Quantum Electrodynamics of Intense Photon Beams

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A new formula for induced transition amplitudes in intense photon beams is derived which relates the photon-number description to the coherent-field description. According to this formula transition amplitudes describing processes in which m photons are added to the intense one-mode beam or subtracted from that beam are equal to the Fourier coefficients of the corresponding amplitude in the external field, expanded with respect to the phase of the classical field. The same formula is used to study the expectation values of dynamical variables in n-photon states leading to the conclusion that for large n such states are very much like classical statistical states with a given amplitude and evenly distributed phase.

I. INTRODUCTION

The present paper is devoted to the theoretical study of intense photon beams in interaction with atomic systems. We derive below a new representation for induced absorption and emission amplitudes and for expectation values of dynamical variables, which is valid when the number of photons in the beam is large. In this representation an average over the phase of the classical field appears, giving rise to a rather simple connection between the photon-number description and the coherent-field description of the electromagnetic field.

Our representation can be used in quantum theory of any many-boson system (photons, phonons, helium atoms, etc.), but we expect that it will be most helpful in quantum optics.

In Sec. II we derive the phase-average representation in a simple exactly soluble model in order to exhibit its main features. In Sec. III we use the phase-average representation to show that for large n the photon number states become very much like classical statistical states with given amplitude and evenly distributed phase. In Sec. IV we extend our method to a general case, and finally in Sec. V we compare our results with those of other authors and discuss an illustrative example.

II. SIMPLE MODEL

In order to introduce the main features of our approach, we shall consider the simplest possible model of a radiating system: the forced harmonic oscillator. This model describes a single mode of the electromagnetic field coupled to a given external current. In the quantized version it is most easily described in terms of annihilation and creation operators a and $a^\dagger$. The time evolution of this system is determined by the following Hamiltonian:

$$H(t)=\hbar\omega a^\dagger a+\alpha^*(t)a+\alpha(t)a^\dagger. \quad (1)$$

The evolution operator $U(t)$ in the Dirac picture obeys the equation

$$i\hbar\frac{\partial}{\partial t}U(t)=[\alpha^*(t)e^{-i\omega t}a+\alpha(t)e^{i\omega t}a^\dagger]U(t). \quad (2)$$

This equation is to be compared with the evolution equation in the Dirac picture in full quantum electrodynamics:

$$i\hbar\frac{\partial}{\partial t}U(t)=\int d^3r\,A_\mu(\vec{r},t)j^\mu(\vec{r},t)U(t), \quad (3)$$

where $A_\mu(\vec{r},t)$ is the field operator of the electromagnetic potential and $j^\mu(\vec{r},t)$ is the current operator. After expanding the potential operator into a series of annihilation and creation operators,

$$A_\mu(\vec{r},t)=\sum_k[f_\mu^{(k)}(\vec{r},t)a_k+f_\mu^{(k)*}(\vec{r},t)a_k^\dagger], \quad (4)$$

and substituting a given external current $J^\mu(\vec{r},t)$ for the current operator $j^\mu(\vec{r},t)$, we can identify the coefficients $\alpha^*(t)e^{-i\omega t}$ and $\alpha(t)e^{i\omega t}$ in Eq. (2) as the projections of the current on the given mode function $f_\mu(\vec{r},t)$ and its complex conjugate:

$$\alpha^*(t)e^{-i\omega t}=\int d^3r\,f_\mu(\vec{r},t)J^\mu(\vec{r},t), \quad (5a)$$

$$\alpha(t)e^{i\omega t}=\int d^3r\, f_\mu^*(\vec{r},t)J^\mu(\vec{r},t). \tag{5b}$$

Vanishing of all projections of the current on the remaining mode functions may be thought to be a special property of a radiating system. If the mode functions $f_\mu^{(k)}$ are, for example, monochromatic plane waves (normalized to one photon per quantization volume),

$$f_\mu^{(k)}(\vec{r},t)=\left(\frac{\hbar c^2}{2\omega V}\right)^{1/2}\epsilon_\mu e^{-i\omega t+i\vec{k}\cdot\vec{r}}, \tag{6}$$

the coefficients α and α^* become simply Fourier components of the current.

It can be checked by a direct calculation that the solution of Eq. (2) obeying the initial condition $U(0)=1$ is

$$U(t)=\exp\left(-\frac{1}{\hbar^2}\int_0^t dt'\int_0^{t'}dt''\,\alpha^*(t')e^{-i\omega t'}\alpha(t'')e^{i\omega t''}\right)\exp\left(-\frac{i}{\hbar}\int_0^t dt'\,\alpha(t')e^{i\omega t'}a^\dagger\right)$$
$$\times\exp\left(-\frac{i}{\hbar}\int_0^t dt''\,\alpha^*(t'')e^{-i\omega t''}a\right). \tag{7}$$

The double integral appearing in the first exponent can be split into its real and imaginary parts according to the formula

$$\int_0^t dt'\int_0^{t'}dt''g^*(t')g(t'')=\frac{1}{2}\int_0^t dt'\int_0^t dt''[1+\epsilon(t'-t'')]\,g^*(t')g(t'')$$
$$=\frac{1}{2}\left|\int_0^t dt' g(t')\right|^2+\frac{i}{2}\int_0^t dt'\int_0^t dt''\,\epsilon(t'-t'')\,\mathrm{Re}g(t')\,\mathrm{Im}g(t'').$$

The imaginary part contributes only the phase factor $e^{i\psi(t)}$ to the evolution operator. The evolution operator can thus be written in the form

$$U=e^{i\psi}e^{-|\eta|^2/2}:e^{i\eta a^\dagger+i\eta^*a}:, \tag{8}$$

where

$$\eta\equiv-\frac{1}{\hbar}\int_0^t dt'\,\alpha(t')e^{i\omega t}. \tag{9}$$

The phase factor $e^{i\psi}$ is clearly unobservable and will be disregarded in further calculations.

We shall now evaluate matrix elements $\langle n+m|U|n\rangle$ and $\langle n|U|n+m\rangle$ of the evolution operator U between eigenstates $|n\rangle$ of the photon-number operator. Using Eq. (8), we obtain

$$[(n+m)!n!]^{1/2}\langle n+m|U|n\rangle$$
$$=e^{-|\eta|^2/2}\langle 0|a^{n+m}e^{i\eta a^\dagger}e^{i\eta^*a}(a^\dagger)^n|0\rangle$$
$$=e^{-|\eta|^2/2}\langle 0|(a+i\eta)^{n+m}(a^\dagger+i\eta^*)^n|0\rangle$$
$$=e^{-|\eta|^2/2}(i\eta)^m\sum_{k=0}^n\binom{n+m}{k}\binom{n}{k}k!\,(i\eta)^{n-k}(i\eta^*)^{n-k}.$$

As has been already observed by Feynman,[1] the sum on the right-hand side is the associated Laguerre polynomial, so that finally

$$\langle n+m|U|n\rangle=[n!/(n+m)!]^{1/2}e^{-|\eta|^2/2}(i\eta)^m L_n^m(|\eta|^2). \tag{10}$$

A similar result is obtained for the absorption amplitude

$$\langle n|U|n+m\rangle=[n!/(n+m)!]^{1/2}e^{-|\eta|^2/2}(i\eta^*)^m L_n^m(|\eta|^2). \tag{11}$$

We will now study the behavior of the induced emission and absorption amplitudes for large values of n. To this end we shall expand the Laguerre polynomial into a series of Bessel functions (see the Appendix)

$$[n!/(n+m)!]e^{-z/2}L_n^m(z)=2^m\sum_{k=0}^\infty p_k^m(z)[2(Nz)^{1/2}]^{-m-k}$$
$$\times J_{m+k}(2(Nz)^{1/2}), \tag{12}$$

where $N=n+\frac{1}{2}(m+1)$. For large N the first term of this series is a good approximation to the left-hand side provided z is not too large:

$$[n!/(n+m)!]e^{-z/2}L_n^m(z)\approx(Nz)^{-m/2}J_m(2(Nz)^{1/2}). \tag{13}$$

Dropping terms of the order of $m^3/3N^2$, we can also write

$$[(n+m)!/n!]^{1/2}\approx N^{m/2}.$$

In that manner we arrive at the following approximate formulas for the transition amplitudes[2]:

$$\langle n+m|U|n\rangle\approx(iN^{1/2}\eta)^m(N|\eta|^2)^{-m/2}$$
$$\times J_m(2(N|\eta|^2)^{1/2}), \tag{14a}$$

$$\langle n|U|n+m\rangle\approx(iN^{1/2}\eta^*)^m(N|\eta|^2)^{-m/2}$$
$$\times J_m(2(N|\eta|^2)^{1/2}). \tag{14b}$$

Finally we use an integral representation for the Bessel function to express our result in the form which has an obvious physical interpretation:

$$(iw)^m|w|^{-m}J_m(2|w|)\equiv i^m\sum_{k=0}^{\infty}(-1)^k\frac{w^{m+k}w^{*k}}{(m+k)!k!}$$
$$=\frac{1}{2\pi}\int_0^{2\pi}d\phi\, e^{-im\phi}\times\exp(iwe^{i\phi}+iw^*e^{-i\phi}). \quad (15)$$

Transition amplitudes can therefore be represented as the following phase averages

$$\langle n+m|U|n\rangle\approx\frac{1}{2\pi}\int_0^{2\pi}d\phi\, e^{-im\phi}\times\exp(iN^{1/2}\eta e^{i\phi}+iN^{1/2}\eta^*e^{-i\phi}), \quad (16a)$$

$$\langle n|U|n+m\rangle\approx\frac{1}{2\pi}\int_0^{2\pi}d\phi\, e^{im\phi}\times\exp(iN^{1/2}\eta e^{i\phi}+iN^{1/2}\eta^*e^{-i\phi}). \quad (16b)$$

In order to see more clearly the meaning of these phase-average representations, we will rewrite them with the help of formulas (5) and (9) in terms of the current:

$$\left.\begin{matrix}\langle n+m|U(t)|n\rangle\\ \langle n|U(t)|n+m\rangle\end{matrix}\right\}\approx\frac{1}{2\pi}\int_0^{2\pi}d\phi\, e^{\mp im\phi}\exp\left(-\frac{i}{\hbar}\int_0^t dt'\int d^3r\,\mathcal{A}_\mu^{(\phi)}(\vec{r},t)J^\mu(\vec{r},t)\right), \quad (17)$$

where

$$\mathcal{A}_\mu^{(\phi)}(\vec{r},t)=N^{1/2}f_\mu(\vec{r},t)e^{-i\phi}+N^{1/2}f_\mu^*(\vec{r},t)e^{i\phi}. \quad (18)$$

For intense photon beams the stimulated emission and absorption can therefore be described in terms of the corresponding classical field. The intensity of the photon beam reflects itself in the amplitude of this classical potential. Owing to the well-known complementarity between the number of photons and the phase of the field, the transition amplitudes between the photon-number eigenstates must not depend on the phase, and that is why the phase average appears. What is surprising however, is that the averaging over the phase must be applied to the transition amplitude rather than to the transition probabilities.

III. PHOTON-NUMBER STATES AS CLASSICAL STATISTICAL STATES

In this section we will show that the n-photon state for large n behaves like the classical statistical state with the field amplitude equal to $(n+\frac{1}{2})^{1/2}$ and evenly distributed phase. In order to describe more precisely in what sense a quantum state may be like a classical state, we will formulate the quantum theory of electromagnetic field in terms of Wigner functions[3] and Weyl representations.[4] For simplicity we will again restrict ourselves to one mode of the field only.

The Wigner-Weyl (WW) transforms $\rho_w(\alpha,\alpha^*)$ of the density operator ρ (or an operator representing a dynamical variable) will be defined as follows[5]:

$$\rho_w(\alpha,\alpha^*)\equiv\int\frac{d^2\beta}{\pi}\mathrm{Tr}\{e^{i(\beta a+\beta^*a^\dagger)}\rho\}\times e^{-i(\alpha\beta+\alpha^*\beta^*)}. \quad (19)$$

This formula can be inverted with the following result

$$\rho=\int\frac{d^2\alpha}{\pi}\int\frac{d^2\beta}{\pi}e^{-i(\beta a+\beta^*a^\dagger)}\times e^{i(\alpha\beta+\alpha^*\beta^*)}\rho_w(\alpha,\alpha^*). \quad (20)$$

The expectation value of an operator F in a given state ρ can be expressed in terms of their WW transforms $F_w(\alpha,\alpha^*)$ and $\rho_w(\alpha,\alpha^*)$ as follows:

$$\mathrm{Tr}\{F\rho\}=\int\frac{d^2\alpha}{\pi}\rho_w(\alpha,\alpha^*)F_w(\alpha,\alpha^*). \quad (21)$$

We shall study now the WW transforms of the photon-number states $|n\rangle$ for large values of n. The Fourier transform of the WW function $\rho_w^{(n)}(\alpha,\alpha^*)$ for such a state can be expressed by the Laguerre polynomial [cf. Eq. (10)]:

$$\tilde{\rho}_w^{(n)}(\beta,\beta^*)\equiv\mathrm{Tr}\{e^{i(\beta a+\beta^*a^\dagger)}|n\rangle\langle n|\}=e^{-|\beta|^2/2}L_n(|\beta|^2). \quad (22)$$

Following our procedure described in Sec. II, we will approximate $\tilde{\rho}_w^{(n)}$ for large n by the Bessel function

$$\tilde{\rho}_w^{(n)}(\beta,\beta^*)\approx J_0(2(n+\tfrac{1}{2})^{1/2}|\beta|). \quad (23)$$

This approximation is valid when $|\beta|$ is not too large, which means that the WW transform $\rho_w^{(n)}(\alpha,\alpha^*)$ will be well approximated for large values of $|\alpha|$. Using the integral representation (15) for the Bessel function, we obtain

$$\rho_w^{(n)}(\alpha,\alpha^*)\approx\int\frac{d^2\beta}{\pi}J_0(2(n+\tfrac{1}{2})^{1/2}|\beta|)e^{-i(\alpha\beta+\alpha^*\beta^*)}=\frac{1}{2\pi}\int_0^{2\pi}d\phi\,\pi\delta^{(2)}[\alpha-(n+\tfrac{1}{2})^{1/2}e^{-i\phi}]. \quad (24)$$

This leads to the following formula for expectation values:

$$\langle n|F|n\rangle \approx \frac{1}{2\pi}\int_0^{2\pi} d\phi\, F_w((n+\tfrac{1}{2})^{1/2}e^{-i\phi}, (n+\tfrac{1}{2})^{1/2}e^{i\phi}). \tag{25}$$

As is well known the WW transforms can be also interpreted as quasiprobability distributions in the classical statistical theory. In such a theory pure classical states are represented by the distributions $\pi\delta^{(2)}(\alpha-\alpha_0)$, and therefore the WW transforms of the photon-number state for large n behave like a classical statistical state with a fixed amplitude equal to $(n+\frac{1}{2})^{1/2}$ and evenly distributed phase. This result can be compared with the properties of ideal laser states $\rho^{(L)}$ which are constructed from coherent states $|\gamma\rangle$ with given absolute value of γ:

$$\rho^{(L)} = \frac{1}{2\pi}\int_0^{2\pi} d\phi |\sqrt{N}e^{i\phi}\rangle\langle\sqrt{N}e^{i\phi}|. \tag{26}$$

The WW transform of $\rho^{(L)}$ is

$$\rho_w^{(L)}(\alpha,\alpha^*) = \frac{1}{\pi}\int_0^{2\pi} d\phi \exp(-2|\alpha-\sqrt{N}e^{i\phi}|^2). \tag{27}$$

For large N this distribution, in contrast to $\rho_w^{(n)}$, is characterized always by a finite spread in the absolute value of the amplitude.

Therefore as far as the modulus of the amplitude is concerned the photon-number states are for large intensities more classical than the coherent states. One may use the following argument (due essentially to E. T. Jaynes, private communication) to explain in simple terms what is meant by this statement. Due to the uncertainty principle there is a limit to localization in phase space for all quantum states. The minimal area which any state can occupy on the (α, α^*) plane is equal to π. Quantum states occupying this minimal area are the closest to classical states. Coherent states belong to this category, but they are smeared evenly in every direction. Formula (24) shows that the region of phase space occupied by the n-photon state has for large n the form of the ring of radius $(n+\frac{1}{2})^{1/2}$ and (negligible) width equal to $\frac{1}{2}(n+\frac{1}{2})^{-1/2}$ and is therefore the closest state to a classical state which has a well-defined amplitude.

IV. GENERAL THEORY

We shall now extend our technique to the full quantum theory of electromagnetic processes. We will not assume any specific form of the Hamiltonian for the charged particles and for their interaction with the electromagnetic field. Our results will therefore be valid in nonrelativistic quantum theory of radiation, in relativistic quantum electrodynamics, or in any other quantum theory provided only that the coupling of the electromagnetic field to the rest of the system is described in terms of electromagnetic potentials.

Our starting point will be the Dyson formula in which the evolution operator $U(t, t_0)$ is represented as a time-ordered exponential of the interaction Hamiltonian $H_I(t)$ in the Dirac picture,

$$U(t,t_0) = T\exp\left(-\frac{i}{\hbar}\int_{t_0}^{t} dt' H_I(t')\right). \tag{28}$$

The interaction Hamiltonian will depend in general on the field operators of the electromagnetic potential $A_\mu(z)$ and also on some operators describing charged particles (or more generally, the rest of the system). The time-ordering operation T applies to all operators in H_I, and we can always think of it as being the product of T_F and T_P, where T_F time-orders the potential field operators and T_P time-orders the particle operators. The time-ordered product of any number of potential field operators can be converted into the normally-ordered product of those operators with the use of the Wick theorem. We will use a compact form of this theorem due to Hori[6]:

$$T(A_{\mu_1}(z_1)\cdots A_{\mu_k}(z_k)) = :\exp\left(\int A\frac{\delta}{\delta\mathcal{A}}\right):\exp\left(\frac{\hbar}{2i}\int\frac{\delta}{\delta\mathcal{A}}D^F\frac{\delta}{\delta\mathcal{A}}\right) \times \mathcal{A}_{\mu_1}(z_1)\cdots\mathcal{A}_{\mu_k}(t_k)|_{\mathcal{A}=0}, \tag{29}$$

where

$$\int A\frac{\delta}{\delta\mathcal{A}} \equiv \int d^4z\, A_\mu(z)\frac{\delta}{\delta\mathcal{A}_\mu(z)},$$

$$\int\frac{\delta}{\delta\mathcal{A}}D^F\frac{\delta}{\delta\mathcal{A}} \equiv \int d^4z\, d^4z'\,\frac{\delta}{\delta\mathcal{A}_\mu(z)} \times D^F_{\mu\nu}(z-z')\frac{\delta}{\delta\mathcal{A}_\nu(z')},$$

and $D^F_{\mu\nu}$ is the Feynman propagator,

$$D^F_{\mu\nu}(z-z') = -g_{\mu\nu}D_F(z-z').$$

In the nonrelativistic quantum theory of radiation, it is usually more convenient to use the radiation gauge in order to separate out the instantaneous Coulomb interaction. The remaining part of the electromagnetic interaction, due to the exchange of transverse photons is described by the following propagator:

$$D^F_{\mu\nu}(z-z') = 0, \quad \text{when either } \mu=0 \text{ or } \nu=0$$
$$= (\delta_{ij} - \Delta^{-1}\partial_i\partial_j)D_F(z-z'),$$
$$\text{when } \mu=i \text{ and } \nu=j.$$

We will not treat this case separately since it only amounts to a different choice of the propagator D^F in the formula (29).

The evolution operator (28) can now be written in the form

$$U(t,t_0)=:\exp\left(\int A\frac{\delta}{\delta\mathfrak{A}}\right):\exp\left(\frac{\hbar}{2i}\int\frac{\delta}{\delta\mathfrak{A}}D^F\frac{\delta}{\delta\mathfrak{A}}\right)\times U[\mathfrak{A}]|_{\mathfrak{A}=0}, \tag{30}$$

where

$$U[\mathfrak{A}]\equiv T_p\exp\left(-\frac{i}{\hbar}\int_{t_0}^{t}dt'\,H_I[t'|\mathfrak{A}]\right) \tag{31}$$

is the evolution operator for the charged particles interacting with an external, classical electromagnetic field described by a c-number potential $\mathfrak{A}_\mu(z)$ and $H_I[t|\mathfrak{A}]$ is the interaction energy of the particles as a functional of the external field and a function of time.

Matrix elements $\langle\psi_f|U[\mathfrak{A}]|\psi_i\rangle$ of the evolution operator $U[\mathfrak{A}]$ between the initial ψ_i and final ψ_f states of the particle system give transition amplitudes for charged particles interacting with a classical electromagnetic field $\mathfrak{A}$. Formula (30) shows two well-known effects which the quantization of the electromagnetic field has on transition amplitudes. First, there are corrections to all electromagnetic processes due to virtual photons (which are represented by internal photon lines on Feynman diagrams) and virtual particle-antiparticle pairs (represented on diagrams by closed particle loops). We will refer to them jointly as radiative corrections. Radiative corrections are generated by the exponential operation

$$\exp\left(\frac{\hbar}{2i}\int\frac{\delta}{\delta\mathfrak{A}}D^F\frac{\delta}{\delta\mathfrak{A}}\right)$$

when it acts on the external field transition amplitude $\langle\psi_f|U[\mathfrak{A}]|\psi_i\rangle$. Second, there are effects of the electromagnetic field quantization on the initial and final states of the electromagnetic field. These effects are generated by the exponential operation $:\exp(\int A\delta/\delta\mathfrak{A}):$. Owing to the normal ordering, this exponential operation is responsible only for the appearance of real photons in the initial and final states of the whole system. Real photons are represented by external photon lines on Feynman diagrams.

Since all photon creation and annihilation operators appear now, like in formula (8), in a normally ordered exponential, we can apply without any essential change the same technique as described in Sec. II.

Let the initial and final states of the full system be product states describing n (and $n+m$) photons in one single mode specified by the photon wave function $f_\mu(z)$ and charged particles in the states ψ_i and ψ_f.

The transition amplitude $A_{fi}(n+m,n)$ between such states with the help of (30) can be written

$$A_{fi}(n+m,n)=\left\langle n+m\left|\exp\left(\int A^{(-)}\frac{\delta}{\delta\mathfrak{A}}\right)\exp\left(\int A^{(+)}\frac{\delta}{\delta\mathfrak{A}}\right)\right|n\right\rangle\exp\left(\frac{\hbar}{2i}\int\frac{\delta}{\delta\mathfrak{A}}D^F\frac{\delta}{\delta\mathfrak{A}}\right)\langle\psi_f|U[\mathfrak{A}]|\psi_i\rangle|_{\mathfrak{A}=0}. \tag{32}$$

Repeating the calculations which led us to formulas (16) and (17), we now obtain

$$\left\langle n+m\left|\exp\left(a^\dagger\int f^*\frac{\delta}{\delta\mathfrak{A}}\right)\exp\left(a\int f\frac{\delta}{\delta\mathfrak{A}}\right)\right|n\right\rangle=\left(\frac{n!}{(n+m)!}\right)^{1/2}\left(\int f\frac{\delta}{\delta\mathfrak{A}}\right)^m L_n^m\left(-\int f^*\frac{\delta}{\delta\mathfrak{A}}\int f\frac{\delta}{\delta\mathfrak{A}}\right)$$
$$\approx\frac{1}{2\pi}\int_0^{2\pi}d\phi\,e^{-im\phi}\exp\left(\int\mathfrak{A}^{(\phi)}\frac{\delta}{\delta\mathfrak{A}}\right)\exp\left(-\frac{1}{2}\int f\frac{\delta}{\delta\mathfrak{A}}\int f^*\frac{\delta}{\delta\mathfrak{A}}\right), \tag{33}$$

where

$$\mathfrak{A}_\mu^{(\phi)}(z)=\sqrt{N}f_\mu(z)e^{-i\phi}+\sqrt{N}f_\mu^*(z)e^{i\phi}. \tag{34}$$

The transition amplitude $A_{fi}(n+m,n)$ can therefore be approximately written in the form

$$A_{fi}(n+m,n)\approx\frac{1}{2\pi}\int_0^{2\pi}d\phi\,e^{-im\phi}\exp\left(\int\mathfrak{A}^{(\phi)}\frac{\delta}{\delta\mathfrak{A}}\right)\exp\left(\frac{\hbar}{2i}\int\frac{\delta}{\delta\mathfrak{A}}\tilde{D}^F\frac{\delta}{\delta\mathfrak{A}}\right)\langle\psi_f|U[\mathfrak{A}]\psi_i\rangle|_{\mathfrak{A}=0}, \tag{35}$$

where the modified propagator $\tilde{D}^F$ differs from D^F by having the part describing the free propagation in the mode f removed:

$$\tilde{D}^F_{\mu\nu}(z-z')=D^F_{\mu\nu}(z-z') - (i/2\hbar)[f_\mu(z)f^*_\nu(z')+f^*_\mu(z)f_\nu(z')]. \quad (36)$$

Both exponential operations in the formula (35) produce changes in the amplitude $\langle\psi_f|U[\mathfrak{A}]|\psi_i\rangle$ which can easily be described. As we have already explained before, the operation

$$\exp\left(\frac{\hbar}{2i}\int\frac{\delta}{\delta\mathfrak{A}}\tilde{D}^F\frac{\delta}{\delta\mathfrak{A}}\right)$$

introduces radiative corrections. This time however, the virtual-photon propagators are modified in accordance with the definition (36) of the propagator $\tilde{D}^F_{\mu\nu}$. The second exponential operation shifts the argument $\mathfrak{A}$ in the original amplitude by the amount $\mathfrak{A}^{(\phi)}$.

The final formulas for the induced emission and absorption amplitudes read therefore,

$$\left.\begin{matrix}A_{fi}(n+m,n)\\ A_{fi}(n,n+m)\end{matrix}\right\}\approx\frac{1}{2\pi}\int_0^{2\pi}d\phi\, e^{\mp im\phi}\langle\psi_f|\tilde{U}[\mathfrak{A}^{(\phi)}]|\psi_i\rangle, \quad (37)$$

where the tilde over U means that all radiative corrections are to be evaluated with the use of the modified photon propagator $\tilde{D}^F$.

For low-energy photons, when $\hbar\omega/mc^2 \ll 1$, radiative corrections have a rather small effect (beyond producing, of course, mass and charge renormalization). When these corrections are neglected, formula (37) gives a very simple relationship between two descriptions of electromagnetic field: the one in terms of a classical field $\mathfrak{A}$ and the other in terms of quantum states $|n\rangle$. According to (37), if the amplitude $\langle\psi_f|U[\mathfrak{A}^{(\phi)}]|\psi_i\rangle$ describing a transition under the influence of an external electromagnetic field $\mathfrak{A}^{(\phi)}$ is expanded into the Fourier series with respect to the phase ϕ of this field:

$$\langle\psi_f|U[\mathfrak{A}^{(\phi)}]|\psi_i\rangle=\sum_{m=-\infty}^{\infty}a_m e^{im\phi}, \quad (38)$$

the expansion coefficients a_m approximate transition amplitudes between photon-number states evaluated for large values of n.

$$a_m\approx\begin{cases}A_{fi}(n+m,n), & m\geq 0\\ A_{fi}(n,n+m), & m<0.\end{cases} \quad (39)$$

It has been observed before by Frantz[7] that the transition probabilities for photon-number states can be related to transition probabilities in an external classical field. What we believe, however, is new in our study is that such a simple and universal relation as (37) holds for the transition amplitudes.

V. PROPERTIES OF THE PHASE-AVERAGE REPRESENTATION

Formula (37) for induced emission and absorption amplitudes can be used as a convenient starting point to derive the rules for the diagrammatic representation of these amplitudes.

First, we may apply standard procedure to represent all contributions to the transition amplitude in an external field $\mathfrak{A}^{(\phi)}$ by Feynman diagrams. We must only remember that internal photon lines represent now not ordinary photon propagators $D^F_{\mu\nu}$ but photon propagators $\tilde{D}^F_{\mu\nu}$ modified according to the formula (36).

Next we should find out in what way the integration over ϕ modifies those amplitudes. Radiative corrections do not depend on the phase ϕ so that they will not be affected by this integration and we may restrict ourselves to the study of the classical field $\mathfrak{A}^{(\phi)}$ only. The nth-order term in perturbation expansion with respect to $\mathfrak{A}^{(\phi)}$ breaks into a sum of 2^n terms, each term containing a particular combination of photon wave functions $f_\mu(z)e^{-i\phi}$ and complex-conjugate photon wave functions $f^*_\mu(z)e^{i\phi}$. Each term has therefore an overall phase factor $\exp[i(n_E-n_A)\phi]$, where n_E is the number of functions f^* and n_A is the number of functions f. After multiplication by $e^{\mp im\phi}$ and integration over ϕ, only those terms will survive for which $n_E-n_A=\pm m$. This result can be easily described in terms of Feynman diagrams if we represent each wave function $\sqrt{N}f_\mu(z)$ on a diagram by the incoming photon line attached to a charged-particle line at the point z and each complex-conjugate wave function $\sqrt{N}f^*_\mu(z)$ by the outgoing photon line. In this way we obtain similar diagrammatic rules to those discovered before by Fried and Eberly[8] and by Ehlotzky.[9] There are, however, several important differences.

First, we discover that virtual photon corrections which have been neglected in previous publications must also be modified according to the formula (36) in the presence of an intense photon beam. Second, we find a slightly different coefficient multiplying the photon wave function (or the vertex); we have $[n+\frac{1}{2}(m+1)]^{1/2}$ instead of $\sqrt{n}$. Owing to this modification, not only the leading term, when $n\to\infty$, is obtained but also the next to the leading term is correctly reproduced. Third, our approach is quite general and can be applied to arbitrary photon states (arbitrary mode function not necessarily a plane wave), to arbitrary form of the coupling of the electromagnetic field to the charged particles, and finally to an arbitrary process. Fourth, we have full control of all the corrections of the order of $N^{-1/2}$ and higher, so that we can estimate their importance should this become necessary.

All these differences will produce, however, only small effects in practical applications of quantum electrodynamics to those processes involving intense photon beams which are presently studied experimentally. For that reason, we have nothing new to add to recent very successful applications by Chang and Stehle[10] of the original simple version of the diagramatic technique to calculate the intensity-dependent level shifts, lifetimes and transition rates.

To end our paper we will apply the phase-average representation to the calculation of transition amplitudes in a model in which an electron bound by the harmonic force interacts with a single mode of radiation (for example, in a cavity). This model is exactly soluble in the dipole approximation even for the quantized electromagnetic field but expressions for transition amplitudes involving a large number N of photons are very complicated. Our method leads to compact and simple expressions for these amplitudes. We will study here only the leading term in N and therefore disregard also radiative corrections. The interaction Hamiltonian in the Dirac picture for this model can be written in the form

$$H_I(t)=-\frac{e}{mc}p(t)A(t)+\frac{e^2}{2mc^2}A^2(t), \tag{40}$$

where

$$p(t)=i(\hbar m\Omega/2)^{1/2}(be^{-i\Omega t}-b^\dagger e^{i\Omega t}), \tag{41}$$

$$A(t)=(\hbar c^2/2\omega V)^{1/2}(fae^{-i\omega t}+f^*a^\dagger e^{i\omega t}), \tag{42}$$

Ω and ω are the characteristic frequencies of the harmonically bound electron and the electromagnetic field, and b and $b^\dagger$ are annihilation and creation operators for harmonic excitations of electronic states.

In order to apply the phase-average representation, we need the evolution operator $U(t,\phi)$ for the electron in the presence of the classical field $\mathcal{A}^{(\phi)}$,

$$\mathcal{A}^{(\phi)}=\left(\frac{\hbar c^2 N}{2\omega V}\right)^{1/2}(fe^{-i(\omega t+\phi)}+f^*e^{i(\omega t+\phi)}). \tag{43}$$

This problem is very similar to the forced-harmonic-oscillator model discussed in Sec. II and can be solved by methods described before. The formula for the evolution operator $U(t,\phi)$ as a function of time and phase reads

$$U(t,\phi)=\exp\{-g^2N[\alpha_+(t)f^2e^{-2i\phi}+\alpha_-(t)f^{*2}e^{2i\phi}+\alpha_0(t)|f|^2]\}\exp\{ig\sqrt{N}[\beta_-(t)fe^{-i\phi}+\beta_+(t)f^*e^{i\phi}]b^\dagger\}$$
$$\times\exp\{ig\sqrt{N}[\beta_+(-t)fe^{-i\phi}+\beta_-(-t)f^*e^{i\phi}]b\}, \tag{44}$$

where

$$g^2=\frac{e^2\Omega}{4mV\omega},$$

$$\beta_\pm(t)=\frac{1-e^{i(\Omega\pm\omega)t}}{\Omega\pm\omega},$$

$$\alpha_\pm(t)=\pm\frac{1-e^{\mp 2i\omega t}}{2\Omega\omega}\pm\frac{e^{\mp 2i\omega t}-1}{2\omega(\Omega\mp\omega)}+\frac{1-e^{-i(\Omega\pm\omega)t}}{(\Omega-\omega)(\Omega+\omega)},$$

$$\alpha_0(t)=\frac{2it}{\Omega}-\frac{it}{\Omega+\omega}-\frac{it}{\Omega-\omega}+\frac{1-e^{-i(\Omega+\omega)t}}{(\Omega+\omega)^2}+\frac{1-e^{-i(\Omega-\omega)t}}{(\Omega-\omega)^2}.$$

Since the evolution operator as given by this expression is normally ordered with respect to electron creation and annihilation operators, its matrix elements between harmonic-oscillator states can be easily found. Finally, by expanding these matrix elements into Fourier series with respect to ϕ, we obtain quantum-mechanical transition amplitudes between photon-number states.

For example, for the induced decay of the first excited electron state with the emission of one additional photon, we obtain

$$\begin{aligned}\langle N+1,0|U(t)|N,1\rangle &\approx ig\sqrt{N}f^*\exp[-g^2N\alpha_0(t)|f|^2]\\ &\times(\beta_-(-t)I_0\{2g^2N|f|[\alpha_-(t)\alpha_+(t)]^{1/2}\}\\ &-\beta_+(-t)[\alpha_-(t)/\alpha_+(t)]^{1/2}\\ &\times I_1\{2g^2N|f|[\alpha_-(t)\alpha_+(t)]^{1/2}\}).\end{aligned} \tag{45}$$

It would have been much more difficult to obtain this formula by the straightforward application of combinatorial methods to transition amplitudes for two coupled harmonic oscillators.

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APPENDIX

We have taken the expansion (12) of the Laguerre polynomial into a series of Bessel functions from the monograph by Buchholz.[11] Since his proof is rather involved, we give below a simple derivation of our own.

First, we will write the associate Laguerre polynomial $L_n^m(z)$ in the form of a Cauchy integral

$$\frac{n!}{(n+m)!}L_n^m(z) \equiv \sum_{k=0}^{n} \frac{n!}{(n-k)!(m+k)!k!}(-z)^k = \frac{(-z)^{-m}}{2\pi i}\oint \frac{du}{u^{m+1}}(1+u)^n e^{-zu}. \quad \text{(A1)}$$

The validity of this formula can be verified by expanding $(1+u)^n$ and e^{-zu} into powers of u and by integrating the result term by term.

Making the following change of variables,

$$u = \tfrac{1}{2}(\coth t - 1),$$

we obtain a new contour integral

$$\frac{n!}{(n+m)!}L_n^m(z) = \left(\frac{-2}{z}\right)^m e^{z/2}\frac{1}{2\pi i}\oint dt\,(\sinh t)^{m-1} \times \exp(-\tfrac{1}{2}z\coth t)\,e^{2Nt}, \quad \text{(A2)}$$

where

$$N = n + \tfrac{1}{2}(m+1).$$

Next we use the identity

$$e^{2Nt} \equiv e^{z/2t}\exp\left[(Nz)^{1/2}\left(-\frac{z}{2t(Nz)^{1/2}} + \frac{2t(Nz)^{1/2}}{z}\right)\right],$$

and we recognize in the second factor the generating function for the Bessel functions

$$\exp\left[\frac{x}{2}\left(y - \frac{1}{y}\right)\right] = \sum_{k=-\infty}^{\infty} y^k J_k(x).$$

This leads to the formula

$$\frac{n!}{(n+m)!}L_n^m(z) = 2^m e^{z/2}\sum_{k=-\infty}^{\infty} p_k^m(z)\frac{J_{m+k}(2(Nz)^{1/2})}{[2(Nz)^{1/2}]^{m+k}}, \quad \text{(A3)}$$

where

$$p_k^m(z) \equiv \frac{(-z)^k}{2\pi i}\oint \frac{dt}{t^{k+1}}\left(\frac{\sinh t}{t}\right)^{m-1} \times \exp[-\tfrac{1}{2}z(\coth t - t^{-1})]. \quad \text{(A4)}$$

For negative integer values of k the integrand in (A4) is analytic at $t=0$ so that the contour integral vanishes. Therefore the summation in formula (A3) effectively extends only from 0 to ∞.

For non-negative integer values of k the coefficient functions $p_k^n(z)$ are polynomials in the variable z^2 of the order k. The lowest power of z^2 in the kth polynomial is $\frac{1}{2}(k+1)$.

We give below the first four terms of the expansion (A3):

$$\frac{n!}{(n+m)!}L_n^m(z) = 2^m e^{z/2}\left[\frac{J_m(2(Nz)^{1/2})}{[2(Nz)^{1/2}]^m} + \frac{z^2}{6}\frac{J_{m+1}(2(Nz)^{1/2})}{[2(Nz)^{1/2}]^{m+1}} + \frac{z^2}{6}\left(\frac{z^2}{16}+m-1\right)\frac{J_{m+2}(2(Nz)^{1/2})}{[2(Nz)^{1/2}]^{m+2}} + \frac{z^4}{36}\left(\frac{z^2}{36}+m-\frac{7}{5}\right)\frac{J_{m+3}(2(Nz)^{1/2})}{[2(Nz)^{1/2}]^{m+3}} + \cdots\right]. \quad \text{(A5)}$$

Since the Bessel functions are bounded, the effective expansion parameter here is $(z/N)^{1/2}$.

[1]R. P. Feynman, Phys. Rev. 80, 440 (1950). On p. 451 in this paper Feynman wrote: "The sum can be expressed as a Laguerre polynomial but there is no advantage in this."

[2]A similar approximation has been used before by N. Polonsky and C. Cohen-Tannoudji [J. Phys. (Paris) 26, 409 (1965)].

[3]E. P. Wigner, Phys. Rev. 40, 749 (1932).

[4]H. Weyl, *The Theory of Groups and Quantum Mechanics* (Dover, New York, 1950) pp. 272–276.

[5]We will use here the complex-plane description of the phase space. The detailed properties of this description can be found, for example, in two papers by K. E. Cahill and R. J. Glauber [Phys. Rev. 177, 1857 (1969); Phys. Rev. 177, 1882 (1969)].

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[10]C. S. Chang and P. Stehle, Phys. Rev. A 4, 641 (1971); Phys. Rev. A 5, 1087 (1972); Phys. Rev. Lett. 30, 1283 (1973). This simplified version of the diagrammatic technique has been given the name of the forward-scattering method by Stehle and his co-workers. We do not use this name in our more general context because if photons do not have well-defined four-momenta, then the photon absorption followed by its subsequent emission does not necessarily result in the electron returning to its initial momentum state.

[11]H. Buchholz, *The Confluent Hypergeometric Function* (Springer-Verlag, Berlin, 1969), p. 97.