Comment on the Momentum-Translation Approximation

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Some objections concerning the validity of a new nonperturbative treatment of multiphoton processes ("momentum-translation approximation") are presented and discussed. In the present state of the question, it seems that the multiphoton amplitudes given by this theory cannot be considered as reliable.

I. INTRODUCTION

A nonperturbative treatment of multiphoton processes has been recently proposed.[1] Its main advantage is to give the multiphoton-transition amplitude in a closed form, only involving the initial and the final states of the transition. This method, which is known as momentum-translation approximation (mt approximation), has been applied to several problems.[2]

However, we feel that some objections concerning the validity of this approximation may be raised. The purpose of this paper is to present and to discuss some of these objections.

II. MOMENTUM-TRANSLATION APPROXIMATION

The Hamiltonian of an atomic system interacting with an electromagnetic wave described by the vector potential $\vec{A}(\vec{r}, t)$ is

$$H=(2m)^{-1}(\vec{p}-e\vec{A})^2+V(\vec{r})=H_0+H', \tag{1}$$

where $H_0=\vec{p}^2/2m+V(\vec{r})$ is the field-free Hamiltonian and $H'=-(e/m)\vec{A}\cdot\vec{p}+(e^2/2m)\vec{A}^2$ is the coupling between the atom and the field (e, m, and $\vec{p}$ are the charge, mass, and impulsion of the electron, respectively; throughout this paper, we take $\hbar=c=1$).

Starting at time t_1 from the field-free eigenstate $|\phi_i(t_1)\rangle$ and switching on $\vec{A}$ adiabatically, one gets at time t_2 a state $|\psi(t_2)\rangle$ which is given by the following integral equation:

$$|\psi(t_2)\rangle=|\phi_i(t_2)\rangle-i\int_{t_1}^{t_2}e^{-iH_0(t_2-\tau)}H'(\tau)|\psi(\tau)\rangle\,d\tau. \tag{2}$$

If H' is too large to be treated as a perturbation, it seems hazardous to solve Eq. (2) by iteration.

The following unitary transformation is then introduced:

$$T=e^{-ie\vec{A}\cdot\vec{r}}. \tag{3}$$

The transformed state

$$|\bar{\psi}(t)\rangle=T|\psi(t)\rangle \tag{4}$$

satisfies the Schrödinger equation

$$i\frac{\partial}{\partial t}|\bar{\psi}(t)\rangle=\left[i\left(\frac{\partial}{\partial t}T\right)T^{\dagger}+THT^{\dagger}\right]|\bar{\psi}(t)\rangle. \tag{5}$$

In the dipole approximation (electromagnetic wavelength large compared to atomic dimension), the variations of $\vec{A}$ with $\vec{r}$ are neglected, and T reduces to the translation in the impulsion space $\vec{p}\to\vec{p}+e\vec{A}$ so that

$$THT^{\dagger}=H_0. \tag{6}$$

The Schrödinger equation (5) becomes

$$i\frac{\partial}{\partial t}|\bar{\psi}(t)\rangle=(H_0+H_I)|\bar{\psi}(t)\rangle. \tag{7}$$

where

$$H_I=i\left(\frac{\partial}{\partial t}T\right)T^{\dagger}=e\left(\frac{\partial}{\partial t}\vec{A}\right)\cdot\vec{r}=-e\vec{E}\cdot\vec{r}. \tag{8}$$

H_I describes the coupling of the electric dipole $e\vec{r}$ with the electric field $\vec{E}$ of the wave.

It may be shown that

$$H_I/H'\approx\omega/\omega_0, \tag{9}$$

where $\omega/2\pi$ is the frequency of the electromagnetic wave and $\omega_0/2\pi$ is a typical atomic frequency. H_I is therefore small compared to H' when

$$\omega\ll\omega_0. \tag{10a}$$

Even if H' cannot be treated as a perturbation with respect to H_0, it is possible for H_I, provided that

$$eaa_0(\omega/\omega_0)\ll 1, \tag{10b}$$

where a is the amplitude of $\vec{A}$ and a_0 is the Bohr radius. The two conditions (10a) and (10b) are the conditions of validity of the mt approximation and will be supposed to be fulfilled in the following.

The idea of the mt approximation is the following: one solves Eq. (7) to zeroth order, which gives

$$|\bar{\psi}^{(0)}(t)\rangle=e^{-iH_0(t-t_1)}|\phi_i(t_1)\rangle; \tag{11}$$

by inverting Eq. (4), one gets the approximate ex-

pression

$$T^{\dagger}(\tau)|\bar{\psi}^{(0)}(\tau)\rangle \tag{12}$$

for the state $|\psi(\tau)\rangle$ which appears in the integral of Eq. (2). Equation (2) then becomes

$$|\psi(t_2)\rangle \approx |\phi_i(t_2)\rangle - i\int_{t_1}^{t_2} e^{-iH_0(t_2-\tau)}H'(\tau)T^{\dagger}(\tau) \times e^{-iH_0(\tau-t_1)}|\phi_i(t_1)\rangle\, d\tau. \tag{13}$$

The transition amplitude to the final field-free state $|\phi_f\rangle$ (the limit $t_2 \to +\infty$ is taken, the field being switched off adiabatically) is

$$\tau_{fi} = \langle\phi_f|\psi(t_2)\rangle = \delta_{fi} - i\int_{t_1}^{t_2}\langle\phi_f|\, e^{-iH_0(t_2-\tau)}H'(\tau) \times T^{\dagger}(\tau)e^{-iH_0(\tau-t_1)}|\phi_i(t_1)\rangle\, d\tau. \tag{14}$$

After a few calculations, Eq. (14) leads to a transition probability per unit time given by

$$W_{fi} = 2\pi\sum_N |T_{fi}^{(N)}|^2\,\delta(E_f - E_i - N\omega), \tag{15}$$

where

$$T_{fi}^{(N)} = i^N(E_i - E_f)\langle\phi_f|J_N(ea\vec{\epsilon}\cdot\vec{r})|\phi_i\rangle \tag{16}$$

(J_N is the N^{th}-order Bessel function and $\vec{\epsilon}$ is the polarization of the vector potential $\vec{A}$).

Thus, the mt approximation yields an explicit expression for the multiphoton transition probability between two atomic levels [according to condition (10a), this expression is valid for $N \gg 1$].

III. GENERAL REMARKS

(a) The N-quanta amplitude (16) fails in reproducing two well-known high-intensity effects: (i) it does not seem to exhibit an enhancement when an intermediate state is quasiresonant for a p-quanta process ($p < N$); (ii) the resonance is always centered at $\omega_0 = N\omega$, whatever the amplitude of the electromagnetic field is. In ordinary perturbation theory, corrections to the lowest-order resonance amplitude and radiative shift appear simultaneously in higher-order calculations.

(b) In the dipole approximation, the unitary transformation (3) corresponds to a gauge transformation.

The incident electromagnetic wave may be described either by the vector potential $\vec{A}$ and the scalar potential $U = 0$ (gauge later referred to as gauge 1), or by

$$\vec{A}' = \vec{A} + \vec{\nabla}\chi, \tag{17a}$$

$$U' = 0 - \frac{\partial}{\partial t}\chi, \tag{17b}$$

χ being an arbitrary function of $\vec{r}$ and t.

For the particular choice[3]

$$\chi = -\vec{A}(\vec{r}, t)\cdot\vec{r}, \tag{18}$$

the spatial variation of $\vec{A}$ being neglected (dipole approximation), one gets

$$\vec{A}' = \vec{0}, \tag{19}$$

$$U' = \frac{\partial}{\partial t}\vec{A}\cdot\vec{r} = -\vec{E}\cdot\vec{r} \quad \text{(gauge 2)}. \tag{20}$$

The Hamiltonians corresponding to gauges 1 and 2 are, respectively, $H_0 + H'$ and $H_0 + H_I$; the same physical state is described by the vector $|\psi(t)\rangle$ in gauge 1, and in gauge 2 by

$$|\psi'(t)\rangle = e^{ie\chi(\vec{r},t)}|\psi(t)\rangle, \tag{21}$$

which gives, according to (18),

$$|\psi'(t)\rangle = e^{-ie\vec{A}\cdot\vec{r}}|\psi(t)\rangle = |\bar{\psi}(t)\rangle. \tag{22}$$

(c) We can then understand the physical meaning of the state $T^{\dagger}(\tau)|\bar{\psi}^{(0)}(\tau)\rangle$, which is introduced as an approximate expression of $|\psi(\tau)\rangle$ in Eq. (2). Since quantum mechanics is gauge invariant, this state has the same physical content in gauge 1 as $|\bar{\psi}^{(0)}(\tau)\rangle$ in gauge 2. But $|\bar{\psi}^{(0)}(\tau)\rangle$ is obtained in gauge 2 by treating H_I to zeroth order, i.e., by neglecting any interaction between the atom and the field; it follows that, in gauge 1, $T^{\dagger}(\tau)|\bar{\psi}^{(0)}(\tau)\rangle$ represents nothing but an unperturbed atomic state and cannot lead to any transition probability.

This statement may also be proved in the following way: the transition amplitude to the field-free state $|\phi_f\rangle$ is written

$$\lim_{t_2\to+\infty}\langle\phi_f|\psi(t_2)\rangle, \tag{23}$$

where $|\psi(t_2)\rangle$ is the exact solution of the Schrödinger equation corresponding to Eq. (2). If, in Eq. (23), $|\psi(t_2)\rangle$ is replaced by the approximate solution $T^{\dagger}(t_2)|\bar{\psi}^{(0)}(t_2)\rangle$, one gets, for the transition amplitude,

$$\lim_{t_2\to+\infty}\langle\phi_f|T^{\dagger}(t_2)e^{-iH_0(t_2-t_1)}|\phi_i\rangle. \tag{24}$$

When $t_2 \to +\infty$, $T^{\dagger}(t_2) \to 1$ [the field is switched off adiabatically and consequently $\lim_{t_2\to+\infty}\vec{A}(t_2) = \vec{0}$]. The transition amplitude therefore reduces to

$$\langle\phi_f|e^{-iH_0(t_2-t_1)}|\phi_i\rangle = e^{-iE_f(t_2-t_1)}\delta_{fi}. \tag{25}$$

So, treating H_I to zeroth order in gauge 2 also leads to no transition probability in gauge 1.

(d) Actually, in the mt approximation, one calculates the transition amplitude not directly from Eq. (24), but from Eq. (2), $|\psi(\tau)\rangle$ being replaced by the approximate solution $T^{\dagger}(\tau)|\bar{\psi}^{(0)}(\tau)\rangle$. Since this state vector, as we have seen before, cannot yield any transition probability, one would expect

to obtain by this procedure only one photon process [$H'(\tau)$ appearing only once in Eq. (2)] and multiphoton amplitudes (16) seem rather puzzling.

IV. COMPARISON WITH AN EXACT SOLUTION FOR A PARTICULAR MODEL

In the mt approximation, no restriction is imposed on the form of the atomic potential $V(\vec{r})$ in the Hamiltonian H_0. It is interesting to choose $V(\vec{r})$ in such a way that an exact solution can be obtained; this allows a test of the multiphoton amplitudes (16) given by the mt approximation.

Let us consider a three-dimensional isotropic harmonic oscillator, of frequency $\omega_0/2\pi$, for which

$$V(\vec{r}) = \tfrac{1}{2}m\omega_0^2\vec{r}^{\,2}, \tag{26}$$

and let us suppose that the electric field $\vec{E}(t)$ of the incident electromagnetic wave is polarized along the x axis. In the dipole approximation, this problem can be solved exactly.[4] One has to find the solution of the Schrödinger equation corresponding to the time-dependent one-dimensional Hamiltonian

$$H = p_x^2/2m + \tfrac{1}{2}m\omega_0^2 x^2 - eE(t)x. \tag{27}$$

The transition probability from the ground state $|\phi_0\rangle$ of the oscillator to the excited state $|\phi_n\rangle$ is given by

$$W_{n0} = e^{-\alpha}\alpha^n/n!, \tag{28}$$

where

$$\alpha = \frac{e^2}{2m\omega_0}\left|\int_{-\infty}^{+\infty} E(t)e^{i\omega_0 t}\,dt\right|^2. \tag{29}$$

α appears to be proportional to the square modulus of the Fourier transform of the excitation $E(t)$, taken at the frequency ω_0 of the oscillator. If we assume a sinusoidal time dependence for $E(t)$,

$$E(t) = E_0 \sin\omega t, \tag{30}$$

this Fourier transform exhibits resonant variations only for $\omega = \omega_0$ and $\omega = -\omega_0$ (the width of the resonance being proportional to the inverse of the interaction time[5]).

It follows that we cannot get multiphoton transitions, for instance, between the ground state and the first excited state $|\phi_1\rangle$: $W_{10} = \alpha e^{-\alpha}$ does not vary rapidly near $\omega_0 = 2\omega, 3\omega, \ldots, n\omega, \ldots$.[6]

Let us now consider the results of the mt approximation applied to this problem. From Eq. (16), we get, for the N-photons transition probability per unit time between $|\phi_0\rangle$ and $|\phi_1\rangle$,

$$T_{10}^{(N)} = 2\pi\omega_0^2\left|\langle\phi_1|J_N\left(\frac{eE_0x}{\omega}\right)|\phi_0\rangle\right|^2. \tag{31}$$

$T_{10}^{(N)}$ is nonzero for N odd.

Therefore, this shows evidence for a case where there is no agreement between the prediction of the mt approximation and the exact solution.

V. WHERE THE MOMENTUM-TRANSLATION APPROXIMATION IS QUESTIONABLE

In the derivation of the mt approximation, no estimation is given for the error introduced by replacing in Eq. (2), $|\psi(\tau)\rangle$ by $T^\dagger(\tau)|\bar{\psi}^{(0)}(\tau)\rangle$. This error is

$$\delta|\psi\rangle = -i\int_{t_1}^{t_2} e^{-iH_0(t_2-\tau)}H'(\tau)T^\dagger(\tau) \times[|\bar{\psi}(\tau)\rangle - |\bar{\psi}^{(0)}(\tau)\rangle]\,d\tau. \tag{32}$$

As $|\bar{\psi}^{(0)}(\tau)\rangle$ is the zeroth-order approximation (in H_I) for the solution $|\bar{\psi}(\tau)\rangle$ of Eq. (7), the square-bracket term of Eq. (32) is of the order of H_I/ω_0, which gives for the integrand an order of magnitude of $H'(H_I/\omega_0)$, i.e., H_I when $H'/\omega_0 \approx 1$.

This error seems to be small compared to the mt approximation solution which, according to Eq. (13), is the integral of a term of the order of H'.

Actually, this is not true; as we will show later, Eq. (13) may be written

$$|\psi(t_2)\rangle = |\phi_i(t_2)\rangle - i\int_{t_1}^{t_2} e^{-iH_0(t_2-\tau)} \times H_I(\tau)T^\dagger(\tau)e^{-iH_0(\tau-t_1)}|\phi_i(t_1)\rangle\,d\tau. \tag{33}$$

It is possible to replace, in Eq. (13), H' by H_I, which gives an integrand of the order of H_I. Therefore, the integral of Eq. (13) is of the same order as the error (32).

The derivation of Eq. (33) lies on the relations

$$H'T^\dagger = [T^\dagger, H_0], \tag{34a}$$

$$i\frac{\partial}{\partial t}T^\dagger = -T^\dagger H_I = -H_I T^\dagger, \tag{34b}$$

which may be easily derived from Eqs. (5)–(7).

According to Eqs. (34a) and (34b), the τ-dependent operator appearing in the integral of Eq. (13) may be written

$$e^{iH_0\tau}H'(\tau)T^\dagger(\tau)e^{-iH_0\tau} = e^{iH_0\tau}[T^\dagger(\tau)H_0 - H_0T^\dagger(\tau)]e^{-iH_0\tau} = i\frac{\partial}{\partial\tau}[e^{iH_0\tau}T^\dagger(\tau)e^{-iH_0\tau}] - e^{iH_0\tau}\left(i\frac{\partial}{\partial\tau}T^\dagger(\tau)\right)e^{-iH_0\tau}$$

$$= i\frac{\partial}{\partial\tau}[e^{iH_0\tau}T^\dagger(\tau)e^{-iH_0\tau}] + e^{iH_0\tau}H_I(\tau)T^\dagger(\tau)e^{-iH_0\tau};$$

Eq. (13) then becomes

$$|\psi(t_2)\rangle = |\phi_i(t_2)\rangle + e^{-iH_0 t_2}\left[e^{iH_0\tau}T^\dagger(\tau)e^{-iH_0\tau}\Big|_{t_1}^{t_2}\right]e^{iH_0 t_1}|\phi_i(t_1)\rangle - i\int_{t_1}^{t_2} e^{-iH_0(t_2-\tau)}H_I(\tau)T^\dagger(\tau)e^{-iH_0(\tau-t_1)}|\phi_i(t_1)\rangle d\tau .$$

When the limit $t_1 \to -\infty$, $t_2 \to +\infty$ is taken, the integrated term vanishes (as the field is zero at these two limits) and we finally get Eq. (33).

One might think that this substitution of H_I to H' can also be performed on the expression (32) which gives the error made in the mt approximation; if this substitution was possible, the error (32) would be negligible. But this is impossible: replacing the bracket of Eq. (32) by the leading term $|\bar{\psi}^{(1)}(\tau)\rangle$ [first-order term in the perturbation expansion of $|\bar{\psi}(\tau)\rangle$ in powers of H_I/ω_0], one gets for the new expression of $\delta|\psi\rangle$,

$$\delta|\psi\rangle = -i\int_{t_1}^{t_2} e^{-iH_0(t_2-\tau)}H'(\tau)T^\dagger(\tau)|\bar{\psi}^{(1)}(\tau)\rangle d\tau , \tag{35}$$

where

$$|\bar{\psi}^{(1)}(\tau)\rangle = -i\int_{t_1}^{\tau} e^{-iH_0(\tau-\tau')}H_I(\tau') \times e^{-iH_0(\tau'-t_1)}|\phi_i(t_1)\rangle d\tau' . \tag{36}$$

$|\bar{\psi}^{(1)}(\tau)\rangle$ can be expressed as

$$|\bar{\psi}^{(1)}(\tau)\rangle = U^{(1)}(\tau)e^{iH_0 t_1}|\phi_i(t_1)\rangle , \tag{37}$$

where

$$U^{(1)}(\tau) = -i\int_{t_i}^{\tau} e^{-iH_0(\tau-\tau')}H_I(\tau')e^{-iH_0\tau'}d\tau' \tag{38}$$

satisfies the differential equation

$$i\frac{d}{d\tau}U^{(1)}(\tau) = H_0U^{(1)}(\tau) + H_I(\tau)e^{-iH_0\tau} . \tag{39}$$

Using the relations (34a), (37), and (39), the τ-dependent operator appearing in the integral of Eq. (35) can then be written

$$\begin{aligned} e^{iH_0\tau}H'(\tau)T^\dagger(\tau)U^{(1)}(\tau) &= e^{iH_0\tau}[T^\dagger(\tau)H_0 - H_0T^\dagger(\tau)]U^{(1)}(\tau) \\ &= e^{iH_0\tau}T^\dagger(\tau)\left(i\frac{d}{d\tau}U^{(1)}(\tau) - H_I(\tau)e^{-iH_0\tau}\right) - e^{iH_0\tau}H_0T^\dagger(\tau)U^{(1)}(\tau) \\ &= i\frac{\partial}{\partial\tau}[e^{iH_0\tau}T^\dagger(\tau)U^{(1)}(\tau)] - e^{iH_0\tau}\left(i\frac{\partial}{\partial\tau}T^\dagger(\tau)\right)U^{(1)}(\tau) - e^{iH_0\tau}T^\dagger(\tau)H_I(\tau)e^{-iH_0\tau} . \end{aligned}$$

According to Eq. (34b), the error (35) is then given by

$$\delta|\psi\rangle = e^{-iH_0t_2}\left[e^{iH_0\tau}T^\dagger(\tau)U^{(1)}(\tau)\Big|_{t_1}^{t_2}\right]e^{iH_0t_1}|\phi_i(t_1)\rangle - i\int_{t_1}^{t_2} e^{-iH_0(t_2-\tau)}H_I(\tau)T^\dagger(\tau)U^{(1)}(\tau)e^{iH_0t_1}|\phi_i(t_1)\rangle d\tau + i\int_{t_1}^{t_2} e^{-iH_0(t_2-\tau)}H_I(\tau)T^\dagger(\tau)e^{-iH_0(\tau-t_1)}|\phi_i(t_1)\rangle d\tau , \tag{40}$$

i.e., by

$$\delta|\psi\rangle = T^\dagger(t_2)|\bar{\psi}^{(1)}(t_2)\rangle + i\int_{t_1}^{t_2} e^{-iH_0(t_2-\tau)}H_I(\tau)T^\dagger(\tau)e^{-iH_0(\tau-t_1)}|\phi_i(t_1)\rangle d\tau - i\int_{t_1}^{t_2} e^{-iH_0(t_2-\tau)}H_I(\tau)T^\dagger(\tau)|\bar{\psi}^{(1)}(\tau)\rangle d\tau . \tag{41}$$

The third term of Eq. (41) is negligible, as the corresponding integrand is of the order of $H_I(H_I/\omega_0)$. Taking as usual $T^\dagger(t_2) = 1(t_2 \to +\infty)$, and using Eq. (36), the first two terms of Eq. (41) may be written in a unique integral:

$$i\int_{t_1}^{t_2} e^{-iH_0(t_2-\tau)}[T^\dagger(\tau) - 1]H_I(\tau) \times e^{-iH_0(\tau-t_1)}|\phi_i(t_1)\rangle d\tau . \tag{42}$$

$T^\dagger(\tau) - 1$ is of the order of $eAr \approx eAp/m\omega_0 \approx H'/\omega_0$ ≈ 1. It follows that the integral of the expression (42) is of the same order of magnitude as the integral of Eq. (33). *It is therefore nonconsistent to neglect the first-order correction* $|\bar{\psi}^{(1)}(\tau)\rangle$ *in the calculation, since it leads to terms comparable with those given by the mt approximation which only keeps* $|\bar{\psi}^{(0)}(\tau)\rangle$. This is due to the fact that the integral of Eq. (13) has an order of magnitude smaller than expected [transformation of Eq. (13) into Eq. (33)].

To improve the mt approximation, we must re-

place, in Eq. (2), $|\psi(\tau)\rangle$ by $T^\dagger(\tau)[|\bar{\psi}^{(0)}(\tau)\rangle + |\bar{\psi}^{(1)}(\tau)\rangle]$. This amounts to add Eq. (41) to Eq. (33), which gives, for $|\psi(t_2)\rangle$,

$$|\psi(t_2)\rangle = |\phi_i(t_2)\rangle + T^\dagger(t_2)|\bar{\psi}^{(1)}(t_2)\rangle - i\int_{t_1}^{t_2} e^{-iH_0(t_2-\tau)} H_I(\tau) T^\dagger(\tau)|\bar{\psi}^{(1)}(\tau)\rangle\, d\tau . \quad (43)$$

The integral of Eq. (33), which is at the origin of the multiphoton transition amplitudes (16), disappears, and the only remaining first-order term is the $T^\dagger(t_2)$ transformed of the first-order solution of Eq. (7). The third term of Eq. (43) does not represent the whole second-order correction to $|\psi(t_2)\rangle$, since it does not include the contribution of the second order term $T^\dagger(\tau)|\bar{\psi}^{(2)}(\tau)\rangle$.

More generally, for a consistent calculation up to the nth order in H_I/ω_0 of $|\psi(t_2)\rangle$, one must replace in the integral of Eq. (2) $|\psi(\tau)\rangle$ by

$$T^\dagger(\tau)\sum_{p=0}^{n}|\bar{\psi}^{(p)}(\tau)\rangle,$$

where $|\bar{\psi}^{(p)}(\tau)\rangle$ is the pth-order contribution in the perturbative solution of Eq. (7). Calculations similar to the previous ones give, for $|\psi(t_2)\rangle$,

$$|\psi(t_2)\rangle = T^\dagger(t_2)\sum_{p=0}^{n}|\bar{\psi}^{(p)}(t_2)\rangle - i\int_{t_1}^{t_2} e^{-iH_0(t_2-\tau)} \times H_I(\tau)T^\dagger(\tau)|\bar{\psi}^{(n)}(\tau)\rangle\, d\tau . \quad (44)$$

As for the first order, a consistent nth-order calculation (in H_I/ω_0) gives nothing but the transformed of the perturbation expression of $|\bar{\psi}(t_2)\rangle$.

We see, therefore, that the integral of Eq. (13), which leads to the compact explicit amplitudes (16) of the mt approximation for multiphoton processes, disappears when we try to improve the precision of the method by replacing $|\psi(\tau)\rangle$ by a more accurate expression in the integral equation (2).

However, one could reply that the integral of Eq. (13) represents an approximation of the exact solution which, although not perfect in the low-intensity region, represents a fit of this exact solution on a larger interval of variation of the intensity parameter a. But this important point seems never to have been demonstrated, and whether this is exact or not remains an open question.

VI. CONCLUSION

We have raised several objections concerning the mt approximation: (i) absence of well-known radiative effects; (ii) gauge invariance considerations; (iii) disagreement with an exact solution in a particular case; and (iv) inconsistencies in the perturbation treatment of H_I.

Therefore, in the present state of the question, we feel that the multiphoton amplitudes (16) cannot be considered as reliable.

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[1] H. R. Reiss, Phys. Rev. A 1, 803 (1970); Phys. Rev. D 4, 3533 (1971).

[2] H. R. Reiss, Phys. Rev. A 6, 817 (1972); N. K. Rahman and H. R. Reiss, Phys. Rev. A 6, 1252 (1972); S. P. Tewari, Phys. Rev. A 6, 1869 (1972); F. H. M. Faisal, J. Phys. B 5, L196 (1972); J. Phys. B 5, L 233 (1972); H. R. Reiss, Phys. Rev. Lett. 29, 1129 (1972).

[3] This gauge transformation has already been used by several authors; see, for example, M. Goeppert-Mayer, Ann. Phys. (Leipz.) 9, 273 (1931); J. Fiutack, Can. J. Phys. 41, 12 (1963).

[4] See, for instance, I. I. Gol'dman and V. D. Krivchenkov, *Problems in Quantum Mechanics* (Pergamon, New York, 1961), problem 3-13, pp. 103–106.

[5] This result is not surprising, since in classical mechanics, the response of a harmonic oscillator is important only for $\omega = \pm\omega_0$.

[6] The lack of multiphoton processes between $|\phi_0\rangle$ and $|\phi_1\rangle$ also appears on a perturbation treatment; for instance, we have at the lowest order and for $\omega_0 = 3\omega$, two amplitudes which interfere destructively,

$$|\phi_0, n+3\rangle \to |\phi_1, n+2\rangle \to |\phi_0, n+1\rangle \to |\phi_1, n\rangle,$$
$$|\phi_0, n+3\rangle \to |\phi_1, n+2\rangle \to |\phi_2, n+1\rangle \to |\phi_1, n\rangle,$$

(the $|\phi_p, n\rangle$ state represents the oscillator in state $|\phi_p\rangle$ with n photons). This destructive interference still exists at higher orders of perturbation.