Values of the Normal-Fluid Density and ^{3}He Inertial Mass in Dilute Solutions of ^{3}He in Superfluid ^{4}He†

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Values of the inertial mass of ^{3}He in superfluid ^{4}He calculated from experimental measurements of the velocity of second sound and osmotic pressure, and depending only on thermodynamic and hydrodynamic arguments and not on any microscopic model of ^{3}He quasiparticle interactions, are presented. These values cover the range of ^{3}He concentrations $1 < X < 6\%$ and temperatures $0.03 < T < 0.6$ K at saturated vapor pressure and at hydrostatic pressures of 10 and 20 atm.

In this comment we present the results of calculations of the ^{3}He inertial mass m_i in superfluid ^{4}He, m_i being defined by the relation $m_i = \rho_n / n_3$, where ρ_n is the normal-fluid density and n_3 is the ^{3}He number density. (In the range of temperature and ^{3}He concentration considered, the contribution to ρ_n from thermally excited phonons and rotons is negligible.) The calculations are based on the most recent experimental measurements of second-sound velocity[1] and osmotic pressure[2] and cover temperatures 30 mK $< T < 0.6$ K and ^{3}He atomic concentrations $0.13\% < X < 6.3\%$ at saturated vapor pressure (SVP) and at hydrostatic pressures $P = 10$ and 20 atm. In the regime where the two sets of experimental data overlap, namely $X > 1\%$, the calculated values of m_i are largely independent of any detailed model of ^{3}He quasiparticle interactions and thus should be of considerable value in the analysis of other experimental data, such as viscosity measurements, which actually measure the product $\rho_n \eta$.[3] In addition we give, at the same three pressures, the extrapolated values of m_i in the limit $T \to 0$, $X \to 0$, quantities which are of fundamental importance in microscopic theories of dilute ^{3}He-^{4}He solutions. Some of these results at SVP have already been published in a preliminary letter[4] where their implications for microscopic theories of the ^{3}He quasiparticle energy spectrum and interaction potential are also discussed.

In the low-frequency hydrodynamic limit an expression for the velocities of both first and second sounds has been given by Khalatnikov.[5] His theory depends only on thermodynamic and Galilean-invariance arguments and involves no assumptions about the ^{3}He quasiparticle interaction or spectrum. After making some approximations, the most important of which is to neglect the effect of thermal expansion, Khalatnikov gives an expression for u_2, which can be written in the form

$$m_4 u_2^2 = (1 - f\xi)\left\{ -\left(\frac{\partial \mu_4}{\partial \ln\xi}\right)_{T,P} + \frac{\xi T}{C_P}\left(\frac{\partial S}{\partial \ln \xi}\right)^2_{T,P}\right\} \Big/ \left(\frac{\rho_n}{\rho_s} + f^2\xi^2\right), \tag{1}$$

where $\xi = n_3/n_{40} = X/(1+\alpha X)$ and $f = 1 + \alpha - m_3/m_4$. In these equations ρ_s is the superfluid density, μ_4 is the ^{4}He chemical potential, S and C_P are the entropy and specific heat per ^{3}He atom, m_3 and m_4 are the ^{3}He and ^{4}He atomic masses, and $X = n_3/(n_3 + n_4)$, where n_4 is the ^{4}He number density. The quantity α is the fractional difference in the volume occupied by a ^{3}He atom and by a ^{4}He atom in the limit of small X at $T = 0$. Values of α, which is often referred to as the Bardeen-Baym-Pines[6] (BBP) parameter, are given in the recent review by Ebner and Edwards.[7] Note that Eq. (1) is obtained under the assumption that $n_3 = n_{40}X/(1+\alpha X)$, where n_{40} is the number density in pure ^{4}He so that thermal expansion, among other things has been neglected. The various approximations have been investigated and have been justified in this temperature and concentration regime.

Rearranging Eq. (1) to get an explicit expression for m_i gives

$$\frac{m_i}{m_3} = \frac{m_4}{m_3}\left(\frac{(1/\xi - f)F - m_4 \xi f^2 u_2^2}{F + m_4 u_2^2 (1 + f\xi)}\right) \tag{2}$$

where

$$F = -\left(\frac{\partial \mu_4}{\partial \ln \xi}\right)_{T,P} + \frac{\xi T}{C_P}\left(\frac{\partial S}{\partial \ln\xi}\right)^2_{T,P}. \tag{3}$$

The principal problem in using Eq. (2) to calculate m_i/m_3 from the measured values of u_2 is the evaluation of the function $F(T, P, \xi)$ given by Eq. (3). This involves differentiation of the experimentally determined osmotic pressure π with respect to concentration, since from Eq. (3) $F \simeq v_{40}(\partial \pi/$

TABLE I. Smoothed experimental values of the ^{3}He inertial mass m_i/m_3.

^{3}He atomic concentration X	Error range	Temperature (K) 0.03	0.06	0.10	0.20	0.40	0.60
		Saturated vapor pressure					
0.143%	a	(2.26)	(2.26)	(2.26)	(2.26)	(2.27)	(2.27)
0.355%	a	(2.27)	(2.27)	(2.28)	(2.29)	(2.30)	(2.30)
0.672%	a	(2.29)	(2.29)	(2.29)	(2.30)	(2.31)	(2.34)
1.313%	±2%	2.28	2.29	2.29	2.30	2.32	2.36
2.700%	±2%	2.20	2.21	2.22	2.24	2.28	2.33
4.473%	±2%	2.13	2.15	2.17	2.20	2.23	2.26
6.278%	±2%	2.14	2.16	2.17	2.18	2.19	2.20
		$p = 10$ atm					
0.138%	a	(2.48)	(2.48)	(2.48)	(2.48)	(2.48)	(2.48)
1.103%	±5%	2.53	2.57	2.58	2.59	2.61	2.64
5.762%	±5%	2.36	2.38	2.39	2.40	2.40	2.42
		$p = 20$ atm					
0.130%	a	(2.84)	(2.84)	(2.84)	(2.84)	(2.84)	(2.84)
1.039%	±5%	2.86	2.87	2.88	2.90	2.91	2.93
5.409%	±5%	2.57	2.60	2.63	2.64	2.68	2.74

[a] See text.

$\partial \ln\xi)_{S,P}$, where v_{40} is the molar volume of pure ^{4}He. Thus an analytic function representing the osmotic pressure must be found, and clearly serious systematic errors could arise from an inappropriately chosen function. We have attempted to get an estimate of the range of these systematic effects by analyzing the second-sound data with a variety of different functional representations of the experimental osmotic-pressure data.

To obtain reasonable functional forms $\pi(P,T,X)$ for the osmotic pressure, we made use of the semiempirical model of Landau *et al.*,[2] which was known to be capable of representing the experimental data to within the estimated experimental error (±0.4%) over the full range of X and T investigated. This model leads, at each pressure, to an expression for $\pi(T,X)$ containing ten arbitrary constants. We developed a Fortran least-squares data-fitting program in which each of these ten constants could be either fixed at any chosen value (including zero) or allowed to vary to give a best fit to the osmotic-pressure data. The same program then read in all of the second-sound data at the same pressure, and for each point calculated the value of m_i/m_3 using the exact form of Eq. (2). In this way we could easily investigate the systematic effects of different osmotic-pressure-fitting functions on the calculated m_i/m_3 values by allowing different subsets of the ten osmotic pressure-fitting parameters to vary.

In all cases it was found that the calculated values of m_i/m_3 were fairly insensitive to the functional form representing the osmotic-pressure data, as long as it was capable of giving a reasonable representation of that data. Osmotic-pressure fits with as few as five adjustable parameters were obtained and, in all cases used for the second-

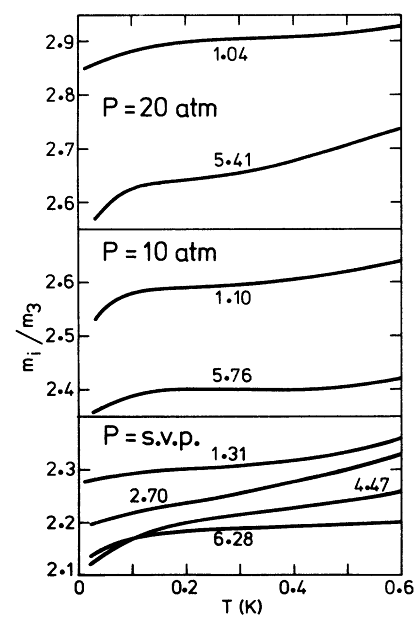

FIG. 1. Smoothed experimental values of the ^{3}He inertial mass m_i/m_3 as a function of temperature. The curves are labeled with ^{3}He atomic concentrations expressed as percentages.

TABLE II. Extrapolated values of the ^{3}He inertial mass m_i/m_3 in the limit $T \to 0$, $X \to 0$.

	svp	10 atm	20 atm
m_i/m_3	2.28 ± 0.04	2.57 ± 0.15	2.85 ± 0.15

sound data analysis, the rms deviation of the osmotic-pressure data from the fitted form was $\leq 1\%$.

In Table I we give smoothed values of the inertial masses m_i/m_3 calculated using the best fit to the osmotic-pressure data. This fit was made with five adjustable parameters and the rms deviation of the data was 0.4%. The error range quoted for the m_i/m_3 values embraces both the random experimental errors (typically $\sim\pm 1\%$) and the range of systematic deviations produced by different functional representations of the osmotic-pressure data. (The larger errors on the 10- and 20-atm values are a consequence of the much smaller amount of data available at these high pressures.)

Thus for ^{3}He concentrations $X > 1\%$, where both the second-sound velocity and the osmotic pressure have been measured, we have considerable confidence that these values of m_i/m_3 can be considered as experimentally determined quantities which are independent of any detailed model of ^{3}He quasiparticle interactions. These values are also shown graphically in Fig. 1.

For the values at lower concentrations, given in Table I in parentheses, the situation is not so clear. Here the calculation of m_i/m_3 is dependent on values of the osmotic-pressure derivative extrapolated to concentrations as much as an order of magnitude lower than have been measured. The resulting values of m_i/m_3 cannot therefore be considered as model independent and may not be correct. Our only reason for including them is that no other experimental values exist and these are the best that can be calculated from existing data.

Finally, in Table II we give the values of m_i/m_3 extrapolated to the limit of zero ^{3}He concentration and temperature. Again the error ranges quoted embrace both random experimental errors as well as systematic errors arising from the osmotic-pressure data-fitting procedure.

Note added in proof. Landau and Rosenbaum have recently measured the osmotic pressure for ^{3}He concentrations in the range $0.08\% \leq X \leq 0.6\%$; however their preliminary publication [J. Low Temp. Phys. 11, 483 (1973)] does not give any numerical values for π except at $T = 0$, so that we have been unable to use their data to calculate values of m_i from the velocity of second sound.

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