)$ is extremely close to unity and, because γ is very small, one may replace $|\epsilon(\omega)|^2$ in the second term on the right-hand side of Eq. (18) by $|1+\epsilon_j(\omega)|^2$.

For these reasons, all constituents of the plasma contribute separately to the stopping power for $\hbar\omega_p\ll 1$ eV, and

$$n\ln I=n_e\ln I_e+\sum_j n_j\ln I_j\,. \quad (19)$$

III. EXTRAPOLATION TO LOWER VELOCITIES, $v<\sqrt{2}ze^2/\hbar$

For nonrelativistic velocities of the primary particles, the limit $\beta^2\to 0$ may be taken, which results in a simplification of the preceding formulas. For still smaller primary velocities additional complications arise. For a "free" electron gas these complications can be well defined, and adequate solutions can be given, which is the purpose of this and the following section.

The standard procedure used so far for the high-Q range is to neglect the motion of the plasma electrons. Hence, $\langle v_e^2\rangle$ is taken equal to zero, v_e being the thermal plasma electron velocity, and Q is taken equal to E_n. For unlike interacting particles, $2mQ_{\max}$ is thus taken equal to $(2m^*v)^2$. When v^2 is not much larger than $\langle v_e^2\rangle$ the effects of recoil and initial motion of the plasma electrons must be accounted for. In the high-Q, the binary-collision, range we first derive the separate contributions to $-dE/ds$ of the plasma electrons with one velocity v_e, and next integrate over the speed distribution of these electrons. The plasma electrons are assumed to have an isotropic velocity distribution.

A. Unlike Interacting Particles

For unlike primary particles scattered by electrons with velocity v_e and an isotropic angular distribution, the double differential cross section per unit E and per unit Q range is given by Eq. (6-3-29) of Ref. 25. In the present notation,

$$d^2\sigma = (4\pi z^2 e^4/v^2 v_e q^4)dq dE\,, \tag{20}$$

where E_n is replaced by E and q is used instead of $Q = q^2/2m$. From energy and momentum conservation it follows that

$$q^2/2m - v_e q \leq E \leq q^2/2m + v_e q \tag{21}$$

and

$$-q^2/2M - vq \leq E \leq vq - q^2/2M\,. \tag{22}$$

These limits are more general than those of Ref. 25, in that the possibility of negative E values, superelastic collisions, is included. If we multiply Eq. (20) by E and integrate with respect to E between the limits given by (21) and (22), we find

$$d(-dE/ds) = (4\pi z^2 e^4/mv^2 q)dq \tag{23}$$

for $v \geq v_e$ and $q \leq 2m^*(v - v_e)$,

$$d(-dE/ds) = (2\pi z^2 e^4/v^2 v_e)[(v^2 - v_e^2)/q^2 + (v_e/m - v/M)/q + (1/4M^2 - 1/4m^2)]dq \tag{24}$$

for $2m^*|v - v_e| \leq q \leq 2m^*(v + v_e)$, and

$$d(-dE/ds) = -(4\pi z^2 e^4/M v v_e q)dq \tag{25}$$

for $v \leq v_e$ and $q \leq 2m^*(v_e - v)$. When q in Eq. (23) is replaced by Q, and (23) is integrated with respect to Q, one finds again that the intermediate- and high-Q ranges join smoothly provided $q_1 = (2mQ_1)^{1/2} \leq 2m^*(v - v_e)$. Before, we have required that Q_1 be much smaller than I_e. For an ionized gas I_e itself is orders of magnitude smaller than the mean thermal energy of the plasma electrons. Therefore, Eqs. (23) and (24) can still be used, in addition to the low-Q formula, to compute $-dE/ds$ when v is only slightly larger than v_e.

For $v \geq v_e + q_1/2m^*$, Eqs. (23) and (24) yield

$$-\frac{dE}{ds}(Q \geq Q_1) = \frac{2\pi z^2 e^4}{mv^2}\left[\ln\frac{(2m^*)^2(v^2 - v_e^2)}{2mQ_1} - \frac{mv}{Mv_e}\ln\frac{v + v_e}{v - v_e} + \frac{2m}{M}\right]. \tag{26}$$

In Eq. (26) we still have to average over the velocity distribution of the plasma electrons. Because Eq. (26) applies for $v_e \leq v - q_1/2m^*$, accurate results are obtained only when the majority of the plasma electrons have velocities $v_e < v$. In this case it is convenient to expand Eq. (26) first in powers of v_e/v, and next to average over the speed distribution. The result is

$$-\frac{dE}{ds}(Q \geq Q_1) = \frac{2\pi z^2 e^4}{mv^2} n_e\left[\ln\frac{(2m^* v)^2}{2mQ_1} - \left(1 + \frac{2m}{M}\right)\frac{\langle v_e^2\rangle}{v^2} + O\left(\frac{\langle v_e^4\rangle}{v^4}\right)\right]. \tag{27}$$

In the *low*-Q range some modifications are also required. Fano's formula for this range applies only for $v_e/v \to 0$. To find the correction terms for the low-Q range, we make use of some relations given by Pines and Bohm.[14] The first one, Eq. (23) of Ref. 14, is the dispersion relation for collective harmonic undamped oscillations of the plasma electrons,

$$1 = (4\pi e^2/m)\sum_j(\omega - \vec{q}\cdot\vec{v}_j/\hbar)^{-2}\,, \tag{28}$$

where $\vec{v}_j$ is the velocity of the jth plasma electron and ω and $\vec{q}/\hbar$ are the angular frequency and wave vector of the plasma oscillation, respectively. For small q, the denominator in (28) may be expanded in a series of powers of $\vec{q}\cdot\vec{v}_j/\hbar\omega$. The result is

$$(\hbar\omega)^2 = (\hbar\omega_p)^2 + q^2\langle v_e^2\rangle + O(q^4\langle v_e^4\rangle/(\hbar\omega_p)^2)\,. \tag{29}$$

From energy and momentum conservation of the primary particle it follows that the minimum momentum transfer is given by

$$q_{\min} = (\hbar\omega/v)[1 + \hbar\omega/2Mv^2 + O((\hbar\omega/Mv^2)^2)]\,. \tag{30}$$

For $\hbar\omega \ll 2Mv^2$, Eqs. (29) and (30) yield

$$q_{\min}^2 = (\hbar\omega_p)^2/(v^2 - \langle v_e^2\rangle) + O((\hbar\omega_p)^2\langle v_e^4\rangle/(v^2 - \langle v_e^2\rangle)^3)\,, \tag{31}$$

which is equivalent to Eq. (51) of Ref. 14. In the integration over the low-Q range, Fano has used the $q_{\min}$ given by the first term of Eq. (30). When damping of the plasma oscillations is neglected, this corresponds to $q_{\min}^2 = (\hbar\omega_p)^2/v^2$, whose relation must be compared with the first term of Eq. (31). For $v^2 \gg \langle v_e^2\rangle$ the two limits are identical with each other. In the extrapolation to lower primary velocities, we suggest that the combination of Eqs. (30) and (31),

$$q_{\min}^2 = (\hbar\omega)^2/(v^2 - \langle v_e^2\rangle) \tag{32}$$

should preferably be used to compute dE/ds for $Q \leq Q_1$. This new lower integration limit leads to the correct results both for $v^2 \gg \langle v_e^2\rangle$ and non-negligible damping of collective oscillations, as for much smaller values of v and negligible damping, $\Gamma \to 0$. We have to impose two further restrictions,

$$q_{\min}^2 < 2mQ_1 \tag{33}$$

for all relevant excitation energies, and

$$q_{\min}^2 \ll 3(\hbar\omega_p)^2/\langle v_e^2\rangle\,. \tag{34}$$

The latter restriction, Eq. (13) of Ref. 14, is equivalent to $(q_{\min}/\hbar)^2 \ll \lambda_D^{-2}$, for an electron gas with an isotropic Maxwellian velocity distribution. Here, λ_D is the Debye length defined by

$$\lambda_D^2 = kT_e/4\pi n_e e^2 = \langle v_e^2\rangle/3\omega_p^2 \tag{35}$$

and k is Boltzmann's constant. In Sec. II we have taken Q_1 such that

$$I_e^2/v^2 \ll 2mQ_1 = q_1^2 \ll 2mI_e . \quad (36)$$

For $I_e \approx \hbar\omega_p \ll m\langle v_e^2\rangle$, as is the case for ionized gases, Eqs. (31)–(36) can simultaneously be complied with when $v^2 \gg 4\langle v_e^2\rangle/3$.

For $v^2 \gg 4\langle v_e^2\rangle/3$ we thus find, using Fano's procedure for the low-Q range with the more accurate lower integration limit (32),

$$-\frac{dE}{ds}(Q \leq Q_1) = \frac{2\pi z^2 e^4}{mv^2} n_e \left[\ln \frac{2mQ_1(v^2 - \langle v_e^2\rangle)}{I_e^2} + O(\langle v_e^4\rangle / v^4)\right]. \quad (37)$$

The total stopping power for unlike incident particles is given by the sum of Eqs. (27) and (37),

$$-\frac{dE}{ds} = \frac{4\pi z^2 e^4}{mv^2} n_e \left[\ln \frac{2m^*(v^2 - \langle v_e^2\rangle)}{I_e} - \frac{m\langle v_e^2\rangle}{Mv^2} + O\left(\frac{\langle v_e^4\rangle}{v^4}\right)\right]. \quad (38)$$

Equation (38) should replace Eq. (10) for small nonrelativistic values of $v > \sqrt{2}\, ze^2/\hbar$. For incident protons and other heavy particles, $M \gg m$, and the small terms containing m/M are comparable in magnitude with those of the nuclear stopping, which have not been included in the analysis.

B. Primary Electrons

The preceding high-Q formulas, with $M = m$ and $z^2 = 1$, do not apply for primary electrons. The two interacting electrons in a binary collision are indistinguishable from each other and Eq. (6-3-30) of Ref. 25 is to be used. In the present notation,

$$d^2\sigma = \frac{4\pi e^4}{v^2 v_e}\left[\frac{1}{q^4} + \frac{v^2 + v_e^2 - q^2/2m^2 - 2E^2/q^2}{m^4 |v^2 - v_e^2 - 2E/m|^3} - \frac{\Phi}{m^2 q^2 |v^2 - v_e^2 - 2E/m|}\right] dq\,dE , \quad (39)$$

where Φ is a function[25] of v, v_e, q, and E, whose value lies between zero for v and $v_e \ll e^2/\hbar$, and unity for v and/or $v_e \gg e^2/\hbar$. In Eq. (39) we have assumed that the plasma electrons have an isotropic spin orientation. Equations (21) and (22) with M replaced by m, also apply for primary electrons. The procedure described for unlike particles can be used to obtain $-dE/ds$ for $Q \geq Q_1$ from Eq. (39). We use this procedure, however, only for the first (the direct) term on the right-hand side of Eq. (39). The contributions from the second (exchange) and third (interference) terms are obtained more conveniently by reversing the order of the E and q integrations.

For primary electrons with kinetic energy $\frac{1}{2}mv^2$ and target electrons with zero kinetic energy, it is not possible after the collision to tell which of the two is the primary one. The convention is to put the label primary on the faster of the two. Consequently, $E_{\max} = \frac{1}{4}mv^2$ and, because $Q = E$, $Q_{\max} = \frac{1}{4}mv^2$. If we adopt the same convention when the target electrons have a kinetic energy $\frac{1}{2}mv_e^2$, the one with kinetic energy exceeding $\frac{1}{4}m(v^2 + v_e^2)$ should, for $v > v_e$, be labeled the primary one, and

$$E \leq \tfrac{1}{4}m(v^2 - v_e^2) . \quad (40)$$

Equation (40) leads to some arbitrariness when v approaches v_e. In particular, for $v = v_e$ only negative energy transfers would be allowed. However, in this limit each of the two electrons can be the primary one. To remove this arbitrariness, Eq. (40) can be replaced by $|E| \leq \frac{1}{4}m(v^2 - v_e^2)$. The formulas resulting for $v^2 < 3v_e^2$ are lengthy and unattractive for practical applications, also because one further integration over the velocity distribution of the plasma electrons has to be made. Therefore, we confine ourselves to values of $v^2 \geq 3v_e^2$, where the combination of Eqs. (21), (22), and (40) also ensures $|E| \leq \frac{1}{4}m(v^2 - v_e^2)$.

If we multiply the direct term of Eq. (39) by E and integrate with respect to E between the limits given by (21), (22), and (40), we obtain

$$d(-dE/ds) = (4\pi e^4 / mv^2 q)\, dq \quad (41)$$

for

$$q \leq m[(v^2 + v_e^2)/2]^{1/2} - mv_e ,$$

and

$$d(-dE/ds) = (2\pi e^4 / v^2 v_e)[m^2(v^2 - v_e^2)/16q^4 - (1/2m - v_e/q)^2]\, dq \quad (42)$$

for

$$m[(v^2 + v_e^2)/2]^{1/2} - mv_e \leq q \leq m[(v^2 + v_e^2)/2]^{1/2} + mv_e .$$

A subsequent integration with respect to q leads to

$$-\frac{dE}{ds}(Q \geq Q_1) = \frac{2\pi e^4}{mv^2}\left[\ln \frac{m(v^2 - v_e^2)}{4Q_1} - \frac{8v_e^2}{3(v^2 - v_e^2)}\right]. \quad (43)$$

The contribution of the exchange and interference terms in Eq. (39) to $-dE/ds$ is obtained by first integrating with respect to q and next multiplying by E and integrating with respect to E from $-\frac{1}{2}mv_e^2$ up to $\frac{1}{4}m(v^2 - v_e^2)$. The result is

$$-\frac{dE}{ds} = \frac{2\pi e^4}{mv^2}\left[1 + \frac{2v_e^2}{v^2 - v_e^2} + 2\frac{v_e^2}{v^2} + 2\frac{v_e^4}{v^4} - (1 + \Phi') \ln \frac{2v^2}{v^2 - v_e^2}\right]. \quad (44)$$

The sum of Eqs. (43) and (44) can be expanded in powers of $(v_e/v)^2$, averaged over the electron velocity distribution and added to the low-Q contribution to $-dE/ds$, which for primary electrons is also given by Eq. (37). We find

$$-\frac{dE}{ds}=\frac{2\pi e^4}{mv^2}n_e\left[\ln\frac{m^2(v^2-\langle v_e^2\rangle)^2}{2I_e^2}+1-(1+\Phi')\ln 2+(\tfrac{1}{3}-\Phi')\frac{\langle v_e^2\rangle}{v^2}+O\left(\frac{\langle v_e^4\rangle}{v^4}\right)\right], \quad (45)$$

where Φ' is unity for $v \gg e^2/\hbar$. Equation (45) should be used in place of Eq. (11) for small nonrelativistic velocities of the primary electrons.

IV. PRIMARY VELOCITIES $v<\sqrt{2}ze^2/\hbar$

In Sec. II we have dealt with primary velocities $v \gg ze^2/\hbar$ and simultaneously $v \gg \langle v_e^2\rangle^{1/2}$. In Sec. III we have made an extrapolation to values of $v > \sqrt{2}\,ze^2/\hbar$. Here we consider still smaller values of v, for which the preceding analysis is no longer completely valid. Firstly, as explained in detail by Williams,[28] the Born approximation is valid only for $v \gg ze^2/\hbar$. In Sec. II, the Born approximation was used for the low- and intermediate-Q ranges. In Sec. III, the exact binary cross-section formulas were used in the intermediate- and high-Q ranges. These formulas are certainly valid for $v > \sqrt{2}\,ze^2/\hbar$ and also for $v < \sqrt{2}\,ze^2/\hbar$. In the low-Q range, the momentum transfers are small compared with the momentum of the primary particle and the relevant energy transfers $\hbar\omega \approx \hbar\omega_p$ are small compared with $\frac{1}{2}mv^2$. For this reason one should not expect to find significant deviations from the Born approximation. However, one further condition for validity of the Born approximation is that the collision be sudden. For stopping of heavy particles by atoms this leads to the classical limit of the Bohr theory,[4,5,28]

$$q_{\min}=\sqrt{2}(ze^2/\hbar v)(\hbar\omega/v)\,, \quad (46)$$

which implies that the reaction time of the atomic electrons $1/\omega$ must be larger than the duration of the collision. Equation (46) applies directly to an electron gas, where collisions which last longer than the rearrangement time of the electron gas, the "shielding time," do not lead to energy loss of the primary particle. Equation (46) is more restrictive than Eq. (30) when $v < \sqrt{2}\,ze^2/\hbar$ and should be used in this region. Therefore, the combination of Eqs. (29) and (30), Eq. (32) is to be replaced by the corresponding combination of Eqs. (29) and (46),

$$q_{\min}^2=(\hbar\omega)^2/(\kappa v^2-\langle v_e^2\rangle)\,, \quad (47)$$

where $\kappa=\frac{1}{2}(\hbar v/ze^2)^2$. Instead of Eq. (38) we then find

$$-\frac{dE}{ds}=\frac{4\pi z^2e^4}{mv^2}n_e\left[\ln\frac{2m^*(\kappa v^2-\langle v_e^2\rangle)^{1/2}(v^2-\langle v_e^2\rangle)^{1/2}}{I_e}-\frac{m\langle v_e^2\rangle}{Mv^2}+O\left(\frac{\langle v_e^4\rangle}{v^4}\right)\right]. \quad (48)$$

For primary electrons we find in place of Eq. (45)

$$-\frac{dE}{ds}=\frac{2\pi e^4}{mv^2}n_e\left[\ln\frac{m^2(\kappa v^2-\langle v_e^2\rangle)(v^2-\langle v_e^2\rangle)}{2I_e^2}+1-(1+\Phi')\ln 2+(\tfrac{1}{3}-\Phi')\frac{\langle v_e^2\rangle}{v^2}+O\left(\frac{\langle v_e^4\rangle}{v^4}\right)\right], \quad (49)$$

where $\Phi'=0$ for $v \ll e^2/\hbar$.

Equation (46) is not an exact and unquestionable limit, in contrast to Eq. (30) which is a direct consequence of energy and momentum conservation. For this reason, Eqs. (48) and (49) are not of the same accuracy as Eqs. (10), (11), (38), and (45). If Eq. (47) is combined with Eq. (34) one finds $v^2 \gg 4\langle v_e^2\rangle/3\kappa$ which is more restrictive than the corresponding limitation $v^2 \gg 4\langle v_e^2\rangle/3$ of Sec. III.

A similar argumentation as described so far in this section applies to particle kinetic theories of plasmas, where[19,20] quantum theories are used when the thermal deBroglie wavelength $\lambda\!\!\!^{-} \approx \hbar/(2mkT_e)^{1/2}$ is greater than the classical distance of closest approach $b_c \approx e^2/kT_e$, which is the case when $\langle v_e^2\rangle^{1/2} > e^2/\hbar$. Classical theories are used when $\lambda\!\!\!^{-} < b_c$, hence for $\langle v_e^2\rangle^{1/2} < e^2/\hbar$.

V. COMPARISON WITH PREVIOUS WORK

The present relativistic formulas (10) and (11) agree with those obtained by Tsytovich[11] and Gould.[13] The corresponding relations obtained with the impact-parameter method[8] (see also Refs. 9, 12, and 26) are slightly different in that they contain an additional factor 1.123 in the argument of the logarithmic term. For this reason, we discuss the impact-parameter method in some detail.

A distinction is to be made between the classical and the semiclassical impact-parameter method. In both cases it is assumed that the primary particle moves along a well-defined rectilinear orbit. In the first case, corresponding to Sec. IV, the interaction with the target is also described with classical mechanics, as is done for example in Ref. 26. In the second case, corresponding to Secs. II and III, quantum mechanics is used to describe the interaction with the target. The concept of a well-defined rectilinear orbit is valid when the impact parameters very much exceed the deBroglie wavelength of the primary particle. This condition is always fulfilled for heavy primary particles, $M \gg m$, and $v^2 > \langle v_e^2\rangle$. For primary electrons the concept of a well-defined rectilinear orbit is valid only in the low- and intermediate-Q ranges for large values of v. Therefore, one should prefer to use the impact-parameter method only for heavy incident particles. The impact parameters b can be subdivided into a large-b range, where collective excitations are described by a macroscopic theory, a small-b range, where excitations are described as binary collisions with individual plasma electrons, and an intermediate-b range where collective as well as binary collisions can take place. In contrast to the situation encountered when Q or q is used as a variable, we cannot make use of a sum rule[5] for the intermediate b excitations, because none exists. This results in a small ambiguity in piecing together the contribution to $-dE/ds$ of the three b ranges. This ambiguity is responsible for the additional factor 1.123 in the argument of the logarithm.

Particle kinetic theories have been used[18-24] to obtain $-dE/ds$ for charged test particles with smaller primary velocities. In the standard derivations of the Boltzmann and Fokker-Planck equations use is made of cutoffs for large and small impact parameters and the resulting formulas are not directly comparable with our formulas. However, Eqs. (7-140a) and (7-140b) of Ref. 22 can be compared with Eq. (48) when $\langle v_e^2 \rangle$ in the latter formula is taken equal to zero. In that case the formulas agree with one another except for a constant in the logarithmic term, which is equal to 1.781 in Eq. (7-140b) and is equal to $\sqrt{2}$ in Eq. (48). The results of Sec. IV are strictly comparable only with solutions of the classical Lenard-Balescu equation, in which proper account must be taken of the indistinguishability of alike interacting particles. The results of Sec. III should be compared with solutions of the quantum Lenard-Balescu equation. To our knowledge, a solution of the Lenard-Balescu equation which is fully comparable in accuracy with the present results has not yet been given. One of the more advanced attempts is due to Lampe[18] who claims to have solved the quantum Lenard-Balescu equation. The basic cross-section formula, Eq. (10) of Ref. 18, only contains a direct scattering term, while for alike interacting particles it should also contain exchange and interference terms. Equation (7) of Ref. 13 does not agree with our formulas with $\langle v_e^2 \rangle = 0$ and $\Gamma \to 0$. It should be noticed, however, that the Lenard-Balescu equation has a more accurate dielectric function $\epsilon(\vec{q}, \omega)$ than the one used here, Eq. (4), which includes only static damping.

Finally, we should refer to Pines and Bohm[14] and Pines,[15] who have made a detailed study of, among other things, the influence of the speed distribution of the plasma electrons on the collective excitations. For $\hbar\omega_p \ll m\langle v_e^2 \rangle$, Eq. (48) of Ref. 15 is identical with the present low-Q formula (37). Pines and Bohm have not considered the high-Q range in any detail, and have not studied the piecing together of low- and high-Q ranges. Their final formula for $-dE/ds$, Eq. (51) of Ref. 15, still contains a minimum cut-off impact parameter and is not directly comparable with our results. Equation (48) of Ref. 15 is valid only for $v > \sqrt{2}\, ze^2/\hbar$ although this restriction was not made.

VI. CONCLUDING REMARKS

The final formulas of this paper, Eqs. (10), (38), and (48) for the energy loss of unlike primary particles to the plasma electrons, and Eqs. (11), (45), and (49) for primary electrons, have a wider range of validity than the formulas available so far from literature, in that in the low- and high-Q ranges proper account is taken of the velocity distribution of the plasma electrons. From practical point of view, our results do not differ very much from those available already, because the arguments of the logarithmic terms are very large for $I_e \approx \hbar\omega_p \ll 1$ eV. Therefore, errors in this term or approximations like $\Gamma = 0$ and $\langle v_e^2 \rangle = 0$ have a small effect on $-dE/ds$, which effect is typically of the order of a few percent for $n_e \approx 10^{12}$ cm^{-3}, hence $\hbar\omega_p \approx 3.7 \times 10^{-5}$ eV. The corrections are of greater importance when n_e, and therefore also I_e, becomes larger. It is therefore of interest that the present high-Q formulas are valid also for dense plasmas as long as v is large enough to make a distinction between high-, intermediate-, and low-Q ranges possible. Furthermore, the velocity distribution of the target electrons must be isotropic.

The present formulas do not apply for thermal primary velocities. In that case it is not possible to make a clear distinction between low-, intermediate-, and high-Q ranges, the expansions in terms of v_e/v are not useful anymore, and the approximate dispersion relation (29) is not valid[29] for too large values of q.

The accuracy of the method is intermediate between that of the cut-off Fokker-Planck equation and that of the Lenard-Balescu equation, although complete solutions of this latter equation are not yet available. The accuracy of the present formulas is limited by (i) the use of an approximate dielectric function, (ii) the expansion in terms of v_e/v, and for primary electrons by (iii) the approximations used for the function Φ encountered in the interference term.

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Ordering Transitions in a Quantum Lattice-Gas Model of Adsorbed Systems*

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A quantum lattice-gas model of adsorbed systems is studied to determine the importance of quantum-mechanical processes upon order-disorder transition temperatures. The phase diagram of systems interacting with nearest-neighbor attractive and repulsive potentials is obtained within the Bethe-Peierls approximation. It is found that the transition temperature depends, in order of increasing importance, upon particle statistics, quantum tunneling, and the nearest-neighbor interaction strength. The results are applied to systems of helium adsorbed on graphite.

INTRODUCTION

Recent heat-capacity studies of submonolayer He^4 and He^3 films adsorbed on graphite indicate that near temperatures of 3 K and within a range of densities, an order-disorder transition takes place in the film.[1] The ordered phase is thought to be characterized by a superlattice of adsorbed atoms which is in registry with the triangular array of adsorption sites provided by the graphite. Within this superlattice, there is one helium atom for every three adsorption sites.

The usual approach to the problem of calculating the order-disorder transition temperature of an adsorbed system is to assume that the adsorbate may be treated classically. Within this approximation, the phase diagram depends upon the interparticle interaction and the particular array of adsorption sites. With the further approximation that the adsorbed atoms are well localized at the adsorption sites, the system can be described by a classical lattice gas for which numerous methods for obtaining approximate, and sometimes exact, solutions are known.[2] The phase diagram for the order-disorder transition of helium on graphite has recently been calculated within this scheme.[3]