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Thermodynamics of a One-Dimensional System of Fermions with a Repulsive δ-Function Interaction

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The thermodynamics of fermions in one dimension with a repulsive δ-function interaction is analyzed in some detail. The excitation spectrum at finite temperature is also derived. It is shown that the spectrum can be classified into two types of excitation.

I. INTRODUCTION

In a previous paper[1] [which we shall call I, and whose equations we shall refer to as (I.1), etc.] we have obtained the thermodynamics of fermions with repulsive δ-function interaction in one dimension. The difficulty with this problem lies in finding all the solutions of Eq. (I.2) and the corresponding quantum numbers. It turns out that the ansatz (I.3) resolves this difficulty. The Λ's lie in strings in the complex plane, and they are fermionlike, so that the quantum numbers can be assigned to them by a continuity argument with respect to the interaction strength. In this paper, we will show that the ansatz is indeed consistent with the integral equations thus obtained. Special cases of these equations are worked out in greater detail, since they can be used to make the *Ansatz* plausible. The excitation spectrum at finite temperature is also derived, so that the meaning of the ϵ and ϕ's will become clear.

II. FORMULATION

The Hamiltonian for the system is the same as in I,

$$H=-\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^{\,2}}+2c\sum_{i>j}\delta(x_i-x_j),\quad c>0, \tag{1}$$

and the energy levels are determined by the algebraic equations

$$e^{ipL}=\prod_{\Lambda'}\left(\frac{-p+\Lambda'-ic}{-p+\Lambda'+ic}\right), \tag{2}$$

$$\prod_{p'}\left(\frac{-p'+\Lambda-ic'}{-p'+\Lambda+ic'}\right)=-\prod_{\Lambda'}\left(\frac{-\Lambda'+\Lambda-ic}{-\Lambda'+\Lambda+ic}\right),\qquad c'=\tfrac{1}{2}c. \tag{3}$$

We again make the *Ansatz* as in (I.3) that the Λ's are located in strings in the complex plane. Furthermore, the strings are fermionlike, by which we mean the Λ's in a string $C(\xi,m)$ are of the form

$$C(\xi,n):\ \ \Lambda=\xi_{n\beta}+\tfrac{1}{2}i\mu c+\delta_\mu,\qquad \mu=-(n-1),\ldots,(n-1), \tag{4}$$

and

$$\delta_\mu=O(e^{-\kappa L}).$$

From Eqs. (2)–(4), we derive the following equations:

$$\sum_p\Theta\left(\frac{\xi_{m\alpha}-p}{m}\right)=2\pi J_{m\alpha}+\sum_{C(\xi,n)}\sum_{\beta=1}^{\nu_n}a_{mnl}\times\Theta\left(\frac{\xi_{m\alpha}-\xi_{n\beta}}{n-l}\right), \tag{5a}$$

$$p_iL=2\pi I_i+\sum_{n,\beta}\Theta\left(\frac{\xi_{n\beta}-p_i}{n}\right),\quad i=1,\ldots,N, \tag{5b}$$

where $\Theta(x)=2\tan^{-1}(2x/c)$. In the limit $L,N\to\infty$ proportionally, Eqs. (5) become integral equations, as in (I.7):

$$\int_{-\infty}^{\infty}\Theta'\left(\frac{\xi-p}{m}\right)\rho(p)\,dp=2\pi(\sigma_m+\sigma_{m,h})+\sum a_{mnl}\times\int_{-\infty}^{\infty}\Theta'\left(\frac{\xi-\xi'}{n-l}\right)\sigma_n(\xi')d\xi', \tag{6a}$$

$$1=2\pi(\rho+\rho_h)-\sum_n\int_{-\infty}^{\infty}\Theta'\left(\frac{p-\xi'}{n}\right)\sigma_n(\xi')d\xi'. \tag{6b}$$

The Fourier transform of Eq. (6a) is

$$\tilde\rho(\omega)e^{-\eta m|\omega|}=\tilde\sigma_m+\tilde\sigma_{m,h}+\sum_{nl}a_{mnl}\,\mathrm{sgn}(n-l)\tilde\sigma_m e^{-\eta|(n-l)\omega|}$$
$$=\tilde\sigma_{m,h}+\sum_\nu\coth|\eta\omega|\,(e^{-\eta|(m-\nu)\omega|}-e^{-\eta|(m+\nu)\omega|})\tilde\sigma_\nu. \tag{7}$$

This can easily be converted into

$$(\tilde\sigma_m+\tilde\sigma_{mh})2\cosh\eta\omega=\tilde\sigma_{m+1,h}+\tilde\sigma_{m-1,h}, \tag{8}$$

where $\tilde\sigma_{oh}$ is defined as $\tilde\rho$. The Fourier transform of Eq. 6(b) is

$$\frac{\delta(\omega)}{2\pi} = \tilde{\rho} + \tilde{\rho}_h - \sum_\nu e^{-\nu\eta|\omega|}\tilde{\sigma}_\nu = \tilde{\rho} + \tilde{\rho}_h - \tfrac{1}{2}\cosh\eta\omega(e^{-\eta|\omega|}\tilde{\rho} - \tilde{\sigma}_{1h}), \tag{9}$$

where (8) has been used to obtain the final result. From Eqs. (8) and (9) one can easily obtain the integral equations (I.10):

$$1/2\pi = \rho + \rho_h - (1/2)\int_{-\infty}^{\infty} G_1(p-k)\rho dk + (1/2)\int_{-\infty}^{\infty} G_0(p-k)\sigma_{1,h} dk, \tag{10}$$

$$\sigma_n + \sigma_{n,h} = (1/2)\int_{-\infty}^{\infty} G_0(p-k)(\sigma_{n+1,h} + \sigma_{n-1,h})dk, \quad n \geq 2.$$

Now if one defines $\delta_h/\rho = e^{\epsilon(p)/T}$, $\sigma_{m,h}/\sigma_m = e^{\phi_m(p)/T}$, and minimizes the free energy in the standard way, one obtains (I.15) for the ϵ amd ϕ's:

$$A = p^2 - \epsilon - \frac{1}{2T}G_1 \ln(1+e^{-\epsilon/T}) - \frac{1}{2T}G_0 \times \ln(1+e^{\sigma_1/T}), \tag{11a}$$

$$\sigma_1 = \frac{1}{2T}G_0[\ln(1+e^{\sigma_1/T}) - \ln(1+e^{-\epsilon/T})], \tag{11b}$$

$$\sigma_n = \frac{1}{2T}G_0[\ln(1+e^{\sigma_{n-1}/T}) + \ln(1+e^{\sigma_{n+1}/T})]. \tag{11c}$$

With an asymptotic condition

$$\lim_{n\to\infty} T^{-1}\phi_n/n = \lambda > 0, \tag{12}$$

the G's are operators as defined in (I.11). The thermodynamic quantities can finally be expressed in terms of the ϵ and ϕ's (I.17). In particular, the pressure P is given by

$$P = \frac{1}{2\pi}\int_{\infty}^{\infty} \ln(1+e^{-\epsilon/T})\, dk. \tag{13}$$

It will be seen in Sec. V that the ϵ and ϕ's can be interpreted as elementary excitations.

III. CONSISTENCY OF *ANSATZ*

It is not easy to solve Eqs. (2)–(3) exactly when N is very large, but asymptotically ($L \gg 1$) the *Ansatz* (4) seems to be a correct one. An *Ansatz* of this kind was first used by Bethe[2] in the antiferromagnetic chain problem. A similar assumption was also used by us in obtaining the ground-state energy of the attractive case of the present problem. In this paper, we would like to show that the ansatz is indeed compatible with the equations derived in (6).

Let Λ be in a string $C(\xi, m)$,

$$\Lambda = \xi + i\mu\eta + \delta_\mu.$$

Substitute the above into Eq. (3) and take the absolute value of both sides. One then obtains

$$\sum_p \frac{(\xi-p)^2 + (\mu-1)^2\eta^2}{(\xi-p)^2+(\mu+1)^2\eta^2} = \left|\frac{\delta_\mu - \delta_{\mu-2}}{\delta_\mu - \delta_{\mu+2}}\right|^2 \prod_{\xi'\nu'} \frac{(\xi-\xi')^2 + (\nu'-\mu+1)\eta^2}{(\xi-\xi')^2+(\nu'+\mu-1)^2\eta^2}\,\frac{(\xi-\xi')^2+(\nu'-\mu-1)^2\eta^2}{(\xi-\xi')^2+(\nu'+\mu+1)^2\eta^2}$$
$$= \left|\frac{\delta_\mu - \delta_{\mu-2}}{\delta_\mu - \delta_{\mu+2}}\right|^2 \prod_{\xi'\nu'} H(\xi,\xi',\nu',\mu-1)H(\xi,\xi',\nu',\mu+1). \tag{14}$$

(Here we assume $\mu < m-1$; for $\mu = m-1$ the procedure is similar.) When L is very large, the logarithm of the above can be written

$$\frac{2}{L}\ln\left|\frac{\delta_\mu - \delta_{\mu-2}}{\delta_\mu - \delta_{\mu+2}}\right| = \int \ln\frac{(\xi-p)^2+(\mu-1)^2\eta^2}{(\xi-p)^2+(\mu+1)^2\eta^2}\rho(p)\,dp - \sum_\nu \int [\ln H(\xi,\xi',\nu',\mu-1) + \ln H(\xi,\xi',\nu',\mu+1)]\sigma(\xi')\,d\xi'$$
$$= 2\pi\int \frac{1}{\omega}(e^{-(\mu+1)\eta\omega} - e^{-(\mu-1)\eta\omega})\tilde{\rho}(\omega)e^{i\omega\xi}d\omega$$
$$-2\pi\int\frac{1}{\omega}(e^{-|(\mu+\nu-1)\eta\omega|} + e^{-|(\mu+\nu+1)\eta\omega|} - e^{-|(\mu-\nu-1)\eta\omega|} - e^{-|(\mu-\nu+1)\eta\omega|})\tilde{\sigma}(\omega)e^{i\omega\xi}d\omega. \tag{15}$$

Now by Eq. (9), the right-hand side of Eq. (15) can be simplified to

$$\frac{2}{L}\ln\left|\frac{\delta_\mu - \delta_{\mu-2}}{\delta_\mu - \delta_{\mu+2}}\right| = -\int \frac{2\pi}{\omega}\frac{\sinh\eta\omega}{\cosh\eta\omega}[\tilde{\sigma}_{\mu+1,h}(\omega) + \tilde{\sigma}_{\mu-1,h}(\omega)]e^{-i\omega\xi}d\omega$$
$$= -2\int_{-\infty}^{\infty} \ln\coth\frac{\pi|\xi-\xi'|}{4\eta}[\sigma_{\mu+1,h}(\xi') + \sigma_{\mu-1,h}(\xi')]d\xi' = -2\kappa < 0.$$

This is consistent with the ansatz that δ_μ is of the order $O(e^{-\kappa L})$. It can be proven that the number of solutions is given by $C^N_M - C^N_{M-1}$. This is just the dimension of the irreducible representation $(N-M, M)$, as if all solutions are to be obtained. The proof will not be given here since it has been presented by others[2,3] in similar cases. In the following, we will solve Eq. (11) for special cases; the results will confirm the ansatz in all cases.

IV. SPECIAL CASES

A. $c=0$

As $c\to 0$ one has $G(p)\to\delta(p)$, and (11) becomes

$$A=p^2-\epsilon-\frac{1}{2T}\ln(1+e^{-\epsilon/T})-\frac{1}{2T}\ln(1+e^{\phi_1/T}), \tag{16a}$$

$$\phi_1=\frac{1}{2T}\ln(1+e^{\phi_2/T})-\frac{1}{2T}\ln(1+e^{-\epsilon/T}), \tag{16b}$$

and

$$\phi_n=\frac{1}{2T}\ln(1+e^{\phi_{n+1}/T})+\frac{1}{2T}\ln(1+e^{\phi_{n-1}/T}), \qquad n\geq 2. \tag{16c}$$

One can easily show that (16c) has the solutions

$$1+e^{\phi_n/T}=\sinh^2(\lambda n+\mu)/\sinh^2\lambda, \tag{17}$$

with

$$\lim_{n\to\infty}\phi_n/n=\lambda T=B.$$

Then by (16a) and (16b):

$$\sinh^2\mu/\sinh^2\lambda=(1+e^{-\epsilon/T})^{-1}, \tag{18}$$

and

$$\sinh\mu/\sinh(\lambda-\mu)=e^{(-A+p^2)/T}, \tag{19}$$

which determine μ by

$$\frac{\sinh\mu}{\cosh\mu}=\frac{\sinh\lambda}{\cosh\lambda+e^{(A-p^2)/T}}. \tag{20}$$

Now (I.16) gives

$$\sigma_{mh}=\frac{1}{\pi}\frac{e^{(A-p^2)/T}\sinh\lambda}{[\cosh\lambda+e^{(A-p)/T}]^2-\sinh^2\lambda}\times\frac{\cosh(n\lambda+\mu)}{\sinh(n\lambda+\mu)}, \tag{21}$$

$$\rho=\frac{1}{2\pi}\left\{\left[1+\exp\left(\frac{-A+p^2-B}{T}\right)\right]^{-1}+\left[1+\exp\left(\frac{-A+p^2+B}{T}\right)\right]^{-1}\right\}, \tag{22}$$

and

$$\frac{N-2M}{L}=\lim_{m\to\infty}\int\sigma_{m,h}(\xi)d\xi=\frac{1}{2\pi}\int\left\{\left[1+\exp\left(\frac{-A+p^2-B}{T}\right)\right]^{-1}-\left[1+\exp\left(\frac{-A+p^2+B}{T}\right)\right]^{-1}\right\}dp. \tag{23}$$

Equations (22) and (23) are precisely the distribution function and magnetization for a free fermion gas in a magnetic field.

B. $c=\infty$

As $c\to\infty$, the integrals $\int G_1\ln(1+e^{-\epsilon/T})\,dk$ and $\int G\ln(1+e^{-\epsilon/T})\,dk$ in (11a) and (11b) do not contribute, allowing ϕ_n to be a constant. This leads to

$$(1+e^{\phi_n/T})=\sinh^2[(n+1)\lambda]/\sinh^2\lambda, \tag{24}$$

$$\epsilon=p^2-A-T\ln(2\cosh\lambda) \tag{25}$$

and

$$\rho=(1/2\pi)(1+e^{\epsilon/T})^{-1}, \tag{26}$$

$$\frac{N-2M}{L}=\frac{-1}{2\pi T}\int(1+e^{-\epsilon/T})^{-1}\frac{\partial\epsilon}{\partial\lambda}\,dk=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tanh\lambda(1+e^{\epsilon/T})^{-1}dk. \tag{27}$$

Equations (26) and (27) are precisely the distribution function for free fermions where each energy level can only be occupied by either a spin-up or a spin-down particle. In fact, as the interaction strength $c\to\infty$, the exchange force due to the symmetry of the wave function becomes unimportant and both species behave the same.

C. Limit $T=0$

From Eq. (11), we have $\phi_n\geq 0$ for $n\geq 2$. Assume that ϵ and ϕ_1 are monotonic increasing functions such that ϵ has zeros at $p^2=Q^2$ and ϕ_1 has zeros at $\xi^2=R^2$. This implies that $T\to 0$:

$$\rho=0 \quad \text{for } p^2>Q^2,$$

$$\rho_h=0 \quad \text{for } p^2<Q^2$$

and

$$\sigma_1=0 \quad \text{for } \xi^2>R^2, \tag{28}$$

$$\sigma_{1h}=0 \quad \text{for } \xi^2<R^2,$$

$$\sigma_n\equiv 0 \quad \text{for } n\geq 2.$$

Then Eq. (8) would give

$$\tilde{\sigma}_{mh}2\cosh\eta\omega=\tilde{\sigma}_{m+1h}+\tilde{\sigma}_{m-1h}, \quad m\geq 2, \tag{29}$$

which has solutions

$$\bar{\sigma}_{mh} = e^{-(m-1)\eta|\omega|}\,\bar{\sigma}_{1h} \quad (m \geq 2). \tag{30}$$

For $m=1$, Eq. (8) gives

$$(\bar{\sigma}_1 + \bar{\sigma}_{1h})2\cosh\eta\omega = \bar{\sigma}_{2h} + \bar{\rho} = e^{-\eta|\omega|}\bar{\sigma}_{1h} + \bar{\rho}$$

or

$$\bar{\sigma}_{1h} = -\bar{\sigma}_1 + e^{-\eta|\omega|}\bar{\rho} - e^{-2\eta|\omega|}\bar{\sigma}_1. \tag{31a}$$

Substituting the above into Eq. (9), one obtains

$$\delta(\omega)/2\pi = \bar{\rho} + \bar{\rho}_h - e^{-\eta|\omega|}\bar{\sigma}_1. \tag{31b}$$

Equations (31a) and (31b) are integral equations of the form

$$\sigma_1 = \int_{-Q}^{Q} K_1\rho\, dp - \int_{-R}^{R} K_2\sigma_1\, d\xi \quad (\xi^2 < R^2), \tag{32a}$$

$$\rho = \frac{1}{2\pi} + \int_{-R}^{R} K_1\sigma_1\, d\xi \quad (p^2 < Q^2), \tag{32b}$$

where K_m is defined as

$$K_m(p,q) = \frac{1}{\pi}\frac{2mc}{m^2c^2 + 4(p-q)^2}. \tag{33}$$

Equations (32a) and (32b) are just the equations obtained by Yang[4] for the ground state.

D. Second Virial Coefficient

The fugacity expansion can be obtained as follows. Let

$$e^{-\epsilon/T} = \sum_{n=1}^{\infty} A_n(k,T)z^n, \quad z = e^{A/T},$$
$$e^{\phi_\nu/T} = b_\nu(k,T) + c_\nu(k,T)z + d_\nu(k,T)z^2 + \cdots. \tag{34}$$

Substituting the above into (11) yields

$$\ln A_1 + p^2/T = \tfrac{1}{2}G\ln b_1,$$

$$A_2/A_1 = \tfrac{1}{2}G_1A_1 + \tfrac{1}{2}G(c_1/b_1) \tag{35a}$$

and

$$\ln(b_n - 1) = \tfrac{1}{2}G(\ln b_{n-1} + \ln b_{n+1}), \quad b_0 = 1, \tag{35b}$$

$$\frac{c_n}{b_n - 1} = \tfrac{1}{2}G\left(\frac{c_{n-1}}{b_{n-1}} + \frac{c_{n+1}}{b_{n+1}}\right), \qquad \frac{c_0}{b_0} = -A_1. \tag{35c}$$

Equation (35b) has the solutions

$$b_n = f_n^2 = [\sinh^2(n+1)\lambda]/\sinh^2\lambda. \tag{36}$$

The the Fourier transform of (34c) will be the difference equation

$$\frac{\bar{c}_\nu}{f_{\nu-1}f_{\nu+1}} = \frac{1}{2\cosh\eta\omega}\left(\frac{\bar{c}_{\nu-1}}{f_{\nu-1}^2} + \frac{\bar{c}_{\nu+1}}{f_{\nu+1}^2}\right), \quad \nu \geq 1, \tag{37}$$

whose general solution is

$$\bar{c}_\nu = A(\omega)(f_\nu f_{\nu-1}e^{-(\nu+2)|\eta\omega|} - f_\nu f_{\nu+1}e^{-\nu|\omega\eta|})$$
$$+ B(\omega)(f_\nu f_{\nu-1}e^{(\nu+2)|\eta\omega|} - f_\nu f_{\nu+1}e^{\nu|\omega\eta|}). \tag{38}$$

The initial condition $c_0/b_0 = -A_1$ gives

$$\bar{c}_1 = \bar{A}_1(e^{-3|\eta\omega|} - f_2e^{-|\eta\omega|}),$$

and finally from (35a) one obtains

$$A_1 = f_1e^{-p^2/T}, \quad A_2 = e^{-p^2/T}\int K_2e^{-p^2/T}dp. \tag{39}$$

For simplicity, let $B=0$ $(\lambda=0)$, then the pressure is given by

$$P = (T/2\pi)\int A_1z + (A_2 - \tfrac{1}{2}A_1^2)z^2 + \cdots$$
$$= [(\pi T)^{1/2}/2\pi]\Big[2z + z^2\Big(-2^{1/2} + 2^{-1/2}\int_{-\infty}^{\infty}\frac{\eta}{\pi(\eta^2+p^2)}$$
$$\times e^{-2p^2/T}dp\Big) + \cdots\Big]. \tag{40}$$

This agrees with results obtained by standard methods.

V. EXCITATION SPECTRUM AT FINITE TEMPERATURE

A. Excitations

In the following pages, we shall derive the excitations at finite temperature. The result to be obtained in Eq. (69) shows that the ϵ and ϕ's in (11) can be regarded as the energy of these excitations. Let us first consider a state with primed I's and J's satisfying Eq. (5):

$$p_i'L = 2\pi I_i' + \sum\Theta\left(\frac{\xi_{n\beta}' - p_i'}{n}\right), \tag{41}$$

$$\sum_{p'}\Theta\left(\frac{\xi_{n\alpha}' - p'}{m}\right) = 2\pi J_{n\alpha}' + \sum_n\sum_l a_{mnl}\Theta\left(\frac{\xi_{m\alpha}' - \xi_{n\beta}'}{n-l}\right),$$

but with

$$J_{m\alpha}' = J_{m\alpha},$$
$$I_j' = I_j, \quad \text{except } j = \gamma. \tag{42}$$

We may call such a state a p-type excitation as only the quantum number for a particular p is changed in (5). To find the momentum difference and energy difference between the two states (5) and (41), one may proceed as follows[5]: Assume that P_j, $\xi_{m\alpha}$ and P_j', $\xi_{m\alpha}'$ are approximately the same (except $j=\gamma$). Write

$$(p_j' - p_j)L = f(p_j), \quad j \neq \gamma$$
$$(\xi_{m\alpha}' - \xi_{m\alpha})L = g_m(\xi_{m\alpha}), \tag{43}$$

and subtract (5) from (41), obtaining, as $L \to \infty$,

$$f(p)=\sum\int\Theta'\left(\frac{\xi-p}{n}\right)[g_n(\xi)-f(p)]\sigma_n(\xi)\,d\xi$$
$$\times\int\Theta'\left(\frac{\xi-p}{m}\right)[g_m(\xi)-f(p)]\rho(p)dp$$
$$+\Theta\left(\frac{\xi-p'_\gamma}{m}\right)-\Theta\left(\frac{\xi-p_\gamma}{m}\right)$$
$$=\sum_{nl}a_{mnl}\int\Theta'\left(\frac{\xi-\xi'}{n-l}\right)[g_m(\xi)-g_n(\xi')]\sigma_n(\xi')\,d\xi'. \quad (44)$$

Now define

$$\chi(p)=(\rho+\rho_h)f(p),\quad \alpha_m(\xi)=(\sigma_m+\sigma_{m,h})g_m(\xi),$$

and use (6) to evaluate the coefficient of $f(p)$ and $g_m(\xi)$ in (43). Writing our equations in operator form, we have

$$\begin{pmatrix}\chi\\ \alpha_1\\ \vdots\\ \alpha_n\\ \vdots\end{pmatrix}=G\begin{pmatrix}\chi\\ \alpha_1\\ \vdots\\ \alpha_n\\ \vdots\end{pmatrix}+G_0\begin{pmatrix}\psi_{p_\gamma p'_\gamma}\\ 0\\ \vdots\\ 0\\ \vdots\end{pmatrix}, \quad (45)$$

where $G(p,q)=G_0(p-q)\Phi(q)$,

$$G_0=\begin{pmatrix}0 & K_1 & K_2 & \cdots & K_n\\ K_1 & K_{1,1} & K_{1,2} & \cdots & K_{1,2}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ K_n & K_{n,1} & K_{n,2} & \cdots & K_{n,n}\\ \cdots & \cdots & \cdots & \cdots & \cdots\end{pmatrix}, \quad (46a)$$

$$\Phi=\begin{pmatrix}(1+e^{\epsilon/T})^{-1} & & & \\ & (1+e^{\phi_1/T})^{-1} & & \\ & & \ddots & \\ & & & (1+e^{\phi_n/T})^{-1} \\ & & & & \ddots\end{pmatrix}, \quad (46b)$$

$$K_{n,m}=-\sum_l a_{nml}K_{n-l}, \quad (46c)$$

$\psi_{p_\gamma p'_\gamma}$ is a block function,

$$\psi_{p_\gamma p'_\gamma}(p)=1\quad\text{for } p\in[p_\gamma,p'_\gamma]$$
$$=0\quad\text{otherwise}. \quad (47)$$

Note that $K_{n,m}=K_{m,n}$. Now the momentum difference and energy difference between the states (5) and (41) are given by

$$\Delta P=\sum(p'_j-p_j)=p'_\gamma-p_\gamma+\int_{-\infty}^{\infty}f(p)\rho(p)\,dp, \quad (48)$$

$$\Delta E=\sum(p_j^{2\prime}-p_j^2)$$
$$=p_\gamma^{2\prime}-p_\gamma^2+\int_{-\infty}^{\infty}f(p)\,2p\rho(p)\,dp. \quad (49)$$

Similarly, we may consider a state whose primed I's and J's satisfy

$$I'_l=I_l,$$
$$J'_{m\alpha}=J_{m\alpha},\quad\text{except } \alpha=\gamma \quad (50)$$

for a specific m. The χ's and α's satisfy the integral equation

$$\begin{pmatrix}\chi\\ \alpha_1\\ \vdots\\ \alpha_m\\ \vdots\end{pmatrix}=G\begin{pmatrix}\chi\\ \alpha_1\\ \vdots\\ \alpha_m\\ \vdots\end{pmatrix}+G_0\begin{pmatrix}0\\ \vdots\\ 0\\ \psi_{\xi_{m\gamma}\xi'_{m\gamma}}\\ 0\\ \vdots\end{pmatrix}. \quad (51)$$

Or more generally, we may consider a state where say, two of the ξ_1's become a string of order 2. That is, the J_1's are ν_1-2 in number, and the J_2's are ν_2+1 in number, and

$$J'_{1\alpha}=J_{1\alpha}\quad\text{except } \alpha=\gamma,\gamma',$$
$$J'_{2\alpha}=J_{2\alpha}\quad\text{except } \alpha=\beta. \quad (52)$$

In this case, the χ and α's satisfy

$$\begin{pmatrix}\chi\\ \alpha_1\\ \vdots\\ \alpha_m\\ \vdots\end{pmatrix}=G\begin{pmatrix}\chi\\ \alpha_1\\ \vdots\\ \alpha_m\\ \vdots\end{pmatrix}+\begin{pmatrix}g_0\\ g_1\\ \vdots\\ g_m\\ \vdots\end{pmatrix}, \quad (53)$$

where

$$2\pi g_0=\Theta\left(\frac{\xi'_{2\beta}-p}{2}\right)-\Theta(\xi_{1\gamma'}-p)-\Theta(\xi_{1\gamma}-p),$$
$$2\pi g_m=\sum_l a_{m2l}\Theta\left(\frac{\xi-\xi'_{2\beta}}{2-l}\right)$$
$$-\sum_l a_{m1l}\left[\Theta\left(\frac{\xi-\xi_{1\gamma'}}{1-l}\right)+\Theta\left(\frac{\xi-\xi_{1\gamma}}{1-l}\right)\right]. \quad (54)$$

Excitations for strings of other orders can be written down similarly. One may call states (50) and (52) ξ-type excitations as only quantum numbers for ξ's are changed in (5). Equation (49) may be designated as ξ_m type and Eq. (52) as ξ_1-ξ_2 type. In Sec. VB, we are going to prove that for the p-type excitation,

$$\Delta P=h_0(p'_\gamma)-h_0(p_\gamma), \quad (55)$$
$$\Delta E=\bar\epsilon(p'_\gamma)-\bar\epsilon(p_\gamma), \quad (56)$$

where

$$h_0(p)=p-\sum_n\int\Theta\left(\frac{\xi-p}{n}\right)\sigma_n(\xi)d\xi,$$
$$\bar\epsilon(p)=\epsilon(p)+A-T\ln(2\cosh BT). \quad (57)$$

(B is the magnetic field.)[1] For the ξ_m-type excitation of (50), we will show

$$\Delta P = h_m(\xi'_{n\gamma}) - h_m(\xi_{m\gamma}),\quad \Delta E = \bar{\phi}_m(\xi'_{m\gamma}) - \bar{\phi}_m(\xi_{m\gamma}), \tag{58}$$

where

$$h_m(\xi) = \int \Theta\left(\frac{\xi - p}{m}\right)\rho(p)\,dp - \sum a_{mnl}\int \Theta\left(\frac{\xi-\xi'}{n-l}\right)\sigma_n(\xi')d\xi', \tag{59}$$

$$\bar{\phi}_m(\xi) = \phi_m(\xi) - \phi_m(\infty). \tag{60}$$

Similarly, for the excitations of (52), we will show

$$\Delta P = h_2(\xi'_{2\beta}) - h_1(\xi_{1\gamma}) - h_1(\xi_{1\gamma'}), \tag{61}$$

$$\Delta E = \bar{\phi}_2(\xi'_{2\beta}) - \bar{\phi}_1(\xi_{1\gamma}) - \bar{\phi}_1(\xi_{1\gamma'}). \tag{62}$$

B. Momentum Difference and Energy Difference

To prove Eqs. (55)–(62), we rewrite (4) as

$$\begin{pmatrix} \rho+\rho_h \\ \sigma_1+\sigma_{1h} \\ \vdots \\ \sigma_n+\sigma_{nh} \\ \vdots \end{pmatrix} = G\begin{pmatrix} \rho+\rho_h \\ \sigma_1+\sigma_{1h} \\ \vdots \\ \sigma_n+\sigma_{nh} \\ \vdots \end{pmatrix} + \begin{pmatrix} 1/2\pi \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix}, \tag{63}$$

Let L be the resolvent

$$(1+L)(1-G) = 1. \tag{64}$$

Then Eq. (63) gives

$$\begin{pmatrix} \rho+\rho_h \\ \sigma_1+\sigma_{1h} \\ \vdots \end{pmatrix} = (1+L)\begin{pmatrix} 1/2\pi \\ 0 \\ \vdots \end{pmatrix}. \tag{65}$$

Also it can be readily shown that ϵ and ϕ's satisfy integral equations of the same form as (63) [if one uses Eq. (63) in minimizing the free energy rather than using (I.10)]. One may thus obtain

$$\begin{pmatrix} \frac{\partial\epsilon}{\partial p} \\ \frac{\partial\phi_1}{\partial\xi} \\ \vdots \\ \frac{\partial\phi_n}{\partial\xi} \\ \vdots \end{pmatrix} = G\begin{pmatrix} \frac{\partial\epsilon}{\partial p} \\ \frac{\partial\phi_1}{\partial\xi} \\ \vdots \\ \frac{\partial\phi_n}{\partial\xi} \\ \vdots \end{pmatrix} + \begin{pmatrix} 2p \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix}. \tag{66}$$

Now let us prove (61). Equation (53) gives

$$\begin{pmatrix} \chi \\ \alpha_1 \\ \vdots \\ \alpha_m \\ \vdots \end{pmatrix} = (1+L)\begin{pmatrix} g_0 \\ g_1 \\ \vdots \\ g_m \\ \vdots \end{pmatrix}. \tag{67}$$

As G_0 is symmetric, one can easily show that

$$\Phi(p)L(p,q)\Phi^{-1}(q) = \tilde{L}(q,p), \tag{68}$$

where $\tilde{L}$ is the transpose of L. Then by use of (48), (67), (68), and (65), we have

$$\begin{aligned}\Delta P &= \int_{-\infty}^{\infty}(1+e^{\epsilon/T})^{-1}\Big[g_0 + \sum_j L_{0j}(p,q)g_{j-1}(q)\Big]dq \\ &= \int_{-\infty}^{\infty}(1+e^{\epsilon/T})^{-1}g_0\,dq + \sum_j\int_{-\infty}^{\infty}L_{j0}(q,p)\Phi_{jj}^{-1}(q)g_{j-1}(q)\,dq \\ &= \int_{-\infty}^{\infty}g_0(1+e^{\epsilon/T})^{-1}dq + \int_{-\infty}^{\infty}[2\pi(\rho+\rho_h)-1](1+e^{\epsilon/T})^{-1}g_0\,dq \\ &\quad + \sum_j\int_{-\infty}^{\infty}2\pi(\sigma_j+\sigma_{jh})(1+e^{\phi_j/T})^{-1}g_j\,dq \\ &= \int_{-\infty}^{\infty}2\pi\rho g_0\,dq + \sum_j\int_{-\infty}^{\infty}2\pi\sigma_j g_j\,dq,\end{aligned}$$

which yields Eq. (61) by the definitions of (54) and (59). The other equations (55)–(62) can be derived in the same manner. Now it is easy to see that for a finite number of simultaneous excitation, one has

$$\begin{aligned}\Delta P &= \sum_\alpha[h_0(p'_\alpha) - h_0(p_\alpha)] + \sum_m h_m(\xi_{m\beta}) - \sum_{m'} h_{m'}(\xi_{m'\gamma}), \\ \Delta E &= \sum_\alpha[\bar{\epsilon}(p'_\alpha) - \bar{\epsilon}(p_\alpha)] + \sum_m \bar{\phi}_m(\xi_{m\beta}) - \sum_{m'}\bar{\phi}_{m'}(\xi_{m'\gamma}),\end{aligned} \tag{69}$$

with $\sum m' = \sum m$.

C. Limit $T=0$

We go to the limit $T=0$ to obtain a qualitative picture of the excitations. From (I.15), we have $\phi_n \geq 0$ for $n \geq 2$. Thus $\sigma_n = 0 (n \geq 2)$ in the limit $T=0$. Thus, in the ground state, there are no strings of order higher than or equal to 2. However, for low-lying excited states, such higher-order strings will be present. Then from (66)–(68), we will have

$$\rho = \frac{1}{2\pi} + \int_{-R}^{R} K_1 \sigma_1 \, d\xi \quad (p^2 < Q^2)$$

$$\sigma_1 = \int_{-Q}^{Q} K_1 \rho \, dp - \int_{-R}^{R} K_2 \sigma_1 \, d\xi \quad (\xi^2 < R^2) \tag{70}$$

$$\sigma_{n,h} = \int_{-Q}^{Q} K_n \rho \, dp - \int_{-R}^{R} (K_{n+1} + K_{n-1}) \sigma_1 \, d\xi \quad (n > 2),$$

where K_l are the kernels defined by (46). Similarly (45) becomes

$$\chi = \int_{-R}^{R} K_1 \alpha_1 d\xi ,$$

$$\alpha_1 = \int_{-Q}^{Q} K_1 \chi_1 dp - \int_{-R}^{R} K_2 \alpha_1 d\xi + (2\pi)^{-1} \times [\Theta(p'_\gamma - \xi) - \Theta(p_\gamma - \xi)], \text{ etc.} \tag{71}$$

Let us assume that $R \approx 0$ so that M/L is proportional to R. For a p-type excitation near the fermi level ($p'_\gamma = Q + k, p_\gamma = Q$), Eqs. (70) and (71) yield, to the lowest order in ΔP,

$$\frac{\Delta E}{\Delta P} = 2\pi r - \frac{M}{L} \left(\frac{8\pi c r}{c^2 + 4\pi^2 r^2} + 4 \tan^{-1} \frac{2\pi r}{c} \right) + O\left[\left(\frac{M}{L} \right)^3 \right], \tag{72}$$

where $r = N/L$. Similarly for the ξ_1-type excitation near the Fermi level $\xi = R(\xi_{1\gamma'} = R + k, \xi_{1\gamma} = R)$, one readily obtains, to order $O(\Delta P)$,

$$\frac{\Delta E}{\Delta P} = \frac{M}{L} \pi^2 \left(\tan^{-1} \frac{2\pi r}{c} - \frac{2\pi c r}{c^2 + 4\pi^2 r^2} \right) \left(\tan^{-1} \frac{2\pi r}{c} \right)^{-2}. \tag{73}$$

Equations (72) and (73) are quasi-particle-like excitations near the Fermi level. For the excitations of Eq. (52), let us take $\xi_{1\gamma} = -\xi_{1\gamma'} = R$, and $\xi_{2\beta} = \xi \approx 0$. One then obtains

$$\Delta P = \xi \left[\frac{2}{\pi} \tan^{-1} \frac{\pi r}{c} \frac{2\pi r}{c} + O\left(\frac{M}{L} \right) \right],$$

$$\Delta E = \frac{1}{\pi} \left((c^2 + 4\pi^2 r^2) \tan^{-1} \frac{2\pi r}{c} - (2c^2 + 2\pi^2 r^2) \tan^{-1} \frac{\pi r}{c} \right) + \frac{2}{\pi} \xi^2 \left[\tan^{-1} \frac{\pi r}{c} - \frac{c\pi r}{c^2 + \pi^2 r^2} + O\left(\frac{M}{L} \right) \right]. \tag{74}$$

Eliminating ξ, ΔE is of the form

$$\Delta E = a + b(\Delta P)^2 , \tag{75}$$

where a, b are constants. Thus Eq. (75) corresponds to collective excitations. If one recalls that when $c \to \infty$, (5) is identical to the algebraic equations of the Heisenberg chain,[6] then it is reasonable to identify these as spin waves.

VI. CONCLUSION

We have shown that the *Ansatz* used to derive the integral equations are compatible with these equations. Special cases of these equations are solved and they all give the correct results. Thus, it is very likely that the *Ansatz* is a correct one. We have also computed the excitation spectrum of the repulsive fermion gas at finite temperature. We may classify an excitation as either of two types: p type and ξ type. Then h_0, ϵ and $h_l, \overline{\phi}_l (k \geq 1)$ can be regarded as the momentum and energy of these excitations, respectively. At $T=0$, the p-type and ξ_1-type excitations are like quasiparticle excitations while ξ_1-ξ_2 type are like collective excitations, analogous to spin waves in antiferromagnetic chains. One can apply the same procedure to the attractive case; the result will be published elsewhere.

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