Hydrodynamics of Superfluid Helium below 0.6 °K. II. Velocity and Attenuation of Ultrasonic Waves*

Humphrey J. Maris†

Department of Physics, Brown University, Providence, Rhode Island 02912 ‡

School of Mathematics and Physics, University of East Anglia, Norwich, England

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The attenuation and velocity of an ultrasonic wave in superfluid helium-4 have been calculated for temperatures below 0.6°K. The calculation is based on the kinetic equations given by Khalatnikov. The phonon Boltzmann equation is solved numerically without making simplifying approximations about the form of the collision integral. The theory predicts that the velocity of sound increases with frequency for very low frequencies, passes through a maximum, decreases to a minimum, and finally increases towards a constant value at high frequencies. A simple explanation for this peculiar behavior is given. The theory is in good agreement with the experimental results of Abrahams *et al.*, and Waters *et al*.

I. INTRODUCTION

This paper presents a calculation of the attenuation and velocity of an ultrasonic wave in liquid helium-4 at temperatures below about 0.6 °K. Despite much effort in recent years,[1-21] a theoretical understanding of the attenuation and velocity in this temperature range has been lacking. At first sight this is very surprising. The only thermal excitations with which the sound wave can interact are long-wavelength phonons. The coupling between a sound wave and the phonons is determined by the Grüneisen constant, and this is known accurately.[22] Additional simplifying features of the helium problem are isotropy and the absence of defects or impurities. One important factor in these calculations is the form of the phonon-dispersion relation. Landau and Khalatnikov[23] approximated the dispersion relation for excitations in superfluid helium by

$$\epsilon^2 = A_1 p^2 + A_2 p^4 + A_3 p^6 + A_4 p^8 , \qquad (1)$$

where ϵ is the energy of an excitation of momentum p and A_1, A_2, A_3, and A_4 are constants. These constants were chosen so that Eq. (1) was a reasonable approximation to the dispersion relation in both the phonon and the roton regions. For phonons with small momentum, Eq. (1) reduces to

$$\epsilon = c_0 p(1 - \gamma_{LK} p^2 \ldots), \qquad (2)$$

where $\gamma_{LK} = 2.8 \times 10^{37}$ cgs units and $c_0 = 2.383 \times 10^4$ cm sec^{-1}. With this form of the dispersion relation, it is easy to use the conditions of conservation of energy and momentum to show that three-phonon collisions are unallowed. Recently, however, evidence has been accumulating[24-27] that suggests that the quantity γ_{LK} appearing in Eq. (2) must be negative. If this is true, the three-phonon process is allowed, and the theory of the attenuation and velocity must be substantially revised.

The results of a calculation of the attenuation and velocity have been briefly reported in a previous note,[27] assuming γ_{LK} is negative. This calculation used the kinetic equations for a superfluid as given by Khalatnikov.[28] The phonon Boltzmann equation included a realistic form for the collision term, and was solved numerically. While the results of this calculation were in good agreement with experiment, they did show some surprising features. For example, the velocity of sound had a more complicated frequency dependence than previously assumed. At very low frequencies the velocity increased with frequency, passed through a maximum, decreased to a minimum, and finally tended towards a constant value at high frequency. In this paper we present more complete quantitative calculations of the velocity and attenuation (Sec. II). The results of these calculations are in good agreement with experiment. In Sec. III we show that the peculiar features of the results mentioned above arise from a resonant interaction of the sound wave with second sound in the phonon gas. Finally, in Sec. IV we discuss some further experiments that might be performed to verify the theory.

II. CALCULATION OF ATTENUATION AND VELOCITY

Khalatnikov[28] has given the following kinetic equations for a superfluid containing excitations:

$$\vec{j} = \rho \vec{v}_s + \int \vec{p}\, n_p \, d\tau_p , \qquad (3)$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \vec{j} = 0, \qquad (4)$$

$$G = G_0 + \int \frac{\partial \epsilon}{\partial \rho} n_p \, d\tau_p, \tag{5}$$

$$\frac{\partial \vec{v}_s}{\partial t} + \nabla(G + \tfrac{1}{2}v_s^2) = 0, \tag{6}$$

where $\vec{j}$ is the momentum density, $\vec{v}_s$ is the superfluid velocity, n_p is the number of excitations of momentum $\vec{p}$, G_0 is the Gibb's free energy per unit mass at zero temperature, and G is the corresponding quantity when excitations are present. The distribution function satisfies the Boltzmann equation

$$\frac{\partial n_p}{\partial t} = \left(\frac{\partial n_p}{\partial t}\right)_{\text{coll}} + \frac{\partial n_p}{\partial \vec{p}} \cdot \frac{\partial H_p}{\partial \vec{r}} - \frac{\partial n_p}{\partial \vec{r}} \cdot \frac{\partial H_p}{\partial \vec{p}}, \tag{7}$$

where $H_p = \epsilon + \vec{p} \cdot \vec{v}_s$. The first term on the right-hand side represents the rate of change of n_p due to collisions between excitations.

Consider first the velocity of sound at zero temperature and in the absence of any applied pressure. Let the mean density be ρ_0. If we consider a small-amplitude wave propagating in the z direction, we may write

$$\rho = \rho_0 + \Delta\rho e^{i(2\pi k z - \Omega t)}, \tag{8}$$

$$\vec{v}_s = \hat{k} v_s e^{i(2\pi k z - \Omega t)}, \tag{9}$$

where $\hat{k}$ is a unit vector in the z direction. In the absence of excitations it is easy to solve Eqs. (3)–(7). We find

$$\Omega = 2\pi k c_0 \tag{10}$$

$$c_0^2 = \rho_0 \frac{\partial G_0}{\partial \rho} \tag{11}$$

$$v_s = c_0 \Delta\rho / \rho_0. \tag{12}$$

A phonon with very small momentum will travel at the sound velocity. Hence the c_0 appearing in Eqs. (10)–(12) must be the same as the c_0 occuring in Eq. (2).

Consider now what happens when the temperature is raised, but the mean density is held constant at ρ_0 by applying a suitable pressure. In the absence of a sound wave, the distribution function for the excitations is the Bose-Einstein function

$$n_p^0 = (e^{\beta\epsilon} - 1)^{-1}, \tag{13}$$

where $\beta = 1/k_B T$. When there is a wave propagating through the liquid, we can write

$$n_p = \tilde{n}_p + \Delta n_p e^{i(2\pi k z - \Omega t)}, \tag{14}$$

where $\tilde{n}_p$ is defined as

$$\tilde{n}_p = \{\exp[\beta(\epsilon + \vec{p} \cdot \vec{v}_s)] - 1\}^{-1}. \tag{15}$$

In this definition of $\tilde{n}_p$ it is intended that local values should be used for ϵ and $\vec{v}_s$. When a wave is present, the energy of an excitation varies with position because ϵ depends on the local density. For a small-amplitude wave we may expand $\tilde{n}_p$ as a power series in $\Delta\rho$ and $\vec{v}_s$. To first order we find

$$\tilde{n}_p = n_p^0 - \beta n_p^0(n_p^0 + 1)\left(\frac{\partial \epsilon}{\partial \rho} \Delta\rho + p\cos\theta \, v_s\right) e^{i(2\pi k z - \Omega t)}, \tag{16}$$

where θ is the angle between $\vec{p}$ and the z axis. We now substitute this result and Eq. (14) into Eqs. (3)–(6). If the terms involving the excitations are small, one can use perturbation theory to calculate the corrected velocity and the attenuation. The result is

$$\alpha = -\frac{\rho_0 \Omega}{2c_0^3} \int \left(\frac{\partial \epsilon}{\partial \rho} + \frac{p c_0 \cos\theta}{\rho_0}\right) \text{Im}\left(\frac{\Delta n_p}{\Delta \rho}\right) d\tau_p, \tag{17}$$

$$\begin{aligned}\frac{\Delta c}{c_0} = {} & \frac{\rho_0}{2c_0^2} \int \left(\frac{\partial \epsilon}{\partial \rho} + \frac{p c_0 \cos\theta}{\rho_0}\right) \text{Re}\left(\frac{\Delta n_p}{\Delta \rho}\right) d\tau_p \\ & - \frac{\beta\rho_0}{2c_0^2} \int \left(\frac{\partial \epsilon}{\partial \rho} + \frac{p c_0 \cos\theta}{\rho_0}\right)^2 n_p^0(n_p^0 + 1) \, d\tau_p \\ & + \frac{\rho_0}{2c_0^2} \int \frac{\partial^2 \epsilon}{\partial \rho^2} n_p^0 \, d\tau_p. \end{aligned} \tag{18}$$

The velocity of sound at *zero applied pressure* can be found by adding to the right-hand side of Eq. (18) a term to allow for thermal expansion. This term is just

$$\left(\frac{\Delta c}{c_0}\right)_{\text{exp}} = \frac{1}{c_0} \frac{\partial c_0}{\partial \rho} (\rho_T - \rho_0), \tag{19}$$

where ρ_T is the density at temperature T. It is straightforward to show that

$$\frac{\rho_T - \rho_0}{\rho_0} = -\frac{1}{c_0^2} \int \left(\frac{\partial \epsilon}{\partial \rho} + \frac{1}{3} \frac{\epsilon}{\rho}\right) n_p^0 \, d\tau_p. \tag{20}$$

The temperature and frequency dependence of the velocity correction Δc have been measured by Whitney and Chase,[29] Abraham *et al.*,[30] and most recently by Roach *et al.*[31] The most-accurate measurements appear to be those of Roach *et al.*[31] These data were taken at constant density. This paper will therefore concentrate on calculating $\Delta c / c_0$ at constant density, as given by Eq. (18).

The problem now reduces to solving the Boltzmann equation to find Δn_p. In a linearized theory we may write the collision term as

$$\int C(\vec{p}, \vec{p}\,') \Delta n_{p'} \, d\tau_{p'}. \tag{21}$$

We then find

$$\Delta n_p(\Omega - 2\pi k v_p \cos\theta) - i \int C(\vec{p}, \vec{p}\,') \Delta n_{p'} \, d\tau_{p'} = \beta\Omega\left(\frac{\partial \epsilon}{\partial \rho} \Delta\rho + p\cos\theta \, v_s\right) n_p^0(n_p^0 + 1), \tag{22}$$

where

$$\vec{v}_p = \frac{\partial \epsilon}{\partial \vec{p}}, \quad (23)$$

$\vec{v}_p$ is the group velocity of an excitation with momentum $\vec{p}$. If the correction to the velocity of sound is small, we may still use Eq. (12) to relate v_s to $\Delta\rho$. Similarly, we may set[32]

$$2\pi k \approx \Omega/c_0 . \quad (24)$$

Then,

$$\Delta n_p\left(1 - \frac{v_p \cos\theta}{c_0}\right) - \frac{i}{\Omega}\int C(\vec{p}, \vec{p}\,')\Delta n_{p'}\, d\tau_{p'} = \beta\Delta\rho\left(\frac{\partial\epsilon}{\partial\rho} + \frac{p c_0 \cos\theta}{\rho_0}\right) n_p^0(n_p^0 + 1). \quad (25)$$

We now have to make some assumptions about the form of the phonon-dispersion relation and the kernel of the collision integral. At the temperatures of interest the only thermal excitations present in the liquid are long-wavelength thermal phonons. In a recent paper on the normal-fluid viscosity[33] (hereafter referred to as I), the following approximation to the phonon-dispersion relation has been used:

$$\epsilon = c_0 p\left(1 + \gamma p^2 \frac{1-(p/p_A)^2}{1+(p/p_B)^2}\right), \quad (26)$$

where γ is positive. Two sets of values of the constants γ, p_A, and p_B are listed in Table I. These sets of constants give dispersion curves which we call C and D. Both dispersion curves are consistent with the neutron-scattering results of Woods and Cowley,[34] the specific-heat data of Phillips *et al.*,[25] and the viscosity measurements of Whitworth.[35] In Fig. 1 it is shown how the phonon phase velocity $c(\equiv \epsilon/p)$ and the group velocity vary with wave number $q(\equiv p/\hbar)$. In Paper I, an expression was also given for the kernel of the collision integral. Only three-phonon processes were considered. The kernel has the property that all the collisions are small angle (less than 25°). There are a number of approximations in the derivation of the collision kernel; these are discussed in detail in I.

Consider first the solution of the Boltzmann equation for very low frequencies. In this limit the system of thermal phonons will always be in local equilibrium and will be characterized by a local temperature $T+\Delta T$, and a local drift velocity $\vec{v}_n$. Thus,

$$n_p = \left(\exp\frac{\epsilon + \vec{p}\cdot(\vec{v}_s - \vec{v}_n)}{k_B(T+\Delta T)} - 1\right)^{-1}. \quad (27)$$

Hence

$$\Delta n_p = \beta\epsilon n_p^0(n_p^0+1)\left(\frac{\Delta T}{T} + \frac{\vec{p}\cdot\vec{v}_n}{\epsilon}\right). \quad (28)$$

Energy is conserved in collisions between phonons. Therefore, if we multiply the Boltzmann equation by ϵ and integrate over all $\vec{p}$, the collision term vanishes. We are left with

$$\frac{\Delta T}{T} - \frac{v_n}{3c_0} - \frac{\Delta\rho}{\rho_0}u_0 = 0. \quad (29)$$

We have used the approximations

$$\epsilon \approx c_0 p \quad (30)$$

and

$$\frac{\partial\epsilon}{\partial\rho} \approx \frac{\epsilon}{\rho}u_0, \quad (31)$$

where u_0 is the small-momentum limit of the Grüneisen constant, given by

$$u_0 \equiv \frac{\rho}{c_0}\frac{\partial c_0}{\partial\rho}. \quad (32)$$

A similar calculation may be made using the conservation of momentum. One obtains

$$-\frac{\Delta T}{T} + \frac{v_n}{c_0} - \frac{\Delta\rho}{\rho_0} = 0. \quad (33)$$

Here we have assumed $v_p \approx c_0$. Using (29) and (33), we find

$$\Delta n_p = \tfrac{3}{2}(\Delta\rho/\rho_0)\beta\epsilon n_p^0(n_p^0+1)[u_0 + \tfrac{1}{3} + (u_0+1)\cos\theta]. \quad (34)$$

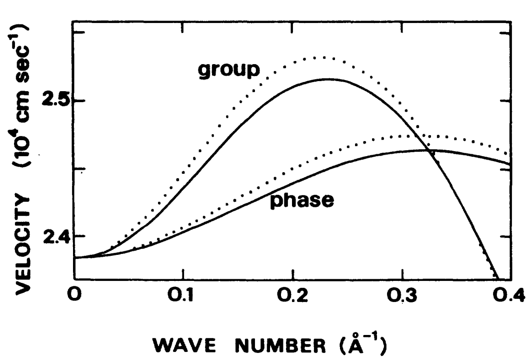

FIG. 1. Phonon group velocity v_p and phase velocity c as a function of wave number. Dispersion curve C, solid line; dispersion curve D, dotted line.

TABLE I. Parameters defining the dispersion curves C and D.

	γ (10^{37} g^{-2} cm^{-2} sec^2)	$p_A/\hbar$ (Å^{-1})	$p_B/\hbar$ (Å^{-1})
C	8	0.5384	0.3727
D	10	0.5418	0.3322

We can now calculate the correction to the velocity of sound. If we use approximation (30) again, the result may be expressed in terms of the normal-fluid density ρ_n, given by[28]

$$\frac{\rho_n}{\rho_0}=\frac{2\pi^2 k_B^4 T^4}{45\hbar^3 c_0^5 \rho_0}=1.22\times 10^{-4}\,T^4\,. \tag{35}$$

Then[3,4]

$$\frac{\Delta c}{c_0}=\frac{3\rho_n}{4\rho_0}\,(u_0^2+2u_0+\tfrac{1}{3}+\tfrac{1}{2}w_0), \tag{36}$$

where we have set

$$\frac{\rho^2}{\epsilon}\,\frac{\partial^2\epsilon}{\partial\rho^2}\approx\frac{\rho^2}{c_0}\,\frac{\partial^2 c_0}{\partial\rho^2}\equiv w_0\,. \tag{37}$$

Abraham *et al.*[22] have measured u_0 and w_0. At zero pressure they find

$$u_0=2.84, \quad w_0=0.19. \tag{38}$$

Thus, in the low-frequency limit,

$$\Delta c/c_0=1.30\times 10^{-3}\,T^4\,. \tag{39}$$

The attenuation is zero in this limit.

Consider now the high-frequency limit. The collision term may be neglected. Then

$$\Delta n_p=\beta\epsilon n_p^0(n_p^0+1)\,\frac{\Delta\rho}{\rho_0}\,\frac{u_0+\cos\theta}{1-v_p\cos\theta/c_0}\,. \tag{40}$$

The correction to the velocity of sound is

$$\frac{\Delta c}{c_0}=(3\rho_n/4\rho_0)[\,(u_0+1)^2 I-2u_0^2-4u_0-\tfrac{8}{3}+\tfrac{1}{2}w_0\,], \tag{41}$$

where

$$I=\frac{15}{16\pi^4}\int_0^\infty \ln\left|\frac{2}{1-v(x)}\right| x^4\sinh^{-2}(x/2)\,dx\,. \tag{42}$$

$v(x)$ is the group velocity of an excitation of energy xk_BT, expressed in units of c_0. The derivation of this result uses the approximations (30), (31), and (37). The attenuation calculation is simpler. For the dispersion curves corresponding to the parameters listed in Table I, the group velocity is greater than c_0 for all phonons of energy up to about 7 °K (see Fig. 1). Thus, at temperatures below 0.6 °K, nearly all the thermal phonons will have $v_p>c_0$. It is then straightforward to show that the high-frequency limit of the attenuation is

$$\alpha=\frac{3\pi(u_0+1)^2\Omega\rho_n}{4c_0\rho_0}\,. \tag{43}$$

In the intermediate-frequency regime it is necessary to solve the Boltzmann equation numerically. We have used a simple iterative method. First, we separate the kernel $C(\vec{p},\vec{p}\,')$ into diagonal and nondiagonal parts:

$$C(\vec{p},\vec{p}\,')=C^{ND}(\vec{p},\vec{p}\,')-\Gamma(p)\delta(\vec{p}-\vec{p}\,') \tag{44}$$

$\Gamma(p)$ is the reciprocal of the lifetime of a phonon with momentum $\vec{p}$. We now write the Boltzmann equation as

$$\Delta n_p=\frac{\beta\epsilon(\Delta\rho/\rho_0)n_p^0(n_p^0+1)(u_0+\cos\theta)+(i/\Omega)[\int C^{ND}(\vec{p},\vec{p}\,')\,\Delta n_{p'}d\tau_{p'}-(1-f)\Gamma(p)\Delta n_p]}{1-v_p\cos\theta/c_0+(if/\Omega)\Gamma(p)}\,. \tag{45}$$

This is true for all values of the constant f. We now iterate the equation to find a solution for Δn_p. For $f=1$, the iteration sequence converges, but slowly. If f is decreased, the rate of convergence increases, but for too small values of f (~0.6 or less), oscillations occur and the sequence eventually diverges. There is a simple way to test for convergence. The right-hand side F_p of Eq. (25) plays the role of "driving force" in the Boltzmann equation. Suppose that after i iterations the approximation to the distribution function is Δn_p^i, whereas the exact solution is Δn_p^∞. We now calculate the left-hand side of the Boltzmann equation using Δn_p^i and call the result F_p^i. We may regard Δn_p^i as the *exact* solution to a Boltzmann equation having the same left-hand side as Eq. (25) but with a driving force F_p^i. Now if F_p and F_p^i differ only slightly for all $\vec{p}$, we expect that Δn_p^i will be a good approximation to Δn_p^∞. Iteration was usually continued until F_p^i and F_p were equal to an accuracy of 1% for all $\vec{p}$. The number of iterations required varied considerably, being only two at the highest frequencies and lowest temperatures but several hundred at the lowest frequencies and highest temperatures. To perform the integration over $\vec{p}\,'$ in Eq. (45), momentum space was first divided into shells such that the energy on shell j was $j\Delta\epsilon$ where $j=1,\ldots,j_{max}$. Most of the calculations were performed using $j_{max}=8$ and $\Delta\epsilon=2k_BT$. Since we consider only a longitudinal sound wave, Δn_p only depends on p and θ, and is independent of ϕ. The range of θ was divided into 36 cells. These were $\frac{1}{2}°$ wide near $\theta=0$ but increased to 15° wide near $\theta=\pi$. This arrangement of the cells was chosen because the distribution function varies rapidly with angle near $\theta=0$, particularly when the sound-wave frequency is high. We have looked very carefully into the effects of the finite mesh of points on the results of the calculation. Two essentially different sorts of errors are introduced. The first of these is the problem at high frequencies already mentioned. This is not very serious, however, since the mesh appears to be adequate up to a frequency where the attenuation and the velocity correction have values that are within 10% of the high-frequency limits derived analytically. The second error is more subtle and occurs at low frequencies. It was possible to choose the mesh approximation to the collision integral so that energy was conserved exactly.[36] It was not possible, however, to ensure that momentum was also exactly conserved. Consider, for example, what happens in the decay of a particle whose momentum vector is in the z direction. Suppose that the momentum of one of the product phonons is at an

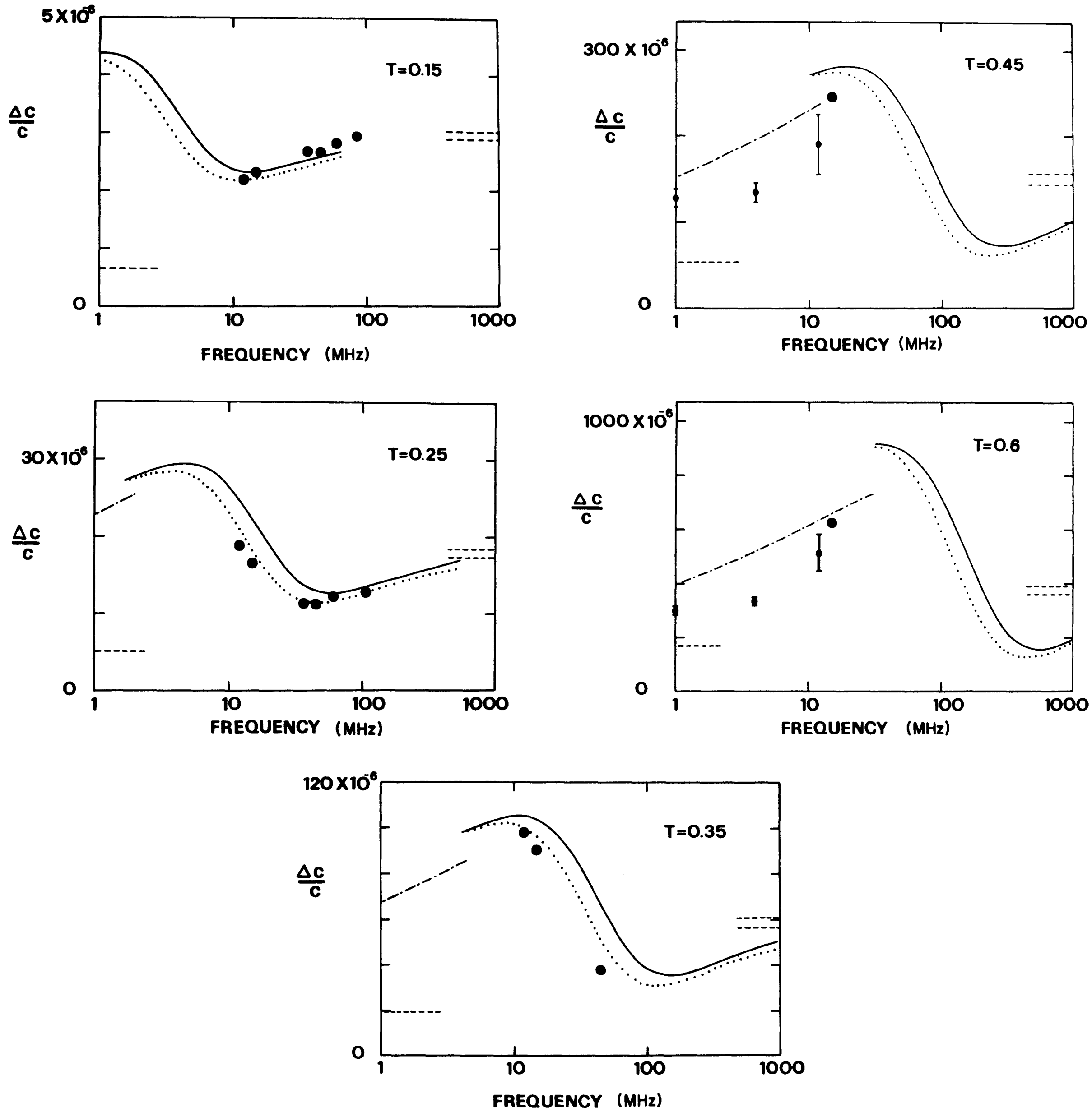

FIG. 2. Frequency dependence of the velocity of sound at five temperatures. Results obtained by the iteration method using dispersion curves C and D are denoted by the solid and dotted lines, respectively. Results obtained by the mode-coupling theory are indicated by the dot-dash line. The high- and low-frequency limiting values of the velocity are denoted by the dashed lines. Experimental points are the measurements of Roach *et al.* (Ref. 31), ●; and Whitney and Chase (Ref. 29), ⫯.

angle θ_1, and that the nearest two θ cells to θ_1 are i and j, with centers at θ_i and θ_j ($\theta_j \geq \theta_1 \geq \theta_i$). This decay has been divided into a fraction

$$(\theta_j - \theta_1)/(\theta_j - \theta_i)$$

into cell i and

$$(\theta_1 - \theta_i)/(\theta_j - \theta_i)$$

into cell j. The effect of this approximation was tested by setting the distribution function proportional to $\cos\theta$ and calculating the rate of change of the z component of momentum P_z. We may then define a "relaxation time" τ_p by

$$\tau_p = -\frac{P_z}{\partial P_z/\partial t}\,. \tag{46}$$

For the mesh used in these calculations, this time is about -9 μsec at 0.35 °K. The minus sign implies that the mesh approximation artificially amplifies momentum in the system. This error becomes more important at low sound-wave frequencies. It is clear that completely spurious results would be obtained if τ_p were comparable to the period of the sound wave. In the present context it is not a serious problem above about 2 MHz at 0.25 °K and 10 MHz at 0.45 °K. Above these frequencies we estimate that uncertainties in α and $\Delta c/c_0$ resulting from the use of a finite mesh of points in momentum space are always less than 15%, and usually only (5–10)%. These figures were arrived at by checking some of the calculations with a finer mesh.

Calculations of the velocity shift and the attenuation were made for temperatures of 0.15, 0.25, 0.35, 0.45, and 0.6 °K. Except in cases where there were convergence or mesh problems, results were obtained for 30 frequencies between 1 and 1000 MHz. The calculations were carried out using the dispersion curves C and D, whose parameters are listed in Table I. The results are shown by the solid and the dotted lines in Figs. 2 and 3. The experimental points shown in Fig. 2 are the velocity measurements of Roach *et al.*[31] and of Whitney and Chase.[29] The measurements of Whitney and Chase were made at the vapor pressure and we have corrected them to allow for thermal expansion. The attenuation measurements are those of Roach *et al.*,[37] Abraham *et al.*,[30] and Waters *et al.*[38] In some cases we have calculated the points shown by making a T^4 extrapolation from the nearest temperature at which a measurement is available. Unfortunately, a detailed error analysis is available only for the data of Whitney and Chase. This error is indicated by the height of the vertical bar on their data points. Figures 2 and 3 also include some theoretical results obtained by a "mode-coupling" theory to be described in the next section.

III. A PHYSICAL PICTURE

The theoretical results for $\Delta c/c_0$ have a surprisingly complicated dependence on frequency. Instead of passing monotonically between its limiting values at low and high frequencies, the velocity passes through a large maximum and then a minimum. We propose here a simple explanation of this effect.

The plots of velocity versus frequency have a shape which suggests that some sort of resonance

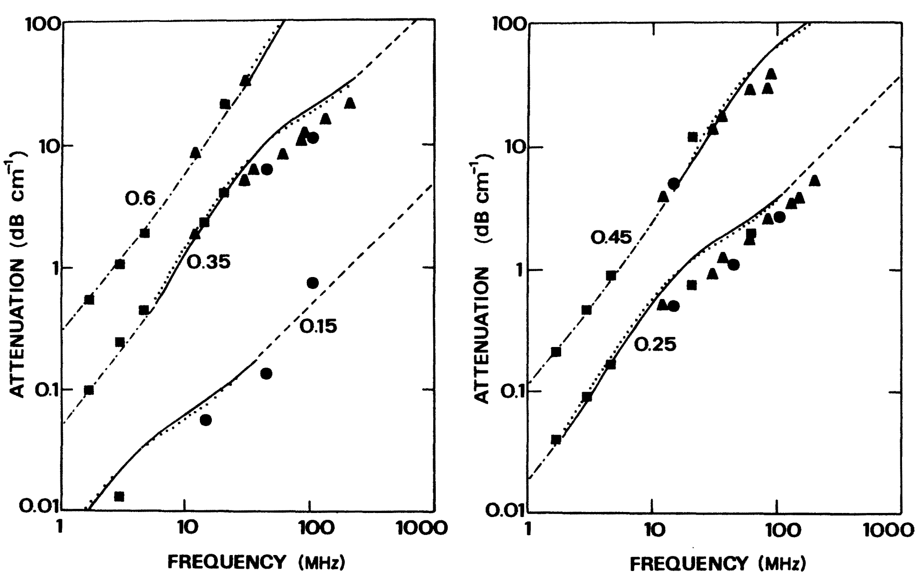

FIG. 3. Frequency dependence of the ultrasonic attenuation at five temperatures. Results obtained by the iteration method using dispersion curves C and D are denoted by the solid and dotted lines, respectively. Results obtained by the mode-coupling theory are indicated by the dot-dash line. The high-frequency limit of the attenuation is indicated by the dashed line. The experimental points are the measurements of Abraham *et al.* (Ref. 30), ▲; Waters *et al.* (Ref. 38), ■; and Roach *et al.* (Ref. 37), ●.

is occurring, and that the resonant frequency increases with increasing temperature. This idea is confirmed by looking at the attenuation results. At each temperature these show a bump at the frequency where the velocity results suggest a resonance is occurring. If we explore this idea further, we find a surprising result. Suppose that the phonon system has a resonant mode at frequency Ω_R with a damping constant Ω_I. If this mode is weakly coupled to the sound wave, one would expect that the dispersion relation for the sound wave should be as shown in Fig. 4a. As the frequency Ω of the sound approaches Ω_R, level repulsion occurs and the velocity of sound decreases. When Ω is greater than Ω_R, the velocity is raised. This is the wrong way round to explain the results we have obtained. One must postulate that there exists some mode of the phonon system whose frequency depends on wave number as shown in Fig. 4b. Level-repulsion arguments then give a correction to the velocity of sound in agreement with the computer results. We will now show that the mode responsible for this effect is second sound.

In the Sec. II conservation laws were used to obtain the low frequency limiting value for $\Delta c/c_0$ [Eq. (36)]. It is possible to derive an equivalent result from the two-fluid equations of motion. One finds[39]

$$c=(B_A/\rho_0)^{1/2}[1+\tfrac{1}{2}(r-1)c_2^2/(c_0^2-c_2^2)], \tag{47}$$

where r is the ratio of the specific heats, B_A is the adiabatic bulk modulus, and c_2 is the velocity

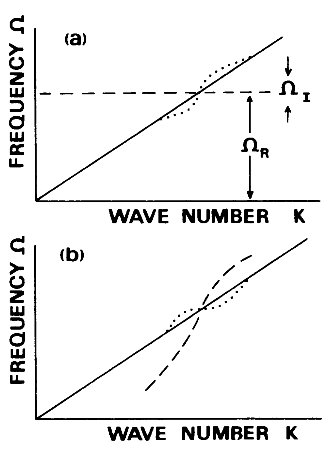

FIG. 4. Frequency Ω vs wave number K for a sound wave interacting with (a) a resonant mode of constant frequency $\Omega_R + i\Omega_I$ and (b) a mode of the phonon system whose frequency increases with wave number K as indicated by the dashed line. The sound-wave dispersion relation $\Omega = 2\pi c_0 K$ in the absence of interaction is shown by the solid line. The effect of a weak-level repulsion is indicated by the dotted line.

of second sound. It is straightforward to show that to first order in ρ_n/ρ_0

$$r=1+(3\rho_n/\rho_0)(u_0+\tfrac{1}{3})^2, \tag{48}$$

$$B_A=\rho_0 c_0^2[1+(\rho_n/8\rho_0)(8u_0+\tfrac{4}{3}+3w_0)]. \tag{49}$$

When the thermal excitations are phonons with velocity c_0, the second-sound velocity is given by[40]

$$c_2=c_0/\sqrt{3}. \tag{50}$$

Using these results, it is easy to verify that the two-fluid result for the velocity of sound [Eq. (46)] is equivalent to the result derived earlier [Eq. (35)]. Consider now the term in Eq. (46):

$$\tfrac{1}{2}(r-1)c_2^2/(c_0^2-c_2^2). \tag{51}$$

This is clearly the contribution to $\Delta c/c_0$ from the interaction between first and second sound.

It remains to be shown that the velocity of second sound depends on wave number in the way indicated in Fig. 4b. The derivation of the result (50) assumes that the frequency of the second sound is sufficiently small that the phonon distribution is always in local equilibrium. In this regard it is essential to distinguish between the small-angle collision time $\tau_\parallel$ and the large-angle time $\tau_\perp$. Equilibrium between phonons traveling at large angles to each other is established only as a result of a large number of small-angle collisions.[33] Thus,

$$\tau_\perp \gg \tau_\parallel . \tag{52}$$

The low-frequency limit (50) applies when

$$\Omega\tau_\perp \ll 1. \tag{53}$$

Recently, it has been shown[41] that when

$$\Omega\tau_\perp \gtrsim 1, \tag{54}$$

the velocity of second sound does indeed increase with increasing frequency. Results of a calculation of this type at 0.35 °K are shown in Fig. 5. The theory as developed so far is only valid if

$$\Omega\tau_\parallel < 1. \tag{55}$$

At 0.35 °K the lifetime of a $4k_BT$ energy phonon is 3.5×10^{-8} sec. Thus, the theory is valid for frequencies of a few MHz or less. Unfortunately, it appears from Fig. 2 that at 0.35 °K the real part of c_2 must become equal to c_0 at a frequency around 30 MHz, and this is therefore beyond the range of the theory. Nevertheless, the results shown in Fig. 5 provide strong evidence that the physical picture we propose is correct. Since second sound is a collective oscillation of the phonon gas, it is clear that it will be heavily damped when $\tau_\parallel$ becomes equal to, or somewhat greater than 1. This is consistent with the fact that the minimum in the first sound velocity is not very pronounced. For $\Omega\tau_{\parallel} \gg 1$, collective oscillations cannot exist, and it makes more sense to think of the sound wave as interacting with individual thermal phonons.

This picture is confirmed by inspecting the details of the phonon distribution function for different sound-wave frequencies. In Fig. 6 we show the angular dependence of the real and imaginary parts of Δn_p at 0.35 °K for frequencies of 1, 10, 100, and 1000 MHz. These results are for phonons of energy $4k_BT$. We have set $\Delta\rho/\rho_0 = 1$ in Eq. (25) and have used curve C for the phonon dispersion. The results at 10, 100, and 1000 MHz were obtained by the iteration method described in the last section. The 1-MHz results were calculated using an alternative approach described below. Between 1 and 10 MHz the amplitude of the real part of the distribution function increases because of the approach to resonance. At 100 MHz resonance has been passed and the sign of the real part of the distribution has changed. Finally, at 1000 MHz the distribution approaches the high-frequency limiting form [Eq. (40)] in which the sound wave interacts with individual phonons.

It is not legitimate to use Eq. (47) to make quantitative calculations of $\Delta c/c_0$, except in the low-frequency limit $\Omega\tau < 1$. This is because the derivation of this result implicitly assumes that the angular dependence of the distribution function for a second-sound wave is

$$\cos\theta + 1/\sqrt{3} \ .$$

This is only true for $\Omega\tau_{\perp} < 1$, and becomes a very bad approximation as $\Omega\tau_{\parallel}$ approaches unity. Another difficulty is that for $\Omega\tau_{\perp} > 1$, extra propagating collective excitations of the phonon gas appear.[41] These are also coupled to first sound and contribute to the velocity correction. We consider these problems in the Appendix and derive an expression analagous to the two-fluid result [Eq. (47)], but valid throughout the $\Omega\tau_{\parallel} < 1$ regime. This "mode-coupling" result provides a convenient method for calculating the attenuation and velocity in the low-frequency regime, since it avoids the iteration and momentum-conservation problems of the straightforward iteration method. Results obtained this way are included in Figs. 2 and 3. It can be seen that the two methods are in agreement to within (10–15)%. Note that in the low-frequency regime the attenuation and velocity are nearly independent of which dispersion relation (C or D) is assumed. The results obtained by the mode-coupling theory have therefore been represented by a single curve.

IV. DISCUSSION

The theory is in good agreement with the available experimental measurements of the velocity correction $\Delta c/c_0$. Although the experimental results are far from complete, most of the features predicted by the theory have been observed. The velocity measurements at 0.15 °K confirm the increase in $\Delta c/c_0$ with frequency in the high-frequency range. The data at 0.25 and 0.35 °K fall in the range near the resonance, and confirm the rapid decrease of $\Delta c/c_0$ in this region. The evidence for the increase of velocity with frequency

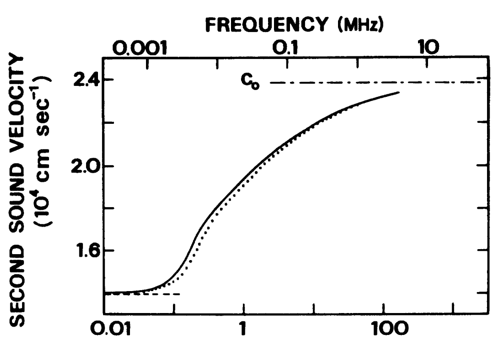

FIG. 5. Velocity of second sound as a function of wave number at 0.35 °K. The calculations were performed using dispersion curves C (solid line) and D (dotted line). The frequencies indicated on the upper horizontal axis correspond to $2\pi c_0$ times the wave number. The low-frequency limit of c_2 is denoted by the dashed line.

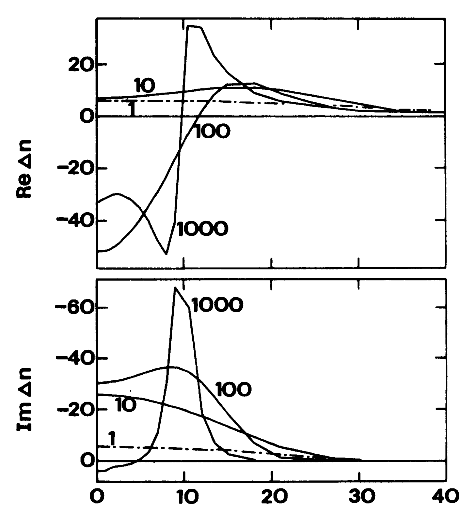

FIG. 6. Angular dependence of the phonon distribution function for $4k_BT$ energy phonons at 0.35 °K. The curves are labeled by the frequency in MHz. Results obtained by the iteration method are denoted by solid line; results obtained by the mode-coupling theory by dot-dash line.

at low frequencies is not so strong. Unfortunately, at temperatures above 0.4 °K Roach *et al.* have made velocity measurements at 15 MHz only. At this frequency their 0.45 and 0.6 °K results are in good agreement with our theory. The only other results are those of Whitney and Chase, who have made measurements at 1, 3.9, and 11.9 MHz. Their results fall considerably below the theory and show almost no frequency dependence between 1 and 3.9 MHz. Further experiments in this frequency range are clearly desirable.

In general, the attenuation results are also in good agreement with experiment. The only discrepancies occur in the frequency range around the resonance and just above. Here the experimental results are as much as 40% below the theory. It seems likely that the resonance between first and second sound actually occurs at a somewhat lower frequency than the theory predicts. This would move the bump on the attenuation curves to lower frequencies and give better agreement with experiment. The same conclusion is suggested by the velocity results, especially at 0.35 °K (Fig. 2). It is very difficult to determine why the theory overestimates the resonant frequency. The results for dispersion curve D give better agreement with experiment in this regard. At first sight this suggests that we should increase the dispersion parameter γ somewhat (see Table I). However, this would worsen the agreement with theory at high frequencies and low temperatures (see Fig. 2 for 0.15 °K).

The attenuation and velocity under pressure have been measured by Roach *et al.*[31,37] We have not attempted to perform quantitative calculations to compare with their results. The difficulty is that there are considerable uncertainties in the phonon-dispersion relation for helium under pressure. The dispersion relations C and D used in this paper are consistent with specific-heat, neutron-scattering, and viscosity measurements. Unfortunately, there are no viscosity results under pressure available, and the only neutron data are at 24 atm.[42] Thus, any assumption about the precise form of the phonon-dispersion relation must be regarded as speculative at the moment.

Finally, we mention that we have not considered boundary scattering of the thermal phonons in this calculation. This problem has been discussed by Dransfeld,[9] who assumed that the dispersion curve had the form proposed by Landau and Khalatnikov [see Eq. (2)]. With this form of the dispersion curve the three-phonon process is unallowed, and the phonon mean-free-path is long. It is therefore reasonable to expect that boundary scattering may be important. For the dispersion curve that we have assumed, the three-phonon process is allowed and the thermal-phonon mean free path Λ is generally much smaller than the dimensions of a typical ultrasonic cell. At 0.2 °K, for example, Λ is only 10^{-2} cm for $4k_BT$ phonons. However, since Λ varies as T^{-5}, boundary scattering should become important below 0.1 °K. Roach *et al.* have made only a few measurements of $\Delta c/c_0$ in this temperature range,[31] and have obtained no results for the attenuation. It would be very interesting to make a detailed experimental study below 0.1 °K to see if the attenuation and velocity depend on the sample size.

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APPENDIX

Consider the Boltzman equation [Eq. (7)] when there is no sound wave present. We look for wave solutions of the form[41]:

$$n_p = n_p^0 + (n_p^0)^{1/2}(n_p^0+1)^{1/2}\,\phi(\vec{p})e^{i(2\pi kz-\Omega t)}\,. \tag{A1}$$

Then

$$\phi(\vec{p})(\Omega - 2\pi k v_p \cos\theta) - i\int \tilde{C}(\vec{p},\vec{p}\,')\phi(\vec{p}\,')\,d\tau_{p'} = 0, \tag{A2}$$

where

$$\tilde{C}(\vec{p},\vec{p}\,') = C(\vec{p},\vec{p}\,')\left(\frac{n_{p'}^0(n_{p'}^0+1)}{n_p^0(n_p^0+1)}\right)^{1/2}. \tag{A3}$$

$\tilde{C}$ is symmetric in $\vec{p}$ and $\vec{p}\,'$. Let the values of Ω for which nontrivial solutions of Eq. (A2) exist be Ω_α, and let $\phi_\alpha(\vec{p})$ be the corresponding solutions. These functions will be orthogonal, and we can choose them to be normalized so that

$$\int \phi_\alpha^2(\vec{p})\,d\tau_p = 1.$$

We look for a solution of the full Boltzmann equation (25) in the form of a linear combination of these modes. Let

$$\Delta n_p = (n_p^0)^{1/2}(n_p^0+1)^{1/2}\sum_\alpha B_\alpha\,\phi_\alpha(\vec{p}). \tag{A4}$$

Therefore,

$$\sum_\alpha B_\alpha\,\phi_\alpha(\vec{p})(\Omega - \Omega_\alpha) = \beta\Omega\epsilon(n_p^0)^{1/2}(n_p^0+1)^{1/2} \times (u_0+\cos\theta)\Delta\rho/\rho_0 \tag{A5}$$

and

$$B_\alpha = \beta\Omega \frac{\Delta\rho}{\rho_0} \frac{1}{\Omega - \Omega_\alpha} \int \epsilon (n_p^0)^{1/2} (n_p^0 + 1)^{1/2} \times (u_0 + \cos\theta) \phi_\alpha(\vec{p}) d\tau_p . \quad \text{(A6)}$$

It follows that the most important modes to consider are those with frequencies close to Ω. For modes whose frequencies Ω_α satisfy the condition $\Omega_\alpha \tau_\parallel < 1$, we may write[41] $\phi_\alpha(\vec{p})$ as a linear combination of the nodeless eigenfunctions $\psi(\vec{p})$ of the operator $\tilde{C}$. These eigenfunctions can be labeled by an "angular momentum" l and a "helicity" m. Thus,

$$\phi_\alpha(\vec{p}) = \sum_{lm} A_{lm}^\alpha \psi_{lm}(\vec{p}). \quad \text{(A7)}$$

We have calculated the frequencies Ω_α and the expansion coefficients A_{lm}^α by the method described previously.[41] Using these results, we can find B_α from Eqs. (A6) and (A7). Equation (A4) can then be used to give Δn_p. The result for Δn_p can be substituted into Eqs. (17) and (18) to give the attenuation and the velocity. It is also possible to derive explicit expressions for α and $\Delta c/c_0$ in terms of the frequencies Ω_α and the A_{lm}^α coefficients. The derivation is straightforward and the result is

$$\alpha = -\frac{3\Omega\rho_n}{2\rho_0 c_0} \sum_\alpha \mathrm{Im}\left(\frac{\Omega}{\Omega - \Omega_\alpha} (u_0 A_{00}^\alpha + A_{10}^\alpha/\sqrt{3})^2\right), \quad \text{(A8)}$$

$$\frac{\Delta c}{c_0} = \frac{3\rho_n}{2\rho_0}\left[\sum_\alpha \mathrm{Re}\left(\frac{\Omega}{\Omega - \Omega_\alpha} (u_0 A_{00}^\alpha + A_{10}^\alpha/\sqrt{3})^2\right) - u_0^2 - \frac{1}{3} + \frac{w_0}{4}\right]. \quad \text{(A9)}$$

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†Science Research Council Senior Visiting Fellow for the year 1972–73.

‡Permanent address.

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