Vertical-Arrow Correlation Length in the Eight-Vertex Model and the Low-Lying Excitations of the X-Y-Z Hamiltonian

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We consider the correlation function G_R for two vertical arrows in the same column. We calculate the critical index ν of the correlation length, and we find the scaling relation $\nu = 1 - \alpha/2$ is satisfied. In the decoupling limit, we prove that the correlation length is not determined by the next-largest eigenvalue. In order to obtain the correct correlation length, it is necessary to integrate over the entire band of complex next-largest eigenvalues. We argue that this is also the situation in the general case of the eight-vertex model. Under certain well-defined assumptions, we compute the correlation length of G_R. We also calculate the low-lying excitation energies of the X-Y-Z Hamiltonian. In addition to free states existing for $0 < \mu < \pi$, there are bound states appearing in the spectrum for $\mu > \pi/2$. In the course of our work, we have rewritten the results of Cheng and Wu for the Ising-model correlation functions in an elegant form, using Baxter's elliptic-function parametrization.

I. INTRODUCTION

Baxter's[1] recent calculation of the partition function of the two-dimensional zero-field eight-vertex model has unexpectedly demonstrated that the critical index α can depend continuously on a parameter which appears linearly in the Hamiltonian. This violates the naive interpretation of the concept of "universality."[2] Therefore, it is desirable to check whether the various linear scaling relations[3,4] between the critical indices are obeyed, even though naive universality is violated. In this paper, we compute the critical index ν of the correlation length between two vertical arrows in the same vertical column. Our principal conclusion is that the scaling law

$$\nu = 1 - \tfrac{1}{2}\alpha \tag{1.1}$$

holds in the eight-vertex model.[5]

Our procedure is based on the use of the equations derived by Baxter,[1] which determine the eigenvalues of the transfer matrix. Let G_R be the correlation function of two vertical arrows in the same vertical column, separated by R vertices. There is a well-known argument[6,7] which implies that the correlation length ξ of G_R is given by

$$\xi^{-1} = -\ln|\Lambda_1/\Lambda_0|\,, \tag{1.2}$$

where $|\Lambda_0|$ is the magnitude of the eigenvalues of the transfer matrix with largest absolute value, and $|\Lambda_1|$ is the magnitude of the next-largest eigenvalue. The relation (1.2) is not a theorem, but it has been verified by Fisher and Burford[7] for the Ising model with the transfer matrix used by Onsager.[6] It is important to recall that the eigenvalues of Onsager's transfer matrix are real. The transfer matrix studied by Baxter[1] for the eight-vertex model is not Hermitian, and, in fact, has complex eigenvalues. In Sec. III, we prove that (1.2) does *not* give the correlation length of G_R in the decoupling limit ($\mu = \frac{1}{2}\pi$), where the eight-vertex model is equivalent to two independent Ising lattices. There are cancellations resulting from the various phases of the complex eigenvalues, which cause the correlation length to be less than that predicted by (1.2). The correct correlation length is obtained only when the entire band of complex "next-largest" eigenvalues are taken into account.

In Sec. VI we present a physical argument indicating that G_R and C_R (the correlation function of two vertical arrows in the same horizontal row, separated by R vertices) have the same correlation length. Since C_R does not depend on Baxter's parameter V, this argument implies that the correlation length ξ of G_R will not depend on V. However, Λ_1/Λ_0 explicitly depends on V, which indicates that (1.2) is incorrect. Instead of (1.2), we find ($\mu \leq \frac{1}{2}\pi$)

$$\xi^{-1} = -\ln k_2\,, \tag{1.3}$$

where k_2 is the modulus of the complete elliptic integral K_2 defined by

$$\pi K_2'/K_2 = 2\lambda. \tag{1.4}$$

Baxter's parameters (e.g., λ, μ, V, α) are discussed in Appendix A. If (1.2) were correct, we would have ($\mu \leq \frac{1}{2}\pi$)

$$\xi^{-1} = -\ln[k_2/\mathrm{dn}^2(K_2\alpha/\pi, k_2')]. \quad (1.5)$$

When $\mu > \frac{1}{2}\pi$, the situation is more complicated, and our results for ξ are given in (6.17) and (6.18). The standard argument (1.2), and the correct answers (1.3), (6.17), and (6.18), yield the same critical index ν, given by (1.1).

There exists a remarkable connection between the exactly soluble lattice models of classical statistical mechanics, and the one-dimensional quantum Hamiltonians of interacting spins $\frac{1}{2}$. The transfer matrix of the zero-field eight-vertex model commutes[8] with a corresponding X-Y-Z Hamiltonian. Baxter[9] extended this relation to the eigenvalues by showing that the X-Y-Z Hamiltonian is essentially a logarithmic derivative of the eight-vertex transfer matrix. The transfer matrices of the planar zero-field Ising models (free fermion models) commute[10] with corresponding X-Y Hamiltonians with magnetic fields.

In Secs. IV and V, we compute the next-largest eigenvalues of the transfer matrix of the eight-vertex model. In Sec. VII we use Baxter's[9] relation between the energy levels of the X-Y-Z Hamiltonian and the eigenvalues of the eight-vertex transfer matrix, to compute the low-lying excitation energies of the X-Y-Z Hamiltonian. The spectrum consists of two classes of excitations. The first class consists of "free states," which exist throughout the region $0 < \mu < \pi$. The free states have a two-parameter dispersion curve (7.8), and have the same qualitative features as the X-Y spectrum (7.10c). There are two quantum numbers[1] (see Sec. II B) corresponding to exact symmetrics of the X-Y-Z Hamiltonian: $\nu' = 0, 1$ and $\nu'' = 0, 1$. There exist four degenerate free-state dispersion curves labeled by these quantum numbers. It appears likely that for fixed values of ν' and ν'', there are two degenerate dispersion curves, which we label $\omega_2 = 0$ and 1. Evidence for the existence of this degeneracy is encountered in Sec. III, when we examine G_R in the decoupling limit. For $\mu \leq \frac{1}{2}\pi$, we believe these are all of the low-lying states.

As μ increases past $\frac{1}{2}\pi$, a new one-parameter "bound-state" curve appears in the spectrum (7.12), each time a point $\mu = \pi - \pi/n$ ($n = 2, 3, 4, \ldots$) is crossed. At the emergence point of a bound-state curve, it is degenerate with the free state. As μ increases, the minimum point decreases in energy and lies below the free-state minimum. The minima of the bound-state curves increase with $n = 2, 3, 4, \ldots$. The bound-state curves go continuously to the known[11] magnon curves (7.13) of the ferromagnetic Heisenberg-Ising model. The free states reduce to the known answers in the X-Y [(7.10c)], and Heisenberg-Ising [(7.10a) and (7.10b)] limits.

Let us outline the contents of this paper. The paper has been organized so that the detailed computations of Secs. IV and V can be omitted upon first reading. The reader need only note the results for the next-largest eigenvalues given in (4.29), (4.33), (5.7), and (5.9).

In Sec. II we review the symmetries[12] of the eight-vertex partition function, and derive the symmetries of G_R. We derive a spectral representation for the asymptotic behavior of G_R in terms of the next largest eigenvalues of the transfer matrix. In Sec. III the correlation function G_R is explicitly evaluated in the decoupling limit, using the known results[13] for the Ising-model correlation functions. We use Baxter's parametrization to rewrite the Ising-model correlation functions as integrals over elliptic functions.

In Sec. IV we study Baxter's[1] equations. In the thermodynamic limit, we approximate these sum equations by a linear integral equation, with a difference kernel, for the density of the zeros. This equation can be solved by Fourier transform. We reproduce Baxter's[1] result for the largest eigenvalue, and then calculate the dispersion curves for the next-largest eigenvalues (for $\mu \leq \frac{1}{2}\pi$). In Sec. V we extend the work on the next-largest eigenvalues to the region $\mu > \frac{1}{2}\pi$. In Sec. VI we use the spectral representation for G_R to study its correlation length and extract the critical index ν. Finally, in Sec. VII the low-lying excitation energies of the X-Y-Z Hamiltonian are obtained.

II. SOME GENERAL PROPERTIES OF THE EIGHT-VERTEX MODEL

A. Symmetries of Partition Function

The zero-field eight-vertex model is defined as follows. Consider a lattice of M rows and N columns, with toroidal boundary conditions. Place arrows on the lattice and allow only those configurations with an even number of arrows pointing into each vertex. The energies $\epsilon_1 = \epsilon_2$, $\epsilon_3 = \epsilon_4$, $\epsilon_5 = \epsilon_6$, $\epsilon_7 = \epsilon_8$ are associated with the eight allowed vertices shown in Fig. 1. The corresponding Boltzmann weights $e^{-\epsilon_i/kT}$ are denoted a, b, c, d. Baxter[1] has shown that it is useful to define

$$w_1 = \tfrac{1}{2}(c + d), \quad w_2 = \tfrac{1}{2}(c - d),$$

$$w_3 = \tfrac{1}{2}(a - b), \quad w_4 = \tfrac{1}{2}(a + b).$$

If no two vertex weights are equal, and all the weights are nonzero, then at low enough temperatures

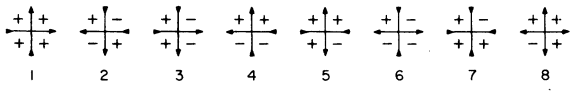

FIG. 1. These are the eight allowed vertices. With each vertex is shown one of the corresponding Ising spin configurations (the other is obtained by reversing all the Ising spins). The correspondence is defined by the relation: an arrow points up (down) or right (left), if the two adjacent Ising spins have the same (opposite) sign.

$$w_4 > |w_3| > w_1 > |w_2| \quad (T < T_c, \text{ ferroelectric})$$

or

$$w_1 > |w_2| > w_4 > |w_3|. \quad (T < T_c, \text{ antiferroelectric})$$

When the temperature is increased, the middle two w's cross ($T = T_c$), and the possible high-temperature configurations are (when $T > T_c$)

$$w_1 > w_4 > |w_2| > |w_3|,$$
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$$w_4 > w_1 > |w_2| > |w_3|,$$
$$w_4 > w_1 > |w_3| > |w_2|.$$

If, for example, $a = b > c, d$, then $w_4 > w_1 > |w_2| > w_3 = 0$ for all temperatures, and there is no critical temperature.

Fan and Wu[12] have derived the symmetry properties of the eight-vertex model partition function. There are two types of symmetries. The first arise from the elementary reflection and bond-hole symmetries of the model. They show equality of partition functions of eight-vertex models specified by different sets of vertex energies. The eight-vertex models thus related, when both have real values for the vertex energies, are either both above, or both below, their respective critical temperatures. These symmetries are

$$Z(w_1, w_2, w_3, w_4) = Z(\pm w_1, \pm w_2, \pm w_3, \pm w_4) = Z(w_2, w_1, w_3, w_4) = Z(w_4, w_3, w_2, w_1). \tag{2.1a}$$

The second type of symmetry is the weak graph (or dual) transformation. The partition functions related in this case correspond to eight-vertex models lying on opposite sides of their critical temperatures, when both have real vertex energies. For the ferroelectric case the dual transformation is

$$Z(w_1, w_2, w_3, w_4) = Z(w_3, w_2, w_1, w_4). \tag{2.1b}$$

In the antiferroelectric case the dual transformation is

$$Z(w_1, w_2, w_3, w_4) = Z(w_1, w_4, w_3, w_2). \tag{2.1c}$$

Together (2.1a)–(2.1c) imply

$$Z(w_1, w_2, w_3, w_4) = Z(\pm w_i, \pm w_j, \pm w_k, \pm w_l), \tag{2.2}$$

where i, j, k, l are any permutation of 1, 2, 3, 4. Hence, it is sufficient to calculate the partition function in the fundamental (but unphysical) region (FR) $w_1 > w_2 > w_3 > w_4 \geq 0$, in order to know it everywhere.

B. Transfer Matrix

Baxter[1] considered the transfer matrix $\underline{T}$ for a row of $N = 2r$ (even) vertices. For a row of vertical bonds in the lattice, let $\alpha_J = +$ or $-$, if there is an up or down arrow in column J, respectively. Let $\alpha = \{\alpha_1, \ldots, \alpha_N\}$ define the configuration of arrows on the whole row of vertical bonds. If α, α' denote the configurations of two successive rows, the transfer matrix $\underline{T}$ is the $2^N \times 2^N$ matrix defined by

$$T_{\alpha|\alpha'} = \sum \exp\left(-\beta \sum_{j=1}^{8} N_j \epsilon_j\right),$$

where the sum is over allowed arrangements of arrows on the intervening row of horizontal bonds and N_j is the number of vertices of type j in this row. Baxter showed that it is useful to consider the following expression for the transfer matrix:

$$T_{\alpha|\alpha'} = \mathrm{Tr}\underline{R}(\alpha_1, \alpha_1')\underline{R}(\alpha_2, \alpha_2') \ldots \underline{R}(\alpha_N, \alpha_N'), \tag{2.3a}$$

where $\underline{R}(\alpha, \alpha')$ are the 2×2 matrices

$$R(\alpha, \alpha')_{\lambda\lambda'} = \sum_{j=1}^{4} w_j \sigma^j_{\alpha\alpha'} \sigma^j_{\lambda\lambda'}, \tag{2.3b}$$

and $\sigma^1, \sigma^2, \sigma^3$ are the Pauli matrices, while σ^4 is the unit matrix.

The transfer matrix has four obvious invariant subspaces. These are specified by the quantum numbers (defined modulo an even integer): $\nu' = 0$ or 1 if the number of down arrows in each row of vertical bonds is even or odd, respectively; $\nu'' = 0$ or 1 in the subspace symmetric or antisymmetric, respectively, with respect to reversing all arrows.

Baxter[1] found the condition for commutation of two transfer matrices with different sets of vertex weights. He proved that

$$[\underline{T}(a, b, c, d), \underline{T}(a', b', c', d')] = 0, \tag{2.4a}$$

provided

$$ab/cd = a'b'/c'd', \tag{2.4b}$$

$$(a^2 + b^2 - c^2 - d^2)/2ab = (a'^2 + b'^2 - c'^2 - d'^2)/2a'b'. \tag{2.4c}$$

He introduced the following parametrization of the vertex weights:

$a: b: c: d = \mathrm{sn}(v+\eta,k): \mathrm{sn}(v-\eta,k): \mathrm{sn}(2\eta,k):$

$$k\,\mathrm{sn}(v+\eta,k)\,\mathrm{sn}(v-\eta,k)\,\mathrm{sn}(2\eta,k). \quad (2.5)$$

This parametrization has the useful property that the ratios in (2.4) depend on k and η, but not v. Hence, all commuting transfer matrices have the same values of k and η, and arbitrary values of v.

Using (2.3), we compute the adjoint of the transfer matrix

$$\underline{T}^{\dagger}(a,b,c,d) = \underline{T}(a,b,d,c). \quad (2.6)$$

Since the transfer matrix is not Hermitian, it can have complex eigenvalues. The transfer matrix is, however, normal, since (2.4) shows

$$[\underline{T},\underline{T}^{\dagger}] = 0. \quad (2.7)$$

This proves the transfer matrix can be diagonalized, and the eigenfunctions do not depend on v.

Sutherland[8] showed that the transfer matrix $\underline{T}(a,b,c,d)$ commutes with the X-Y-Z Hamiltonian

$$H = -\frac{1}{2}\sum_{i=1}^{N}\{(1+\gamma)\sigma_i^x\sigma_{i+1}^x + (1-\gamma)\sigma_i^y\sigma_{i+1}^y + \Delta\sigma_i^z\sigma_{i+1}^z\}, \quad (2.8a)$$

when

$$\gamma = cd/ab, \quad \Delta = (a^2+b^2-c^2-d^2)/2ab. \quad (2.8b)$$

It follows that all transfer matrices corresponding to the same values of k and η, and arbitrary values of v, have the same eigenfunctions as the X-Y-Z Hamiltonian of (2.8). Since H is Hermitian, this proves that the eigenfunctions of the transfer matrix form a complete orthonormal set.

The partition function is related to the transfer matrix by

$$Z = \mathrm{Tr}\underline{T}^{M}, \quad (2.9)$$

where M (even) is the number of rows. One can derive the symmetry relations (2.2) from the explicit expression for the transfer matrix given in (2.3). We shall collect these arguments here.

We rewrite (2.6) as

$$\underline{T}^{\dagger}(w_1,w_2,w_3,w_4) = \underline{T}(w_1,-w_2,w_3,w_4). \quad (2.10a)$$

Since N is even,

$$\underline{T}(w_1,w_2,w_3,w_4) = \underline{T}(-w_1,-w_2,-w_3,-w_4). \quad (2.10b)$$

Using the cyclic invariance of the trace, we find

$$T(w_1,w_2,w_3,w_4)_{\alpha|\alpha'} = \mathrm{Tr}\prod_{J=1}^{N}\underline{\sigma}^x\underline{R}(\alpha_J,\alpha_J')\underline{\sigma}^x = T(w_1,-w_2,-w_3,w_4)_{\alpha|\alpha'}. \quad (2.10c)$$

Define

$$\mathfrak{U} = e^{-(1/4)i\pi\sigma^x},$$

$$U = \prod_{J=1}^{N} e^{-(1/4)i\pi\sigma_J^x},$$

$$(U\underline{R}U^{-1})_{\alpha\alpha'} \equiv \sum_{\beta\gamma}\mathfrak{U}_{\alpha\beta}\underline{R}(\beta,\gamma)\mathfrak{U}^{-1}_{\gamma\alpha'},$$

then

$$(UT(w_1,w_2,w_3,w_4)U^{-1})_{\alpha|\alpha'} = \mathrm{Tr}\prod_{J=1}^{N}\mathfrak{U}(U\underline{R}U^{-1})_{\alpha_J\alpha_J'}\mathfrak{U}^{-1} = T(w_1,w_3,w_2,w_4)_{\alpha|\alpha'}. \quad (2.10d)$$

Additional relations, analogous to (2.10c) and (2.10d) can be obtained by replacing σ^x by σ^y or σ^z, in the above derivation.

Following Baxter,[14] we define

$$P_o = \sigma_1^x\sigma_3^x\cdots\sigma_{N-1}^x,$$

$$P_e = \sigma_2^x\sigma_4^x\cdots\sigma_N^x,$$

then

$$(P_oT(w_1,w_2,w_3,w_4)P_e)_{\alpha|\alpha'} = \mathrm{Tr}\prod_{J=1}^{N}(\sigma^x\underline{R})_{\alpha_{2J-1},\alpha_{2J-1}'}\underline{\sigma^x\sigma^x}(\underline{R}\sigma^x)_{\alpha_{2J},\alpha_{2J}'} = T(w_4,w_3,w_2,w_1)_{\alpha|\alpha'}. \quad (2.10e)$$

Equivalence transformations and taking the transpose leave the trace of a matrix invariant. Using (2.9) and the transformations (2.10), one easily derives the symmetries (2.2) of the partition function.

The eigenstates of the transfer matrix lie in invariant subspaces labeled by ν' and ν''. We shall tabulate the change of ν' and ν'' under the transformations given in (2.10). It is useful to define

$$C' = \prod_{i=1}^{N}\sigma_i^z,$$

$$C'' = \prod_{i=1}^{N}\sigma_i^x = P_oP_e.$$

The operator $C' = (-)^{\nu'}$, when acting on a state with an even or odd number of down arrows. The operator $C'' = (-)^{\nu''}$, when acting on a wave function which is symmetric or antisymmetric under the reversal of all arrows. Clearly,

$$\underline{C}'\,\underline{T}\,\underline{C}' = \underline{T},$$

$$\underline{C}''\,\underline{T}\,C'' = \underline{T}.$$

Let $\Lambda^{\nu',\nu''}(w_1,w_2,w_3,w_4)$ denote an eigenvalue of

$\underline{T}(w_1, w_2, w_3, w_4)$ corresponding to quantum numbers ν' and ν''. Using (2.10), we obtain the following transformation relations (recall $r = \frac{1}{2}N$):

$$\Lambda^{\nu', \nu''}(w_1, w_2, w_3, w_4)$$

$$= \Lambda^{\nu', \nu' + \nu'' + r}(w_2, w_1, w_3, w_4) \quad (2.11a)$$

$$= \Lambda^{\nu' + \nu'' + r, \nu''}(w_1, w_3, w_2, w_4) \quad (2.11b)$$

$$= \Lambda^{\nu'', \nu'}(w_3, w_2, w_1, w_4) \quad (2.11c)$$

$$= (-)^{\nu''} \Lambda^{\nu' + r, \nu''}(w_4, w_3, w_2, w_1) \quad (2.11d)$$

$$= (-)^{\nu'} \Lambda^{\nu', \nu' + \nu''}(w_1, w_2, w_4, w_3). \quad (2.11e)$$

For physical values of the Boltzmann weights (positive), the transfer matrix is of block diagonal form, with each of the two blocks ($\nu' = 0$ and 1) containing strictly positive matrix elements. The Frobenius-Perron[15] theorem implies that each block has its spectral radius as a real positive eigenvalue. The corresponding eigenvectors are nondegenerate, and can be chosen to have all positive components. These eigenvectors cannot be antisymmetric, so they correspond to $\nu'' = 0$.

Baxter[1] has considered the transfer matrix in the fundamental region. Here, the Boltzmann weight b is negative, so the Frobenius theorem does not apply. Baxter[1,14] has shown that in the FR there are two asymptotically degenerate eigenvalues of opposite sign, which are largest in magnitude. These eigenvalues,

$$\Lambda_0^{\nu' \nu''} = (-)^{\nu''} |\Lambda_0^{\nu' \nu''}|$$

correspond to the quantum numbers $\nu' = r$ and $\nu'' = 0, 1$.

It is interesting to note that in the FR the strictly largest eigenvalue has sign $(-)^r$, i.e., it corresponds to $\nu'' = r$. The asymptotically degenerate, smaller eigenvalue corresponds to $\nu'' = r + 1$. To show this, we use (2.11e) to relate the eigenvalues in the FR to those in a physical region:

$$\Lambda_0^{\nu' \nu''}(\mathrm{FR}) = (-)^{\nu'} \Lambda_0^{\nu', \nu' + \nu''} \quad (\text{physical}).$$

In the physical region, the strictly largest eigenvalue is positive, and the corresponding eigenvector is symmetric, i.e., $\nu' + \nu'' = 0$. Since $\nu' = r$, this proves $\nu'' = r$, thus proving our assertion.

In the physical regions corresponding to $T < T_c$, the two asymptotically degenerate eigenvalues correspond to the same value of ν', and opposite value of ν''. In the physical regions where $T > T_c$, the asymptotically degenerate eigenvalues have $\nu'' = 0$, and opposite values of ν'.

The free energy per site of the infinite lattice is given by

$$-\beta f = \lim_{M \to \infty} \lim_{N \to \infty} (MN)^{-1} \ln Z.$$

It follows from (2.9) that f is related to the largest eigenvalue of the transfer matrix by

$$-\beta f = \lim_{N \to \infty} N^{-1} \ln |\Lambda_{\max}(N)|.$$

Baxter[1] has calculated the right-hand side in the FR. Using the symmetries (2.2), this determines f everywhere.

C. Vertical-Arrow Correlation Functions

Consider the correlation function G_R between two vertical arrows in the same column, separated by R vertices. In the usual manner,[7] G_R is related to the transfer matrix:

$$G_R(w_1, w_2, w_3, w_4) = Z^{-1} \operatorname{Tr} \sigma^z T^R(w_1, w_2, w_3, w_4) \times \sigma^z T^{M-R}(w_1, w_2, w_3, w_4). \quad (2.12)$$

Using the cyclic invariance of the trace together with (2.10d), we may also write

$$G_R(w_1, w_2, w_3, w_4) = Z^{-1} \operatorname{Tr} \sigma^y T^R(w_1, w_3, w_2, w_4) \times \sigma^y T^{M-R}(w_1, w_3, w_2, w_4). \quad (2.13)$$

Using (2.12) together with the transformations (2.10), we derive the symmetries of G_R:

$$G_R(w_1, w_2, w_3, w_4) = G_R(\pm w_1, \pm w_2, \pm w_3, \pm w_4) \quad (2.14a)$$

$$= G_R(w_2, w_1, w_3, w_4) \quad (2.14b)$$

$$= (-)^R G_R(w_4, w_3, w_2, w_1). \quad (2.14c)$$

Relations (2.10a)–(2.10c) were used to derive (2.14a). In proving (2.14b), we used the expression analogous to (2.10d), which is obtained by replacing σ^x by σ^z in its derivation. By using (2.10e), we proved (2.14c).

Unlike Z, G_R cannot be calculated everywhere from knowledge of G_R with w_i in the FR. There exist no symmetries of G_R, analogous to (2.1b), relating correlation functions of eight-vertex models above T_c, to correlation functions of eight-vertex models below T_c. It is possible, however, to compute G_R everywhere, from knowledge of the transfer matrix with w_i in the FR. For $T < T_c$, this can be done by using (2.12). The symmetries (2.14) allow us to arrange the arguments of the transfer matrix so that they lie in the FR. For $T > T_c$, we use (2.13). The symmetries (2.14) enable us to place the arguments of the transfer matrix in the FR.

We wish to study the asymptotic decay of G_R for large R. Since the transfer matrix is diagonalizable and it commutes with the Hermitian X-Y-Z Hamiltonian, it has a complete orthonormal set of eigenvectors. For $T < T_c$, upon introducing the complete set into (2.12), we find

$$G_R^{(N,M)} = Z^{-1} \sum_{mn} |\langle \psi_m(N)|\sigma^x|\psi_n(N)\rangle|^2 \times \left(\frac{\Lambda_n(N)}{\Lambda_m(N)}\right)^R \Lambda_m^M(N). \quad (2.15)$$

We first take the limit $M \to \infty$, so all matrices remain finite dimensional, and $\Lambda_0^{\nu',\nu}(N)$ is strictly the largest eigenvalue. The superscripts denote the values of ν' and ν''.

$$(-)^{qR} G_R^{(N)} = \sum_n |\langle \psi_0^{\nu,\nu}(N)|\sigma^x|\psi_n^{\nu,\nu+1}(N)\rangle|^2 \left(\frac{\Lambda_n(N)}{\Lambda_0(N)}\right)^R \quad (2.16)$$

Here, $q=0$ or 1, when $w_4 < w_1$ or $w_1 < w_4$, respectively. All eigenvalues are evaluated in the FR.

For $T > T_c$, we use (2.13) to obtain

$$(-)^{qR} G_R^{(N)} = \sum_n |\langle \psi_0^{\nu,\nu}(N)|\sigma^y| \times \psi_n^{\nu+1,\nu+1}(N)\rangle|^2 \left(\frac{\Lambda_n(N)}{\Lambda_0(N)}\right)^R. \quad (2.17)$$

Again all eigenvalues are evaluated in the FR. The correlation function in the thermodynamic limit is

$$G_R = \lim_{N\to\infty} G_R^{(N)}.$$

The matrix element $\langle \psi_0^{\nu,\nu}(N)|\sigma^x|\psi_0^{\nu,\nu+1}(N)\rangle$, between the eigenvectors of the two asymptotically degenerate largest eigenvalues, is not forced to be zero by symmetry considerations. This corresponds to the possibility of the existence of long-range order below T_c. The matrix element $\langle \psi_0^{\nu,\nu}(N)|\sigma^y|\psi_0^{\nu,\nu+1}(N)\rangle$ vanishes, since σ^y changes the evenness or oddness of the number of down arrows in a row. This corresponds to the absence of long-range order above T_c. Note that $\langle \psi_0^{\nu,\nu''}(N)|\sigma^{y,z}|\psi_0^{\nu,\nu''}(N)\rangle$ vanish, since both σ^y and σ^z change the symmetry of the wave function.

In the FR, the two exponentially degenerate eigenvalues $\pm\Lambda_0$, largest in magnitude, have opposite signs. When $T \neq T_c$, there is a band of complex eigenvalues Λ_e, next largest in magnitude. For N large, these are closely spaced, and the largest of the band Λ_1 is separated in absolute value from Λ_0 by an $O(e^N)$ gap. There exist other bands of complex eigenvalues, but the eigenvalue largest in magnitude in each of these other bands is smaller than Λ_1.

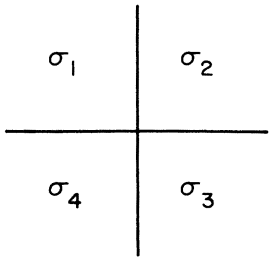

FIG. 2. The configuration of a vertex can be specified by giving the values of the Ising spins σ_1, σ_2, σ_3, and σ_4. The corresponding Boltzmann weight is $\exp(\beta J_1\sigma_1\sigma_3 + \beta J_2\sigma_2\sigma_4 + \beta J_4\sigma_1\sigma_2\sigma_3\sigma_4)$.

In Secs. IV and V we compute the continuous complex dispersion curves to which the next-largest eigenvalues converge as $N \to \infty$. Let us first consider the case $\mu \leq \frac{1}{2}\pi$. (See Appendix A for details of Baxter's[1] parametrization.) We find that for large N, the band of next-largest eigenvalues should lie $O(1/N)$ from the dispersion curve:

$$\frac{\Lambda_e}{\Lambda_0} = k_2 \,\mathrm{sn}\left[\frac{K_2}{\pi}(\phi_1 - i\alpha)\right] \mathrm{sn}\left[\frac{K_2}{\pi}(\phi_2 - i\alpha)\right]. \quad (2.18)$$

The elliptic functions are of modulus k_2, determined by $\pi K_2'/K_2 = 2\lambda$. The quantum numbers ϕ_1 and ϕ_2 lie in the interval $[-\pi, \pi]$.

We expect the matrix elements appearing in (2.16) and (2.17), which correspond to the two-parameter dispersion curve (2.18), to be $O(1/N)$. If in the limit $N \to \infty$, the matrix elements are continuous in ϕ_1 and ϕ_2, then the sum over states can be converted to an integral over the quantum numbers ϕ_1 and ϕ_2. Moreover, only the band of next-largest eigenvalues is expected to contribute to the leading asymptotic behavior of G_R, for large R. When $\mu \leq \frac{1}{2}\pi$, and R is large, we obtain

$$(-)^{qR} G_R - (\text{long-range order}) \approx \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d\phi_1\, d\phi_2 [\rho_1(\phi_1, \phi_2; \lambda, \tau; \pm) + (-)^R \rho_2(\phi_1, \phi_2; \lambda, \tau; \pm)] (\Lambda_e/\Lambda_o)^R. \quad (2.19)$$

The functions ρ_1 and ρ_2 are positive and do not depend on Baxter's parameter α, since the wave functions do not depend on α. The $\pm$ signs indicate above and below T_c. Our study of the decoupling limit ($\mu = \frac{1}{2}\pi$), in Sec. III, indicates that there are two dispersion curves of opposite sign[16] contributing to (2.19). This accounts for the $(-)^R$. Here, Λ_e/Λ_o is given by (2.18).

When $\mu > \frac{1}{2}\pi$, the situation is more complicated. There exist dispersion curves lying higher than the one given in (2.18). These are discussed in Sec. V, and they correspond to bound states appearing in the spectrum of the X-Y-Z model. The correlation functions for $\mu < \frac{1}{2}\pi$ and $\mu > \frac{1}{2}\pi$ are discussed in Sec. VI.

Let us also consider the correlation function C_R between two vertical arrows in a row, separated by R vertices:

$$C_R^{(N,M)} = Z^{-1} \mathrm{Tr}\sigma_0^x \sigma_R^x T^M. \quad (2.20)$$

In the limit $M \to \infty$,

$$C_R^{(N)} = \langle \psi_0^{r,r}(N) | \sigma_0^z \sigma_R^z | \psi_0^{r,r}(N) \rangle . \quad (2.21)$$

Since the eigenvectors do not depend on α, this proves $C_R^{(N)}$ is independent of α. Using the fact that the transfer matrix commutes with the X-Y-Z Hamiltonian of Eq. (2.8), we see that $C_R^{(N)}$ is also the correlation function, in the ground state, between two spins separated by R sites of the X-Y-Z model of (2.8). In the thermodynamic limit

$$C_R = \lim_{N\to\infty} C_R^{(N)}.$$

We note that G_r and C_r correspond to four-spin correlation functions in the Ising model equivalent[17] (see Fig. 2) to the eight-vertex model.

At $T = T_c$, the largest Boltzmann weight equals the sum of the other three. To be specific, we consider $c = a + b + d$. From (2.8), we see that in this case the transfer matrix commutes with the Hamiltonian:

$$H = -\frac{1}{2}\sum_{j=1}^{N} [(1+\gamma)\sigma_j^x \sigma_{j+1}^x + (1-\gamma)\sigma_j^y \sigma_{j+1}^y - (1+\gamma)\sigma_j^z \sigma_{j+1}^z], \quad (2.22a)$$

$$\gamma = d(a+b+d)/ab. \quad (2.22b)$$

At $T = T_c$, $C_R^{(N)}$ is given by (2.21), where ψ_0 is the ground-state eigenfunction of (2.22). The Heisenberg-Ising Hamiltonian,

$$H' = -\tfrac{1}{2}(1+\gamma)\sum_{j=1}^{N}\left[\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \left(\frac{\gamma-1}{\gamma+1}\right)\sigma_j^z \sigma_{j+1}^z\right], \quad (2.23)$$

is unitarily equivalent to (2.22), where γ is still given by (2.22b). If ψ_0' is the ground-state eigenvector of H', then at $T = T_c$,

$$(-)^R C_R^{(N)} = \langle \psi_0'(N) | \sigma_0^y \sigma_R^y | \psi_0'(N) \rangle . \quad (2.24)$$

D. Six-Vertex Model

We conclude this section by contrasting the properties of the six-vertex model[18,19] with those of the eight-vertex model. The six-vertex model corresponds to the Boltzmann weight $d = 0$. In other words, $w_1 = w_2$, which is on the boundary of the regions discussed in Sec. II A. The weak graph transformation (2.1b) and (2.1c), which relates an eight-vertex model above T_c, to an eight-vertex model below T_c, will not, in general, transform a six-vertex model into another six-vertex model. Therefore, in the six-vertex model, the spectra of the transfer matrix for temperatures above, and below T_c, must be separately investigated, and are quite different. Above T_c, the spectrum does not have a gap, and the correlation function G_R falls off algebraically. Below T_c, the spectrum has a gap, and G_R decays exponentially. In the eight-vertex model there is a gap both above and below T_c, and G_R decays algebraically only at T_c.

From (2.2), we see that the partition function for the six-vertex model above T_c is that of an eight-vertex model at T_c. Furthermore, from the commutation relation (2.8) of Sutherland,[8] and McCoy and Wu,[8] we directly see that

$$C_R = \langle \psi_0 | \sigma_0^z \sigma_R^z | \psi_0 \rangle , \quad (2.25)$$

where ψ_0 is the ground-state wave function of the Heisenberg-Ising Hamiltonian

$$H = -\frac{1}{2}\sum_{j=1}^{N}(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z), \quad (2.26a)$$

with

$$\Delta = (a^2 + b^2 - c^2)/2ab. \quad (2.26b)$$

Comparing this with (2.24), we see that even though the partition function of the six-vertex model above T_c is the same as an eight-vertex model at T_c, the correlation functions are not identical.

From (2.26) we see that the six-vertex model at $T = \infty$ corresponds to a Heisenberg-Ising model with $\Delta = +\frac{1}{2}$. For such a Heisenberg-Ising model the correlation function C_R is expected *not* to vanish. In fact, from the result of Yang and Yang[20] for the ground-state energy per spin ($= 2f$), we can compute $C_1 = -4(\partial f/\partial\Delta)$. This is plotted in Fig. 3, where it is clear that C_1 vanishes only at $\Delta = 1^-$. This failure of all the correlation functions of the six-vertex model to vanish at $T = \infty$ is clearly unphysical, and is a defect of the model. This defect is not present in the eight-vertex model ($d \neq 0$), since from (2.8b) we see that as $T \to \infty$, $\gamma \to 1$ and $\Delta \to 0$. Therefore, the Hamiltonian (2.8a) degenerates into $-\frac{1}{2}\sum_{j=1}^{N}\sigma_j^x \sigma_{j+1}^x$, and for this system $\langle \psi_0 | \sigma_0^z \sigma_R^z | \psi_0 \rangle = 0$.

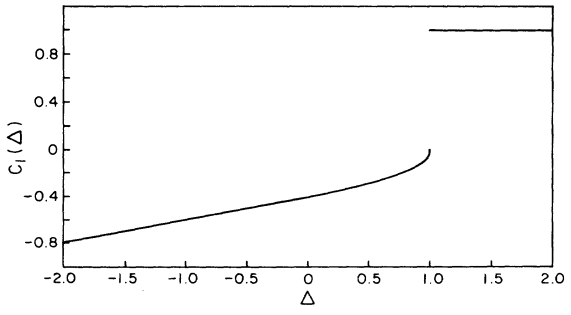

FIG. 3. Short-range order in the ground state $C_1(\Delta)$ of the Heisenberg-Ising model is plotted against Δ. We have used the relation $C_1(\Delta) = -4\partial f/\partial\Delta$, where $2f$ is the ground-state energy per spin of the infinite chain (Ref. 20). Note that $C_1(\frac{1}{2}) \neq 0$. The short-range order is antiferromagnetic for all $\Delta < 1$. Note (Ref. 20) that $\partial C_1(\Delta)/\partial\Delta \to \infty$ as $\Delta \to 1^-$.

III. THE DECOUPLING LIMIT

The eight-vertex model is equivalent[17] to two interpenetrating Ising lattices with nearest-neighbor coupling, interacting with one another via a four-spin coupling (see Figs. 1 and 2). The Boltzmann weights of the eight-vertex model can be written in terms of the corresponding Ising parameters:

$$a = e^{\beta(J_1+J_2+J_4)},\quad b = e^{\beta(-J_1-J_2+J_4)}$$

$$c = e^{\beta(-J_1+J_2-J_4)},\quad d = e^{\beta(J_1-J_2-J_4)}.$$

In the decoupling limit ($J_4 = 0$), the two Ising lattices become independent of each other. In this case, the correlation function G_R between two vertical arrows in a column, and the correlation function C_R between two vertical arrows in a row, can be related to Ising-model correlation functions (see Fig. 4):

$$G_{2R} = C_{2R} = \langle \mu_{00}\mu_{RR}\rangle^2, \tag{3.1}$$

$$G_{2R+1} = \langle \mu_{00}\mu_{R,R+1}\rangle\langle \mu_{00}\mu_{R+1,R}\rangle, \tag{3.2}$$

$$C_{2R+1} = \langle \mu_{00}\mu_{RR}\rangle\langle \mu_{00}\mu_{R+1,R+1}\rangle. \tag{3.3}$$

Here, $\langle \mu_{00}\mu_{MN}\rangle$ is the correlation function between two Ising spins, separated by M rows and N columns. The leading asymptotic behavior of $\langle \mu_{00}\mu_{MN}\rangle$ has been computed by Cheng and Wu[13] for M^2+N^2 large, and J_1, $J_2 > 0$.

Note that (2.21) implies C_R does not depend on Baxter's parameter α. In the decoupling limit, G_{2R} also does not depend on α, since $G_{2R} = C_{2R}$. At first glance, this conclusion seems difficult to reconcile with the expression for G_R given in (2.19). The α dependence of the integrand on the right-hand side must cancel out upon performing the integration. This cancellation is possible, because the eigenvalues are complex. We shall explicitly exhibit this cancellation.

There is another interesting consequence of the eigenvalues being complex. A well-known argument[6,7] indicates that for large R.

$$G_R - (\text{long-range order}) \sim (\Lambda_1/\Lambda_0)^R, \tag{3.4}$$

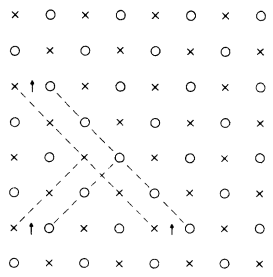

FIG. 4. The two interpenetrating Ising sublattices are represented by ×'s and O's. The vertical arrows correspond to the arrows of the eight-vertex model. An arrow points up (down), if the two adjacent Ising spins have the same (opposite) sign. The correlation function between two vertical arrows in the eight-vertex model corresponds to a four-spin Ising correlation function. For even separations, in the decoupling limit, the correlation function between two vertical arrows in a column equals the correlation function between two vertical arrows in a row. Both are equal to the square of the corresponding diagonal Ising correlation function.

where Λ_1 is the eigenvalue largest in magnitude among the next-largest eigenvalues, and Λ_0 is the largest eigenvalue of the transfer matrix. The meaning of ~ is that there may be additional polynomial behavior in R resulting from the matrix elements. In our case, Λ_1/Λ_0 depends on α, since it is given by (2.18), with $\phi_1 = \phi_2 = \pi$. This proves (3.4) is incorrect, since G_{2R} does not depend on α. The standard argument fails, because the cancellations resulting from the varying phases of the complex eigenvalues cause G_R to fall off faster than predicted by (3.4). Even to determine the leading exponential behavior for R large, it is necessary to integrate over the entire band of next-largest eigenvalues.

In the decoupling limit, we can write down the asymptotic expansion of G_R by using the results of Cheng and Wu[13] for the Ising-model correlation functions. Relating the Ising parameters to Baxter's elliptic-function parametrization (Appendix A), we can rewrite the results of Cheng and Wu in an elegant form. For $T < T_c$,

$$\langle \mu_{00}\mu_{N,N+p}\rangle \approx (1-k_2^2)^{1/4}\left[1 + \frac{K_2^2 k_2^2}{2\pi^4}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d\phi_1\, d\phi_2\, \mathrm{sn}^2\frac{K_2}{\pi}(\phi_1-\phi_2)\right.$$
$$\left.\times\left(k_2\,\mathrm{sn}\frac{K_2}{\pi}(\phi_1+i\lambda)\,\mathrm{sn}\frac{K_2}{\pi}(\phi_2+i\lambda)\right)^p\left(k_2\,\mathrm{sn}\frac{K_2}{\pi}(\phi_1-i\alpha)\,\mathrm{sn}\frac{K_2}{\pi}(\phi_2-i\alpha)\right)^{2N+p}\right]. \tag{3.5}$$

For $T > T_c$,

$$\langle \mu_{00}\mu_{N,N+p}\rangle \approx \frac{K_2}{\pi^2}(1-k_2^2)^{1/4}\int_{-\pi}^{\pi} d\phi\left(k_2^{1/2}\,\mathrm{sn}\frac{K_2}{\pi}(\phi+i\lambda)\right)^p\left(k_2^{1/2}\,\mathrm{sn}\frac{K_2}{\pi}(\phi-i\alpha)\right)^{2N+p}. \tag{3.6}$$

Here the elliptic functions are of the modulus k_2. In the decoupling limit,

$$k_2=(\sinh 2\beta J_1 \sinh 2\beta J_2)^{-1}, \quad (T<T_c) \qquad k_2=\sinh 2\beta J_1 \sinh 2\beta J_2, \quad (T>T_c).$$

Then $\lambda=\pi K_2'/2K_2$ determines λ in terms of J_1 and J_2.

Substituting (3.5) in (3.1) and (3.2), we find for $T<T_c$,

$$G_R \approx (1-k_2^2)^{1/2}\left(1+\frac{K_2^2 k_2^2}{\pi^4}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d\phi_1\, d\phi_2\, \mathrm{sn}^2\frac{K_2}{\pi}(\phi_1-\phi_2)[(-)^R\cos^2\tfrac{1}{2}(\theta_1+\theta_2)+\sin^2\tfrac{1}{2}(\theta_1+\theta_2)](\Lambda_e/\Lambda_0)^R\right). \tag{3.7}$$

For $T>T_c$, we substitute (3.6) into (3.1) and (3.2), and obtain

$$G_R \approx \frac{K_2^2}{\pi^4}(1-k_2^2)^{1/2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d\phi_1\, d\phi_2\,[\cos^2\tfrac{1}{2}(\theta_1-\theta_2)+(-)^R\sin^2\tfrac{1}{2}(\theta_1-\theta_2)](\Lambda_e/\Lambda_0)^R. \tag{3.8}$$

We have defined $\theta_i=\mathrm{am}[(K_1/\pi)\phi_i, k_1]$, the modulus k_1 being determined by $\pi K_1'/K_1=\lambda$. The dispersion curve Λ_e/Λ_0 is defined in (2.18). Note these expressions for G_R are of the form given in (2.19).

We shall explicitly show that G_{2R} does not depend on α. From (3.7) and (3.8), we find

G_{2R} – (long-range order)

$$\approx \frac{K_2^2}{\pi^4}(1-k_2^2)^{1/2}k_2^{2R}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d\phi_1\, d\phi_2\,\rho(\phi_1-\phi_2) \times \mathrm{sn}^{2R}\left[\frac{K_2}{\pi}(\phi_1-i\alpha)\right]\mathrm{sn}^{2R}\left[\frac{K_2}{\pi}(\phi_2-i\alpha)\right], \tag{3.9a}$$

$$\rho(\phi_1-\phi_2)=k_2^2\,\mathrm{sn}^2\frac{K_2}{\pi}(\phi_1-\phi_2), \quad (T<T_c) \tag{3.9b}$$

$$\rho(\phi_1-\phi_2)=1, \quad (T>T_c). \tag{3.9c}$$

The function $\mathrm{sn}(K_2\phi/\pi)$ has poles at $\phi=2\pi m+(2n+1)2i\lambda$, and the function $\mathrm{sn}^2(K_2\phi/\pi)$ has real period 2π. Since $|\alpha|<\lambda$ in the FR, Cauchy's theorem allows us to move the integration contour from the real axes to the lines $\phi_1+i\alpha$ and $\phi_2+i\alpha$ (see Fig. 5). In this way, we find

G_{2R} – (long-range order)

$$\approx \frac{K_2^2}{\pi^4}(1-k_2^2)^{1/2}k_2^{2R}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d\phi_1\, d\phi_2 \times\rho(\phi_1-\phi_2)\,\mathrm{sn}^{2R}\frac{K_2\phi_1}{\pi}\,\mathrm{sn}^{2R}\frac{K_2\phi_2}{\pi}. \tag{3.10}$$

There is no α dependence in this expression.

We can also explicitly demonstrate that the correlation length is not given by the next-largest eigenvalue. From (3.7) and (3.8), it follows that

$$G_R - (\text{long-range order}) \sim k_2^R. \tag{3.11}$$

Equation (2.18) shows that $k_2<\Lambda_1/\Lambda_0$, so the correlation function falls off faster than predicted by (3.4). This demonstrates that when the transfer matrix has a continuous band of complex next-largest eigenvalues, one must be very careful in applying the standard argument of (3.4).

IV. DETERMINATION OF EIGENVALUES OF TRANSFER MATRIX

A. Baxter's Equations

We shall review Baxter's equations for the eigenvalues of the transfer matrix. Baxter's parametrization of the vertex weights is summarized in Appendix A. Some useful properties of elliptic Θ functions are collected in Appendix B. In the following, we shall use Baxter's notation without giving definitions in the text. The reader should consult the appendices for details.

Baxter has shown that the eigenvalues $T(v)$ of the transfer matrix satisfy the equation

$$T(v)Q(v)=\phi(v+\eta)Q(v-2\eta)+\phi(v-\eta)Q(v+2\eta). \tag{4.1}$$

Here

$$\phi(v)=[\rho\Theta(0)h(v)]^N, \tag{4.2a}$$

$$Q(v)=\exp(-(\tfrac{1}{2}i)\nu\pi v/K_k)\prod_{j=1}^{r}h(v-v_j), \tag{4.2b}$$

$$h(v)=H(v)\Theta(v). \tag{4.2c}$$

Recall $N=2r$ is the number of vertices in a row.

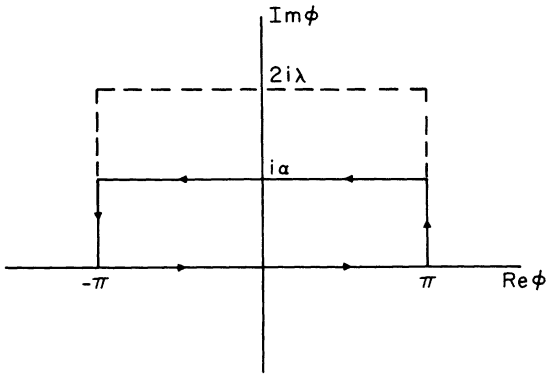

FIG. 5. When there are no singularities of the integrand inside the contour, the integral around it is zero by Cauchy's theorem. When the integrand has 2π as a real period, the two vertical pieces of the contour cancel each other.

Also,

$$\nu+\nu'+r=\text{even integer}, \tag{4.3a}$$

$$K_k^{-1}\left(\tfrac{1}{2}i\nu K_k'-\sum_{j=1}^{r}v_j\right)=\nu''+r+(\text{even integer}). \tag{4.3b}$$

One obtains the eigenvalues $T(v)$ from Eq. (4.1) in the following manner. Although the function $\phi(v)$ is known, the function $Q(v)$ is parametrized by its as yet unknown zeros $v_1,\ldots,v_r$. One obtains a set of r coupled nonlinear equations for the zeros by noting that the left-hand side of Eq. (4.1) vanishes when $v=v_j$, $(j=1,\ldots,r)$:

$$\left(\frac{h(v_j+\eta)}{h(v_j-\eta)}\right)^N=-e^{-2\pi i\nu\eta/K_k}\times\prod_{i=1}^{r}\frac{h(v_j-v_i+2\eta)}{h(v_j-v_i-2\eta)},\quad(j=1,\ldots,r). \tag{4.4}$$

Once the zeros are determined from Eq. (4.4), the function $Q(v)$ is known, and $T(v)$ can be determined from Eq. (4.1). One should note that if $4\eta=2m_1K_k+im_2K_k'$, where m_1 and m_2 are integers, then the equations for $v_1,\ldots,v_r$ split up into r independent equations. This occurs for the Ising, dimer, and free fermion models, which can be solved by the Pfaffian method. Let us observe that, because of the periodicity of (4.4), we may confine our attention to $-K_k\leq \mathrm{Re}v_j\leq K_k$.

B. Integral-Equation Formalism for Calculation of Largest Eigenvalue

Baxter[1] computed the maximum eigenvalues $(\pm\Lambda_0)$ in the thermodynamic limit $N\to\infty$, by developing a new perturbation-expansion technique. He remarked that one could also use the integral-equation method,[20] applied previously to models solvable by Bethe ansatz.[18-21] We adopt the integral equation approach because it is simpler for the computation of the next-largest eigenvalues, for which some of the zeros are complex. To illustrate the integral-equation technique, we use it to reproduce Baxter's result for the largest eigenvalues. We then proceed to the calculation of the next-largest eigenvalues.

Because of the symmetry properties of the partition function, it is sufficient to calculate the largest eigenvalue in the fundamental region

$$0\leq\alpha<\lambda<\tau. \tag{4.5}$$

For the maximum eigenvalues, the quantum number $\nu=0$, and the zeros are all real.[1] In the FR, there are two eigenvalues of maximum modulus, but opposite sign, corresponding to the two values of the quantum number $\nu''=0,1$. The quantum number ν' is fixed by (4.3a), once ν and ν'' are specified.

When $\nu=0$, the logarithm of Eq. (4.4) is

$$NF_1(\phi_j)=2\pi i I_j+\sum_{i=1}^{r}F_2(\phi_j-\phi_i),\quad(j=1,\ldots,r). \tag{4.6}$$

The I_j are half-integers specifying the branches of logarithm. The largest eigenvalues will correspond to

$$I_j-I_{j-1}=1,\quad(j=2,\ldots,r). \tag{4.7}$$

Here, $\phi_j=\pi v_j/K_k$, and $F_p(\phi_j)=\ln[h(v_j+p\eta)/h(v_j-p\eta)]$. As $N\to\infty$, the ϕ_j become infinite in number, but always lie in the interval $[-\pi,\pi]$. We assume that the number of ϕ_j in the interval $[\phi,\phi+d\phi]$ approaches

$$NR(\phi)d\phi.$$

We will use the relation

$$\frac{1}{N}\sum_{i=1}^{r}\to\int_{-\pi}^{\pi}d\phi R(\phi),\quad(N\to\infty). \tag{4.8}$$

Applying standard arguments,[20] the coupled nonlinear equations (4.6) are replaced, for $N\to\infty$, by a linear integral equation for $R(\phi)$:

$$\frac{d}{d\phi}F_1(\phi)=-2\pi iR(\phi)+\int_{-\pi}^{\pi}d\phi'R(\phi')\frac{d}{d\phi}F_2(\phi-\phi'). \tag{4.9}$$

Since this equation has a difference kernel, and the integration is over a full period, it can be solved by Fourier transform. Using (B14) of Appendix B, we find

$$2\pi R_m=\int_{-\pi}^{\pi}d\phi e^{-im\phi}R(\phi)=\frac{1}{2\cosh m\lambda}. \tag{4.10}$$

The minus sign in (4.9) was chosen so that $R(\phi)$ would be positive. We note $2\pi R_0=\frac{1}{2}$, as it must to be consistent with (4.8).

Baxter has shown that when $K_k'/2>\mathrm{Im}v>0$, the first term on the right-hand side of (4.1) exponentially dominates the second for large N. Hence,

$$T(v)\approx\phi(v+\eta)\frac{Q(v-2\eta)}{Q(v)}. \tag{4.11}$$

Employing (A13), we find

$$T(v)\approx c^N\left(\frac{\theta(0)}{\theta(v-\eta)}\,\frac{H(v+\eta)}{H(2\eta)}\right)^N\frac{Q(v-2\eta)}{Q(v)}. \tag{4.12}$$

Using the periodicity property (B6), together with (B7), we Fourier expand the logarithm of both quotients of θ functions appearing in (4.12). Also,

using (4.2b), (4.8), and (B8), we find

$$\frac{1}{N}\ln\frac{Q(v-2\eta)}{Q(v)} \approx 2\pi(\lambda-\alpha-i\pi)R_0 + \sum_{m=1}^{\infty}\frac{\sinh m(\tau-\lambda)\sinh m(\lambda-\alpha)}{m\sinh m\tau}(4\pi R_m). \tag{4.13}$$

Then, combining terms, we obtain

$$\frac{1}{N}\ln[(-)^r T(v)] \approx \ln c + 2\sum_{m=1}^{\infty}\frac{\sinh^2 m(\tau-\lambda)\{\cosh m\lambda - \cosh m\alpha\}}{m\sinh 2m\tau\cosh m\lambda}. \tag{4.14}$$

This is Baxter's result for the maximum eigenvalue, corresponding to $(-)^{r+\nu''}=1$. At the end of Sec. IV C, we show that the eigenvalue corresponding to $\tilde{I}_j = I_j + 1$ has the opposite value of ν'', and the opposite sign, as the eigenvalue corresponding to I_j.

C. Integral Equations Determining Next-Largest Eigenvalues

Let us derive the equations determining the next-largest eigenvalues by studying the effect of changing one half-integer I_κ from its value in the equations determining the maximum eigenvalue, to the new value I'_κ. Consider an eigenvalue Λ_e corresponding to half-integers I'_j, zeros ϕ'_j, and quantum number $\nu = \nu_e$, satisfying

$$NF_1(\phi'_j) = 2\pi i I'_j + 2\nu_e\lambda + \sum_{i=1}^{r} F_2(\phi'_j - \phi'_i), \quad (j=1,\ldots,r). \tag{4.15}$$

We consider the case

$$I'_j = I_j + \omega_1, \quad (j\neq\kappa)$$

where $\omega_1 = 0, 1$ corresponds to the possibility of Λ_e having the same or opposite sign as the eigenvalue specified by I_j, denoted Λ_0. Subtracting (4.6) from (4.15) yields

$$N[F_1(\phi'_j) - F_1(\phi_j)] = 2\pi i\omega_1 + 2\nu_e\lambda + \sum_{i=1}^{r}[F_2(\phi'_j - \phi'_i) - F_2(\phi_j - \phi_i)], \quad (j\neq\kappa). \tag{4.16}$$

We assume $\phi'_j \approx \phi_j$ except for $j=\kappa$. Then, defining

$$\chi(\phi_j) = N(\phi'_j - \phi_j), \quad (j\neq\kappa)$$

we obtain

$$\chi(\phi_j)\frac{d}{d\phi_j}F_1(\phi_j) = 2\pi i\omega_1 + 2\nu_e\lambda + \frac{1}{N}\sum_{i\neq\kappa}[\chi(\phi_j)-\chi(\phi_i)]\frac{d}{d\phi_j}F_2(\phi_j-\phi_i) + F_2(\phi_j-\phi'_\kappa) - F_2(\phi_j-\phi_\kappa) + O\left(\frac{1}{N}\right). \tag{4.17}$$

In the limit $N\to\infty$, (4.17) becomes

$$-2\pi i J(\phi) + \int_{-\pi}^{\pi} d\phi'' J(\phi'')\frac{d}{d\phi}F_2(\phi-\phi'') = 2\pi i\omega_1 + 2\nu_e\lambda + F_2(\phi-\phi'_\kappa) - F_2(\phi-\phi_\kappa). \tag{4.18}$$

Here, $J(\phi) = R(\phi)\chi(\phi)$, and we have used (4.9) to eliminate some terms. We note

$$\sum_{i\neq\kappa}(\phi'_i - \phi_i) = \int_{-\pi}^{\pi} d\phi J(\phi).$$

From (4.3a) and (4.3b), we find

$$\nu_e + \nu'_e - \nu'_0 = \text{even integer}, \tag{4.19a}$$

$$\frac{1}{\pi}\left(i\nu_e\tau - (\phi'_\kappa - \phi_\kappa) - \int_{-\pi}^{\pi} d\phi J(\phi)\right) = \nu''_e - \nu''_0 + (\text{even integer}). \tag{4.19b}$$

The subscripts 0 and e denote quantities corresponding to the largest and next-largest eigenvalues, respectively.

It follows from (4.11) that for $N\to\infty$,

$$\ln\frac{\Lambda_e}{\Lambda_0} \approx \ln\frac{Q(v-2\eta)}{Q(v)}\bigg|_e - \ln\frac{Q(v-2\eta)}{Q(v)}\bigg|_0. \tag{4.20}$$

Using this relation together with (4.2b), (4.8), and (B8), we show

$$\ln\frac{\Lambda_e}{\Lambda_0} = -\nu_e\lambda - 2\pi i J_0 + 2\pi i\sum_{m\neq 0} J_m\frac{\sinh m(\lambda-\tau)}{\sinh m\tau}e^{m(\lambda-\alpha)} + \Omega, \tag{4.21a}$$

where

$$\Omega = \ln\frac{h(v-v'_\kappa-2\eta)}{h(v-v'_\kappa)} - \ln\frac{h(v-v_\kappa-2\eta)}{h(v-v_\kappa)}. \tag{4.21b}$$

Equations (4.18), (4.19), and (4.21) are the basis of our calculation of the next-largest eigenvalues. Before proceeding to this calculation, we wish to make the following comment. Baxter has shown

that in the FR there are two largest eigenvalues, exponentially degenerate in magnitude, but with opposite sign. These correspond to opposite values of ν''. If one of these, say Λ_0, is described by I_j as given in (4.7), then the other $\bar{\Lambda}_0$ corresponds to $\bar{I}_j = I_j + 1$. To see this, consider (4.18) with the right-hand side replaced by $2\pi i$. Then one finds $J_0 = -1/2$, $J_m = 0$ for $m \neq 0$. The sum rule (4.19b) becomes

$$\nu_e'' - \nu_0'' = 1 = \frac{1}{\pi}\sum_{j=1}^{r}(\phi_j' - \phi_j) = -2J_0.$$

Also, (4.21a) and (4.21b) imply $\ln(\bar{\Lambda}_0/\Lambda_0) = i\pi$, hence $\bar{\Lambda}_0 = -\Lambda_0$. The quantum number ω_1 appearing in (4.18) denotes the possibilities of Λ_e having the same or opposite sign as Λ_0.

D. Computation of the Next-Largest Eigenvalues

By arguments[21] which are by now standard one can determine certain qualitative features of the solutions to Eqs. (4.4). A solution $v_1, \ldots, v_r$ must be made up of groups known as p-strings. This means, to order e^{-N}, the solutions can be broken up into p-strings centered either about the real axis

$$v_j = v_e + m\eta, \quad m = -(p-1), -(p-3), \ldots, (p-1);$$

or about the line parallel to the real axis and displaced by $i(K_k'/2)$:

$$v_j = v_e + i(\tfrac{1}{2}K_k') + m\eta,$$
$$m = -(p-1), -(p-3), \ldots, (p-1).$$

Here, $p = 1, 2, 3, \ldots$, and if one member of a p-string appears in a solution of the equations, then the other members must also appear. Baxter[22] has shown that not all types of strings are allowed. In the upper half-plane, a string cannot end in a region periodically equivalent (period iK_k') to $-(\frac{1}{2}K_k') < \mathrm{Im}\, v < 0$. In the lower half-plane, a string cannot end in a region periodically equivalent to $0 < \mathrm{Im}\, v < \frac{1}{2}K_k'$. For example, this rules out 2-strings centered about $i(\frac{1}{2}K_k')$. In this paper, we only consider eigenvalues corresponding to p-strings $[(p-1)\lambda < \tau]$ centered about the real axis, and zeros on the line $\frac{1}{2}iK_k'$ denoted 0-strings.

We wish to calculate the band of next-largest eigenvalues Λ_e of the transfer matrix. To accomplish this, it is necessary to specify the branches of all the logarithms appearing in the equations. For $2\lambda < \tau$, we have done this for the excitations of placing zeros on $\frac{1}{2}iK_k'$, and producing p-strings which do not overlap the period: $(p-1)\lambda < \tau$. In Appendix B we define the branches and give the Fourier transforms of all the logarithms of quotients of θ functions appearing in the equations. In this section we have explicitly computed Λ_e/Λ_0 for $2\lambda \leq \tau$. The dispersion curves corresponding to moving one zero to $\frac{1}{2}iK_k'$, and to creating a 2-string, are found to be degenerate. Exciting more than one zero to $\frac{1}{2}iK_k'$, or producing longer strings yield dispersion curves lying lower. In Sec. V we consider the case $2\lambda > \tau$.

We now write the equations determining $\ln(\Lambda_e/\Lambda_0)$, for the case where we take out p zeros ϕ_j, $j \in \{0\}$ with $\{0\}$ having p elements, from the real line distribution for Λ_0, and place them in a p-string with real abscissa ϕ_e. We denote placing one zero on the line $\frac{1}{2}iK_k'$, a $p = 0$ string. Then, since equations (4.18) are linear, we find

$$-2\pi i J(\phi) + \int_{-\pi}^{\pi} d\phi'' J(\phi'')\frac{d}{d\phi}F_2(\phi - \phi'')$$
$$= 2\pi i(\omega_1 + \omega_2) + (1 - \delta_{p,0})$$
$$\times[F_{p+1}(\phi - \phi_e) + F_{p-1}(\phi - \phi_e)] + \delta_{p,0}$$
$$\times G_2(\phi - \phi_e) - \sum_{t \in \{0\}} F_2(\phi - \phi_t). \quad (4.22)$$

The functions F_p and G_p are defined in Appendix B. There is additional freedom in the choice of branches which allows the introduction of $\omega_2 = 0$ or 1. We shall comment on this quantum number after Eq. (4.33):

$$\nu_e = \text{number of zeros on } \tfrac{1}{2}iK_k', \quad (4.23a)$$

$$\nu_e + \nu_e' - \nu_0' = \text{even integer}, \quad (4.23b)$$

$$\pi^{-1}\Big(\sum_{t \in \{0\}} \phi_t - p\phi_e - \delta_{p,0}\phi_e - \int_{-\pi}^{\pi} d\phi\, J(\phi)\Big)$$
$$= \nu_e'' - \nu_0'' + (\text{even integer}). \quad (4.23c)$$

The eigenvalues Λ_e are computed from (4.21a) with

$$\Omega = (1 - \delta_{p,0})[F_p(\phi_e + i\lambda - i\alpha) - 2\pi n]$$
$$+ \delta_{p,0}G_1(\phi_e + i\lambda - i\alpha) + \nu_e\lambda$$
$$- \sum_{t \in \{0\}} F_1(\phi_t + i\lambda - i\alpha). \quad (4.21b')$$

The integer $n = 0, 1, \ldots, p-1$ takes into account the possibility of choosing different branches of the logarithms appearing in the first term of Ω. This ambiguity arises because we have combined the sum of p logarithms into the logarithm of the product of p factors. It appears that $n = 0$ is sufficient for the consideration of the next-largest eigenvalues.

The first case we consider corresponds to all the zeros ϕ_j' being distributed on the real axis between $-\pi$ and $+\pi$, except for one complex zero $\phi_\kappa' = \phi_e + i\tau$. It follows from the sum rule (4.3b) that $\nu_e = 1$ in this case. Using (B11), (B13), and (B14), we solve (4.22) by Fourier transform, ob-

taining for $\lambda<\frac{1}{2}\tau$:

$$2\pi i J_m=\left(\frac{e^{-im\pi}}{m}+\frac{\sinh m(2\lambda-\tau)}{\sinh m\tau}\,\frac{e^{-im\phi_K}}{m}-\frac{\sinh 2m\lambda}{\sinh m\tau}\,\frac{e^{-im\phi_e}}{m}\right)\Big/\left(1-\frac{\sinh m(2\lambda-\tau)}{\sinh m\tau}\right), \tag{4.24a}$$

$$2\pi J_0=\tfrac{1}{2}(\phi_K-\pi)-\pi(\omega_1+\omega_2). \tag{4.24b}$$

From (4.23), we find

$$\nu_e'-\nu_0'=1+(\text{even integer}) \tag{4.25a}$$

$$\begin{aligned}-2J_0&=\omega_1+\omega_2+(\pi-\phi_K)/2\pi\\&=\nu_e''-\nu_0''+(\phi_e-\phi_K)/\pi+(\text{even integer}).\end{aligned} \tag{4.25b}$$

Finally, (4.21) yields

$$\operatorname{Re}\ln\frac{\Lambda_e}{\Lambda_0}=\alpha-\lambda+\sum_{m=1}^{\infty}\frac{\sinh m(\alpha-\lambda)(\cos m\phi_K+\cos m\pi)}{\cosh m\lambda}, \tag{4.26a}$$

$$\operatorname{Im}\ln\frac{\Lambda_e}{\Lambda_0}=\tfrac{1}{2}\phi_K+\pi(\omega_1+\omega_2)+\tfrac{3}{2}\pi+\sum_{m=1}^{\infty}\frac{\cosh m(\alpha-\lambda)\sin m\phi_K}{m\cosh m\lambda}. \tag{4.26b}$$

From our study of the decoupling limit (Sec. III), we see that it is likely that the next-largest eigenvalues are described by a two-parameter dispersion curve. We can obtain a two-parameter excitation, still corresponding to $\nu_e=1$, in the following manner. In addition to removing a zero from $\phi^{(1)}$ and placing it at $\phi_e+i\tau$, also remove a zero from $\phi^{(2)}$ and place it at π. Then for $2\lambda<\tau$,

$$\operatorname{Re}\ln\frac{\Lambda_e}{\Lambda_0}=\alpha-\lambda+\sum_{m=1}^{\infty}\frac{\sinh m(\alpha-\lambda)(\cos m\phi^{(1)}+\cos m\phi^{(2)})}{m\cosh m\lambda}, \tag{4.27a}$$

$$\operatorname{Im}\ln\frac{\Lambda_e}{\Lambda_0}=\tfrac{1}{2}(\phi^{(1)}+\phi^{(2)})+\pi(\omega_1+\omega_2+1)+\sum_{m=1}^{\infty}\frac{\cosh m(\alpha-\lambda)(\sin m\phi^{(1)}+\sin m\phi^{(2)})}{m\cosh m\lambda}. \tag{4.27b}$$

The sum rule (4.23) becomes

$$\omega_1+\omega_2+\frac{\phi^{(1)}+\phi^{(2)}}{2\pi}=\nu_e''-\nu_0''+\frac{\phi_e}{\pi}+(\text{even integer}). \tag{4.28}$$

The ϕ's satisfy $-\pi\leq\phi^{(1)},\ \phi^{(2)}\leq\pi$.

If we let $\omega_1=\nu_e''-\nu_0''$, then $\omega_2=0$ or 1 corresponds to positions of ϕ_e differing by π. This exhibits two different excitations, both corresponding to $\Delta\nu'=0$ and the same value of $\Delta\nu''$.

Equations (4.27a) and (4.27b) can be written neatly in terms of the elliptic function $\operatorname{sn}(K_2u/\pi,k_2)$, where the modulus k_2 is determined by

$$\pi K_2'/K_2=2\lambda.$$

Recall that $\operatorname{sn}(K_2u/\pi,k_2)$ is related to elliptic θ functions corresponding to $q=e^{-2\lambda}$ by

$$k_2^{1/2}\operatorname{sn}(K_2u/\pi,k_2)=H(K_2u/\pi)/\theta(K_2u/\pi).$$

Using the periodicity property (B6), this becomes

$$k_2^{1/2}\operatorname{sn}(K_2u/\pi,k_2)=-ie^{(iu-\lambda)/2}\,\frac{\theta[(K_2/\pi)(u+2i\lambda)]}{\theta[(K_2/\pi)u]}.$$

Using (B7), we can take the logarithm, obtaining

$$\begin{aligned}&\ln\{k_2^{1/2}\operatorname{sn}[(K_2/\pi)(\phi-i\alpha),k_2]\}\\&\quad=\tfrac{1}{2}[\alpha-\lambda+i(\phi-\pi)]+\sum_{m=1}^{\infty}\frac{\sinh m(\alpha-\lambda+i\phi)}{m\cosh m\lambda}.\end{aligned}$$

Now it is clear that (4.27a) and (4.27b) become ($\omega_1=0,1$ and $\omega_2=0,1$)

$$\begin{aligned}\Lambda_e/\Lambda_0=(-)^{\omega_1+\omega_2}k_2\operatorname{sn}[(K_2/\pi)(\phi^{(1)}-i\alpha),k_2]\\\times\operatorname{sn}[(K_2/\pi)(\phi^{(2)}-i\alpha),k_2].\end{aligned} \tag{4.29}$$

Let us consider the case of a complex conjugate pair of zeros $\phi_K'=\phi_e\pm i\lambda$. The sum rule (4.3b) implies $\nu_e=0$. Using (B13) and (B14), we solve (4.22) by Fourier transform, obtaining for $\lambda<2\tau/3$

$$2\pi i J_m=\left[\frac{\sinh m(2\lambda-\tau)}{m\sinh m\tau}(e^{-im\phi^{(1)}}+e^{-im\phi^{(2)}})-\left(\frac{\sinh m(\lambda-\tau)+\sinh m(3\lambda-\tau)}{m\sinh m\tau}\right)e^{-im\phi_e}\right]\Big/\left(1-\frac{\sinh m(2\lambda-\tau)}{\sinh m\tau}\right), \tag{4.30a}$$

$$2\pi J_0=\tfrac{1}{2}(\phi^{(1)}+\phi^{(2)}-2\phi_e)-\pi(\omega_1+\omega_2). \tag{4.30b}$$

Then (4.23) becomes

$$\nu_e' - \nu_0' = (\text{even integer}), \tag{4.31a}$$

$$\omega_1 + \omega_2 + (\phi^{(1)} + \phi^{(2)})/2\pi = \phi_e/\pi + \nu_e'' - \nu_0'' + (\text{even integer}). \tag{4.31b}$$

Again we let $\omega_1 = \nu_e'' - \nu_0''$, so $\omega_2 = 0$ or 1 corresponds to positions of ϕ_e differing by π. There are two excitations corresponding to $\Delta\nu' = 1$ and the same value of $\Delta\nu''$. For $4\lambda - \alpha < 2\tau$, we find

$$\text{Re}\ \ln\frac{\Lambda_e}{\Lambda_0} = \alpha - \lambda + \sum_{m=1}^{\infty} \frac{\sinh m(\alpha-\lambda)[\cos m\phi^{(1)} + \cos m\phi^{(2)}]}{m\cosh m\lambda}, \tag{4.32a}$$

$$\text{Im}\ \ln\frac{\Lambda_e}{\Lambda_0} = \tfrac{1}{2}(\phi^{(1)} + \phi^{(2)}) + \pi(\omega_1 + \omega_2 + 1) + \sum_{m=1}^{\infty} \frac{\cosh m(\alpha-\lambda)[\sin m\phi^{(1)} + \sin m\phi^{(2)}]}{m\cosh m\lambda}. \tag{4.32b}$$

From (4.32), we obtain ($\omega_1 = 0, 1$ and $\omega_2 = 0, 1$)

$$\frac{\Lambda_e}{\Lambda_0} = (-)^{\omega_1+\omega_2} k_2\, \text{sn}[K_2(\phi^{(1)} - i\alpha)/\pi, k_2] \times \text{sn}[K_2(\phi^{(2)} - i\alpha)/\pi, k_2]. \tag{4.33}$$

Let us note that Λ_e/Λ_0 does not depend on the parameter τ. This provides some insight as to how the modulus k_2 arises, for in the decoupling limit $\pi K_2'/K_2 = 2\lambda = \tau = \pi K_k'/2K_k$.

Let us take this opportunity to comment on the factor $(-)^{\omega_1+\omega_2}$. The freedom $\omega_1 = 0$ or 1 corresponds to excitations for which $\Delta\nu'' = 0$ or 1, respectively. The existence of $\omega_2 = 0$ or 1 is suggested by our study of the decoupling limit in Sec. III. It accounts for the presence of the factor $(-)^R$ in Eqs. (3.7) and (3.8). Although the formalism presented here is flexible enough to allow for $\omega_2 = 0$ or 1, it does not predict unambiguously its existence.

V. CALCULATION OF EIGENVALUES FOR $\mu > \frac{1}{2}\pi$

In Sec. IV we studied Baxter's equations for the eigenvalues of the transfer matrix. We derived integral equations which were sufficient to determine the dispersion curves of the next-largest eigenvalues, once the branches of the logarithms appearing in the equations were specified. For $\mu \leq \frac{1}{2}\pi$ we presented a consistent set of branch cuts. We now present a reformulation of the integral equations, and a choice of branch cuts, which allows the calculation of the next-largest eigenvalues in the entire region $0 < \mu < \pi$.

The new formalism is based upon the symmetry relation (2.11d); i.e.,

$$\Lambda(w_1, w_2, w_3, w_4) = (-)^{\nu''}\Lambda(w_4, w_3, w_2, w_1) \tag{5.1a}$$

Defining $\alpha' = \tau - \alpha, \lambda' = \tau - \lambda, \rho' = \rho\exp(-\tau + \frac{3}{2}\lambda + \frac{1}{2}\alpha)$, this relation becomes

$$\Lambda(\rho, \alpha, \lambda, \tau) = (-)^{\nu''}\Lambda(\rho', \alpha', \lambda', \tau). \tag{5.1b}$$

The unprimed variables will be considered to be in the FR, i.e., $0 \leq \alpha < \lambda < \tau$. Then the primed variables will *not* lie in the FR, but will satisfy $0 < \lambda' < \alpha' \leq \tau$. It turns out to be convenient to calculate the eigenvalues in the region $0 < \lambda' < \alpha' \leq \tau$, and then to use (5.1b) to determine the eigenvalues in the FR.

In the FR the maximum eigenvalue corresponds to all the zeros of $Q(\phi)$ lying on the real axis. We shall show that when $0 < \lambda' < \alpha' \leq \tau$, the maximum eigenvalue corresponds to all zeros lying on the line $[-\pi + i\tau, \pi + i\tau]$. We proceed in a manner very similar to that of Sec. IV B. Following the steps leading to (4.9), we derive an equation determining the density of zeros $\tilde{R}(\phi)$ on the $i\tau$ line. We obtain

$$\frac{d}{d\phi}G_1(\phi) = 2\pi i\tilde{R}(\phi) + \int_{-\pi}^{\pi} d\phi'\,\tilde{R}(\phi')\frac{d}{d\phi}F_2(\phi - \phi'), \tag{5.2a}$$

subject to the constraint

$$\int_{-\pi}^{\pi} d\phi\,\tilde{R}(\phi) = \tfrac{1}{2}. \tag{5.2b}$$

Here λ', and not λ, appears in the definition of the functions $G_1(\phi)$ and $F_2(\phi - \phi')$. Equation (5.2a) differs from (4.9) in two ways. First, we choose a plus sign multiplying the term $2\pi i\tilde{R}(\phi)$ in order to assure that the density of zeros be positive. Also, the function $G_1(\phi)$ appears on the left-hand side, rather than $F_1(\phi)$, since the zeros lie on the $i\tau$ line. The solution to (5.2) is

$$2\pi\tilde{R}_m = (2\cosh m\lambda)^{-1}. \tag{5.3}$$

Using (5.3) together with (4.12) with primed variables, and

$$\frac{1}{N}\ln\frac{Q(v' - 2\eta')}{Q(v')} \approx \sum_{m=1}^{\infty} \frac{\sinh m(\tau-\lambda)\sinh m(\lambda-\alpha)}{m\sinh m\tau}(4\pi\tilde{R}_m), \tag{5.4}$$

we find that $(-)^{\nu''}\Lambda_0(\rho', \alpha', \lambda', \tau)$ is indeed given by Baxter's expression (4.14).

It is natural to expect that the next-largest eigenvalues correspond to removing zeros from the

$i\tau$ line. We denote a string of s-zeros centered about the real axis an "s-string":

$$\phi = \phi_e + im\lambda', \quad m = -(s-1), -(s-3), \ldots, (s-1).$$

We shall consider strings of length $s = 0, 1, 2, \ldots, \gamma'$, where $\gamma' = 1 + [\pi/(\pi - \mu)]$. Here, $[x]$ means largest integer $\leq x$. A 0-string corresponds to removing a zero from one position on the $i\tau$ line, and inserting it in another position on the $i\tau$ line.

Following a procedure similar to that of Secs. IV C and IV D we derive the equation for the change in the density of zeros on the $i\tau$ line, caused by the removal of s zeros, and placing them in an s-string:

$$2\pi i\tilde{J}(\phi) + \int_{-\pi}^{\pi} d\phi' \tilde{J}(\phi') \frac{d}{d\phi} F_2(\phi - \phi') = 2\pi i \omega_2 + (1 - \delta_{s,0})[G_{s+1}(\phi - \phi_e) + G_{s-1}(\phi - \phi_e)]$$
$$+ \delta_{s,0} F_2(\phi - \phi_e) - \sum_{i \epsilon \{0\}} F_2(\phi - \phi_i). \tag{5.5}$$

The next-largest eigenvalues are given by

$$\ln \frac{\Lambda_e(\rho', \alpha', \lambda', \tau)}{\Lambda_0(\rho', \alpha', \lambda', \tau)} = \int_{-\pi}^{\pi} \tilde{J}(\phi') \frac{d}{d\phi'} G_1[\phi' + i(\alpha - \lambda)]\, d\phi' + (1 - \delta_{s,0}) F_s[\phi_e + i(\alpha - \lambda)]$$
$$+ \delta_{s,0} G_1[\phi_e + i(\alpha - \lambda)] - \sum_{i \epsilon \{0\}} G_1[\phi_i + i(\alpha - \lambda)]. \tag{5.6}$$

The sum rules are the same as in (4.23), except $\tilde{J}(\phi)$ replaces $J(\phi)$, and s replaces p. Note $\tilde{J}_0$ is not restricted by (5.5). The $m = 0$ mode of (5.5) provides a constraint on the $\{\phi_i\}$ and ϕ_e. The sum rule then determines $\tilde{J}_0$ in terms of the $\{\phi_i\}$. The roles of the sum rules, and the $m = 0$ mode of the equation for the change in density, have been reversed from those of Sec. IV.

Let us consider the situation dealt with in Sec. IV, $\mu \leq \frac{1}{2}\pi$; i.e., $\mu' \geq \frac{1}{2}\pi$. We consider two types of excitations. The first corresponds to $\Delta\nu' = 1$. We remove zeros from $\phi_1 + i\tau$ and $\phi_2 + i\tau$, and place one on the real axis at ϕ_e, and the other at $i\tau + \pi$. The second excitation corresponds to $\Delta\nu' = 0$. Here, we remove two zeros from $\phi_1 + i\tau$ and $\phi_2 + i\tau$, and place them in a 2-string about the real axis, with real coordinate ϕ_e. In both cases, we obtain the result

$$(-)^{\Delta\nu''} \frac{\Lambda_e(\rho, \alpha, \lambda, \tau)}{\Lambda_0(\rho, \alpha, \lambda, \tau)} = \frac{\Lambda_e(\rho', \alpha', \lambda', \tau)}{\Lambda_0(\rho', \alpha', \lambda', \tau)}$$
$$= (-)^{\omega_2} k_2 \,\mathrm{sn}[(K_2/\pi)(\phi_1 + i\alpha)]\,\mathrm{sn}[(K_2/\pi)(\phi_2 + i\alpha)]. \tag{5.7}$$

Here, the elliptic functions are of modulus k_2. The zero mode of (5.5) relates ϕ_e to ϕ_1 and ϕ_2 by

$$0 = 2\pi\omega_2 + 2\phi_e - \phi_1 - \phi_2 + 2\pi(\text{even integer}). \tag{5.8}$$

In this way, we recover the results of Sec. IV given in Eqs. (4.29) and (4.31).

Now let us deal with the interesting case $\mu > \frac{1}{2}\pi$, i.e., $\mu' < \frac{1}{2}\pi$. We remove $s \leq \gamma' = 1 + [\pi/\mu']$ zeros from the $i\tau$ line, hence s can assume values greater than 2. Consider removing $s = \gamma'$ or $\gamma' - 1$ zeros from the $i\tau$ line. We take $s - 2$ from $\pi + i\tau$, and one each from $\phi_1 + i\tau$ and $\phi_2 + i\tau$. These are placed in an s-string centered about the real line, with real coordinate ϕ_e. The dispersion curves of (5.7) are again obtained. Therefore, these curves exist for $0 < \mu < \pi$. They will be seen to correspond to the free states in the X-Y-Z model.

We shall now derive dispersion curves lying higher than the free-state dispersion curves of (5.7). Removing $s \leq \gamma' - 2$ zeros from $\pi + i\tau$, and placing them in an s-string centered about the real axis, we find

$$(-)^{\Delta\nu''} \frac{\Lambda_e(\rho, \alpha, \lambda, \tau)}{\Lambda_0(\rho, \alpha, \lambda, \tau)} = \frac{\Lambda_e(\rho', \alpha', \lambda', \tau)}{\Lambda_0(\rho', \alpha', \lambda', \tau)}$$
$$= k_2 \,\mathrm{sn}[(K_2/\pi)(\phi_e + i\alpha + is\tau - i(s+1)\lambda)]$$
$$\times \mathrm{sn}[(K_2/\pi)(\phi_e + i\alpha - is\tau + i(s+1)\lambda)]. \tag{5.9}$$

Here, $-\pi \leq \phi_e \leq \pi$ and $s \leq [\pi/\mu'] - 1$. The elliptic functions are of modulus k_2. Note that $\Delta\nu' = 0$, if s even; and $\Delta\nu' = 1$, if s odd. These curves will be seen to correspond to bound states appearing in the spectrum of the X-Y-Z model.

VI. CORRELATION LENGTH OF G_R AND C_R

A. $\mu < \pi/2$

In Sec. II we derived a spectral representation, which should be satisfied asymptotically by G_R, as $R \to \infty$. In Sec. III the unknown functions appearing in this representation were evaluated in the decoupling limit ($\mu = \frac{1}{2}\pi$), using the results of Cheng and Wu[13] for the Ising-model correlation functions. Let us now study the extent to which the spectral representation of (2.19) determines

the correlation length of G_R, when $\mu<\frac{1}{2}\pi$. Recall,

$$(-)^{qR}G_R - (\text{long-range order}) \approx \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d\phi_1 d\phi_2 \rho(\phi_1,\phi_2) \times[k_2 \operatorname{sn}(K_2/\pi)(\phi_1 - i\alpha)\operatorname{sn}(K_2/\pi)(\phi_2 - i\alpha)]^R, \tag{6.1}$$

where we have introduced the short-hand notation

$$\rho(\phi_1,\phi_2)\equiv\rho_1(\phi_1,\phi_2;\lambda,\tau;\pm) + (-)^R\rho_2(\phi_1,\phi_2;\lambda,\tau;\pm). \tag{6.2}$$

The reality of G_R is assured if

$$\rho(\phi_1,\phi_2)=\rho_1(-\phi_1,-\phi_2). \tag{6.3}$$

We shall argue that it is very likely that

$$(-)^R\rho(\phi_1,\phi_2)=\rho(\phi_1+2\pi,\phi_2)=\rho(\phi_1,\phi_2+2\pi). \tag{6.4}$$

This periodicity condition is satisfied in the decoupling limit, as can be seen from Eqs. (3.7) and (3.8).

We wish to deform the contour of integration in (6.1) as indicated in Fig. 5. We make the assumption that $\rho(\phi_1,\phi_2)$ has no singularities in the interior of the closed path. If the periodicity condition (6.4) holds, then the contributions of the vertical lines cancel, and we are left with the integration over the line segment $[-\pi+i\alpha,\pi+i\alpha]$. Then

$$(-)^{qR}G_R - (\text{long-range order}) \approx \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d\phi_1 d\phi_2 \rho(\phi_1+i\alpha,\phi_2+i\alpha) \times[k_2 \operatorname{sn}(K_2\phi_1/\pi)\operatorname{sn}(K_2\phi_2/\pi)]^R. \tag{6.5}$$

Expanding about the saddle points, we find

$$(-)^{qR}G_R - (\text{long-range order}) \sim k_2^R. \tag{6.6}$$

If the periodicity condition (6.4) is incorrect, the leading asymptotic behavior comes from the end point contributions over the verticle segments at $\phi_1=\pm\pi$ and $\phi_2=\pm\pi$. In this case,

$$(-)^{qR}G_R - (\text{long-range order}) \sim (\Lambda_1/\Lambda_0)^R, \tag{6.7}$$

as predicted by the standard argument.[6,7] Here $\Lambda_1/\Lambda_0 = k_2/\mathrm{dn}^2[K_2/\pi)\,\alpha, k_2']$.

In the decoupling limit, we know (6.6) is correct. The following physical argument indicates (6.6) is also correct for $\mu<\frac{1}{2}\pi$. It seems very probable that G_R and C_R have the same correlation length. This would imply that the correlation length of G_R must be independent of α, since C_R does not depend on α, thus ruling out (6.7).

To see why G_R and C_R should have the same correlation length, consider the Ising model with four-spin coupling, which is equivalent to the eight-vertex model. Both G_R and C_R correspond to four-spin correlation functions (Fig. 4). The four spins are grouped such that there are two pairs of nearest neighbors separated (for $R\to\infty$) by a large distance. We expect that the exponential falloff depends only on the "center of mass" separation, and not on the relative orientation of the two pairs of spins. This indicates that the correlation length of G_R should be equal to that of C_R.

It is interesting to note that the same value of the critical index ν is predicted by (6.6) and (6.7). Following Baxter,[1] we define

$$Q=\exp(-\pi K_1'/K_1)\approx(\tfrac{1}{4}l)^2, \quad (T\to T_c).$$

We see that Q is proportional to $|T-T_c|$ as $T\to T_c$. Using the relation $K_2\approx\ln(4/k_2')$ as $k_2\to 1$, we find $(0<\mu<\pi)$

$$-\ln k_2 \approx 8Q^{\pi/2\mu}, \quad (T\to T_c) \tag{6.6a}$$

where at $T=T_c$,

$$\cos\mu=(2w_2^2-w_1^2-w_4^2)/(w_1^2-w_4^2).$$

On the other hand, from (4.27a) and (4.32a) evaluated at $\phi^{(1)}=\phi^{(2)}=\pi$, we find $(\mu\leq\frac{1}{2}\pi)$

$$-\ln\frac{\Lambda_1}{\Lambda_0}=\sum_{m=1}^{\infty}\frac{\sinh m(\lambda-\alpha)\cos m\pi}{m\cosh m\lambda}. \tag{6.8}$$

Following Baxter's[1] procedure for extracting the critical index of the specific heat, we perform a Poisson summation on (6.8). We find

$$-\ln\frac{\Lambda_1}{\Lambda_0}=8\sum_{n=0}^{\infty}\frac{Q^{(n+1/2)\pi/\mu}}{1-Q^{(2n+1)\pi/\mu}} \times\frac{(-)^n\sin\{[\pi(n+\frac{1}{2})/\mu](\mu-U)\}}{2n+1}. \tag{6.9}$$

The slowest approach to zero comes from the term $n=0$:

$$-\ln\frac{\Lambda_1}{\Lambda_0}\approx 8\sin[(\pi/2\mu)(\mu-U)]Q^{\pi/2\mu}, \quad (T\to T_c). \tag{6.7a}$$

Since Baxter[1] has shown that the critical index α of the specific heat is given by $\alpha=2-\pi/\mu$, and (6.6a) and (6.7a) both yield $\nu=\pi/2\mu$, we have the result[5]

$$\nu=1-\tfrac{1}{2}\alpha. \tag{6.10}$$

This confirms a prediction of scaling theory.[3,4]

The exceptional case $\mu=0$ corresponds to the six-vertex limit $w_1=w_2$. To extract the behavior of ξ^{-1} in this case, when $T\to T_c^-$, recall that

$$\Delta=-\cosh\lambda=(a^2+b^2-c^2)/2ab. \tag{6.11}$$

Below T_c, $c > a+b$, so $\Delta < -1$. Assuming that the correlation length in this limit is still given by $\xi^{-1} = -\ln k_2$, we have

$$\xi^{-1} \approx 8e^{-\pi^2/2\lambda}, \quad (\lambda \to 0). \tag{6.12}$$

From (6.11), we see λ is proportional to $(T_c - T)^{1/2}$, as $T \to T_c^-$. This is to be compared with the recent result of Baxter[23] that the spontaneous staggered polarization P_0 behaves as

$$P_0 \simeq (2\pi/\lambda)\, e^{-\pi^2/4\lambda}, \quad (\lambda \to 0). \tag{6.13}$$

B. $\mu > \pi/2$

We now study the correlation length of G_R, when $\mu > \frac{1}{2}\pi$. In this region, there exist the bound-state curves of (5.9) as well as the free states of (5.7). It is necessary to consider $T < T_c$ and $T > T_c$, separately.

For $T < T_c$ and $\mu \leq 2\pi/3$, the free-state curve dominates. This is because the $s=1$ bound-state curve does not contribute. The matrix element vanishes since $s=1$ corresponds to $\Delta\nu' = 1$, and σ^z leaves ν' unchanged. The asymptotic behavior of G_R is given by (6.1).

When $T < T_c$ and $\mu > 2\pi/3$, the $s=2$ bound-state curve yields the leading asymptotic behavior, recall $s=2$ corresponds to $\Delta\nu' = 0$. Then,

$$\begin{aligned}(-)^{aR}G_R - (\text{long-range order}) \\ \approx \int_{-\pi}^{\pi} d\phi\, \rho(\phi;\lambda,\tau;-)[k_2 \operatorname{sn}(K_2/\pi) \\ \times(\phi + i\alpha + 2i\tau - 3i\lambda)\operatorname{sn}(K_2/\pi) \\ \times(\phi + i\alpha - 2i\tau + 3i\lambda)]^R.\end{aligned} \tag{6.14}$$

In the case of $T > T_c$, the $s=1$ bound-state curve dominates in the entire region $\frac{1}{2}\pi < \mu < \pi$, so

$$\begin{aligned}(-)^{aR}G_R \approx \int_{-\pi}^{\pi} d\phi\, \rho(\phi;\lambda,\tau;+) \\ \times[k_2 \operatorname{sn}(K_2/\pi)(\phi + i\alpha + i\tau - 2i\lambda) \\ \times \operatorname{sn}(K_2/\pi)(\phi + i\alpha - i\tau + 2i\lambda)]^R.\end{aligned} \tag{6.15}$$

If G_R and C_R have the same correlation length, one expects that the following periodicity condition will be satisfied:

$$\rho(\phi + 2\pi;\lambda,\tau;\pm) = \rho(\phi;\lambda,\tau;\pm). \tag{6.16}$$

Then, assuming no singularities inside the contour, the integration path can be shifted from the real axis to $[-\pi - i\alpha, \pi - i\alpha]$, without picking up contributions from the vertical pieces of the contour. In this case, we find, for $T < T_c$,

$$(-)^{aR}G_R - (\text{long-range order}) \sim k_2^R, \quad (\mu \leq 2\pi/3), \tag{6.17a}$$

$$(-)^{aR}G_R - (\text{long-range order}) \sim \{k_2/\operatorname{dn}^2[(K_2/\pi)(2\tau - 3\lambda), k_2']\}^R, \quad (2\pi/3 < \mu < \pi). \tag{6.17b}$$

For $T \to T_c$, this goes as

$$\{1 - (1-k_2)[1 - 2\sin^2(\tfrac{1}{4}\pi(2\pi/\mu - 3))]\}^R.$$

When $T > T_c$,

$$(-)^{aR}G_R \sim \{k_2/\operatorname{dn}^2[(K_2/\pi)(\tau - 2\lambda), k_2']\}^R, \quad (\tfrac{1}{2}\pi < \mu < \pi). \tag{6.18}$$

For $T \to T_c$, this goes as

$$\{1 - (1-k_2)[1 - 2\sin^2(\tfrac{1}{4}\pi(\pi/\mu - 2))]\}^R.$$

In all these cases, the critical index $\nu = \pi/2\mu$. This confirms[5] the scaling prediction[3,4]

$$\nu = 1 - \tfrac{1}{2}\alpha. \tag{6.19}$$

If the periodicity condition is not satisfied, then the leading asymptotic behavior comes from the end point contributions to the integrations over the vertical segments. Again, one finds $\nu = \pi/2\mu$.

Throughout this section we have assumed that dispersion curves will contribute to the spectral representation for G_R, as long as the corresponding matrix elements are not forced to vanish by symmetry considerations. In Sec. III it was checked that the free-state dispersion curves reproduce the known results in the decoupling limit $\mu = \frac{1}{2}\pi$. Since the free states are the highest-lying curves for $\mu < \frac{1}{2}\pi$, we expect them to yield the leading asymptotic behavior in this region also. For $\mu > \frac{1}{2}\pi$, the bound-state curves appear and lie above the free states. The bound states appearing earliest (as μ increases), lie highest. Assuming dispersion curves do contribute, when allowed by symmetry considerations, we obtain (6.14) and (6.15). Unfortunately, there is no limit in which we can check the predictions for the correlation lengths of G_R against known results when $\mu > \frac{1}{2}\pi$. The form of the bound-state dispersion curves is subjected to a test in Sec. VII. We show that the corresponding bound-state curves of the X-Y-Z model go over to the known results in the ferromagnetic Heisenberg-Ising limit.

VII. LOW-LYING EXCITED STATES OF X-Y-Z MODEL

A. $\mu \leq \pi/2$

We consider the Hamiltonian

$$H = -\frac{1}{2}\sum_{j=1}^{N}(J_x\sigma_j^x\sigma_{j+1}^x + J_y\sigma_j^y\sigma_{j+1}^y + J_z\sigma_j^z\sigma_{j+1}^z), \tag{7.1}$$

with a periodic boundary condition, and with

$$J_z < J_y < -|J_x|.$$

An H corresponding to arbitrary values of the J's can be shown to be unitarily equivalent to an H with J's in this region. The σ's are Pauli spin matrices and $N=2r$ is an even integer.

Baxter[9] has shown that if one parametrizes the J's by

$$J_x : J_y : J_z = \mathrm{cn}(2\zeta, l) : \mathrm{dn}(2\zeta, l) : 1,$$

then ζ and l lie in the FR for J's restricted as above. The energy eigenvalues E of H, and the total momentum P of the corresponding state are given in terms of the eigenvalues of $\underline{T}(V)$ by[9]

$$E = -J_z \,\mathrm{sn}(2\zeta, l)\left(\frac{d}{dV}\ln T(V)\Big|_{V=\zeta} - \tfrac{1}{2}N[\mathrm{cn}(2\zeta, l) + \mathrm{dn}(2\zeta, l) - 1]/\mathrm{sn}(2\zeta, l)\right), \tag{7.2}$$

and

$$P = -i\ln T(\zeta) + iN\ln 2. \tag{7.3}$$

Recall that $T(\zeta)$ is equal to 2^N times the cyclic shift operator that moves all spins one site to the left.

The difference ΔE between an excited-state energy and the ground-state energy is

$$\Delta E = -J_z \,\mathrm{sn}(2\zeta, l)\frac{d}{dV}\ln\frac{\Lambda_e}{\Lambda_0}\Big|_{V=\zeta}. \tag{7.4}$$

The difference ΔP between an excited-state total momentum and the total momentum of *one* ground state, corresponding to Λ_0, is given by

$$\Delta P = \mathrm{Im}\,\ln\frac{\Lambda_e}{\Lambda_0}\Big|_{V=\zeta}. \tag{7.5}$$

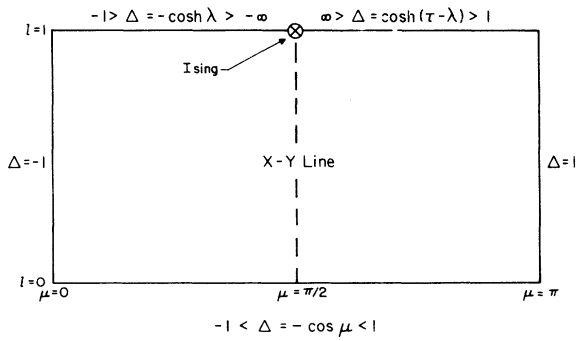

FIG. 6. The interior of the rectangle is $0<l<1$ and $0<\mu<\pi$. This region corresponds to X-Y-Z Hamiltonians with no two J's equal. The boundary of the rectangle represents the limiting cases of the Heisenberg-Ising Hamiltonians.

For $2\lambda<\tau$, we find for the excitation of one root to $\frac{1}{2}iK_l'$

$$\Delta E = -J_z \,\mathrm{sn}(2\zeta, l)(K_1/K_l')\times[\mathrm{dn}(K_1\phi_1/\pi, k_1) + \mathrm{dn}(K_1\phi_2/\pi, k_1)], \tag{7.6}$$

$$\Delta P = \pi(\omega_1+\omega_2) + q_1 + q_2, \tag{7.7}$$

by using (4.27a) and (4.27b). Here, the modulus k_1 is defined by $\pi K_1'/K_1 = \lambda$, and

$$q_i = \int_{-K_1}^{K_1\phi_i/\pi} \mathrm{dn}(\phi, k_1)\,d\phi, \quad (i=1,2).$$

Note $0 \le q_i \le \pi$.

For $\lambda < 2\tau/3$, the excitation energy ΔE and momentum ΔP of a 2-string are again given by (7.6) and (7.7). The excitation energy can be expressed as a function of q_1 and q_2, and for both types of excitations, we find

$$\Delta E = -J_z \,\mathrm{sn}(2\zeta, l)(K_1/K_l')\times[(1-k_1^2\cos^2 q_1)^{1/2} + (1-k_1^2\cos^2 q_2)^{1/2}]. \tag{7.8}$$

Our expression for ΔE reduces to the known results in the Heisenberg-Ising[11] limit with

$$H = -\frac{1}{2}\sum_{j=1}^{N}(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \Delta\sigma_j^z\sigma_{j+1}^z), \tag{7.9a}$$

and the X-Y[24] limit

$$H = -\frac{1}{2}\sum_{j=1}^{N}[(1+\gamma)\sigma_j^x\sigma_{j+1}^x + (1-\gamma)\sigma_j^y\sigma_{j+1}^y]. \tag{7.9b}$$

From the region $0<l<1$, $2\lambda<\tau$, we can continuously reach the X-Y model, and the Heisenberg-Ising model with $-\infty<\Delta<0$. See Fig. 6. In taking the limits, we must use the symmetry properties of the X-Y-Z Hamiltonian to arrange that $J_z < J_y < -|J_x|$:

$-\infty<\Delta<-1$:

$$\begin{aligned} &J_x=-1,\ J_y=-1,\ J_z=\Delta,\\ &l=1,\ K_l=\infty,\ K_l'=\tfrac{1}{2}\pi,\ \lambda=\pi\zeta/K_l'=2\zeta,\\ &\Delta=-\cosh\lambda.\\ &\Delta E=(2K_1/\pi)(\sinh\lambda)[(1-k_1^2\cos^2 q_1)^{1/2} + (1-k_1^2\cos^2 q_2)^{1/2}]; \end{aligned} \tag{7.10a}$$

$-1<\Delta<1$:

$$\begin{aligned} &J_x=\Delta,\ J_y=-1,\ J_z=-1,\\ &l=0,\ K_l=\tfrac{1}{2}\pi,\ K_l'=\infty,\ \mu=\pi\zeta/K_l=2\zeta,\\ &\lambda=0 \text{ implies } k_1=1,\ K_1/K_l'=\pi/\mu,\\ &\Delta=-\cos\mu\\ &\Delta E=(\pi/\mu)(\sin\mu)(\sin q_1+\sin q_2); \end{aligned} \tag{7.10b}$$

X-Y model ($\tau = 2\lambda$):

$$J_x = 0,\ J_y = -(1-\gamma),\ J_z = -(1+\gamma),$$
$$k_1^2 = 4l'/(1+l')^2,\ K_1/K_l' = 1+l',$$
$$k_1^2 = 1-\gamma^2,\ J_y/J_z = l',$$
$$\Delta E = 2[(1-(1-\gamma^2)\cos^2 q_1)^{1/2} + (1-(1-\gamma^2)\cos^2 q_2)^{1/2}]. \quad (7.10c)$$

B. $\mu > \pi/2$

In this region, there exist free-state dispersion curves given by (7.8). In addition, there are bound-state curves which are obtained from (5.9), by using (7.4) and (7.5). See Fig. 7. These are given by

$$\Delta E = -J_z(K_1/K_l')\,\mathrm{sn}(2\zeta, l) \times \{\mathrm{dn}[(K_1/\pi)(\phi_e + is\tau - i(s+1)\lambda), k_1] + \mathrm{dn}[(K_1/\pi)(\phi_e - is\tau + i(s+1)\lambda), k_1]\}, \quad (7.11a)$$

$$\Delta P = -\pi + \pi\Delta\nu'' - \int_0^{K_1\phi_e/\pi} d\phi \times \{\mathrm{dn}[\phi + i(K_1/\pi)((s+1)\lambda - s\tau), k_1] + \mathrm{dn}[\phi - i(K_1/\pi)((s+1)\lambda - s\tau), k_1]\}, \quad (7.11b)$$

with $-\pi \le \phi_e \le \pi$. Defining $\Delta Q = \Delta P - \pi\Delta\nu''$, $-2\pi \le \Delta Q \le 0$; and $y = sK_1\lambda'/\pi$, we obtain

$$\Delta E = -\frac{2J_z K_1}{K_l'}\,\frac{\mathrm{sn}(2K_l - 2\zeta', l)}{\mathrm{sn}(y, k_1')} \times [\sin^2(\tfrac{1}{2}\Delta Q) + \cos^2(\tfrac{1}{2}\Delta Q)\,k_1'^2\,\mathrm{sn}^2(y, k_1')]^{1/2} \times [\sin^2(\tfrac{1}{2}\Delta Q) + \cos^2(\tfrac{1}{2}\Delta Q)\,\mathrm{sn}^2(y, k_1')]^{1/2}. \quad (7.12)$$

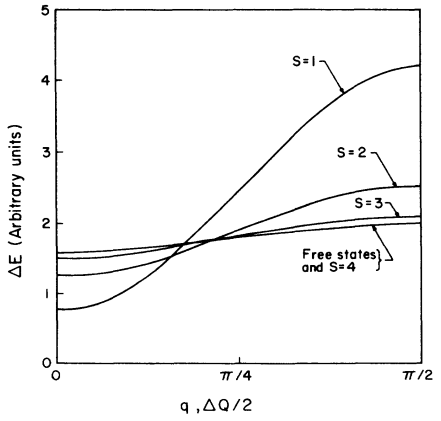

FIG. 7. These are the *X-Y-Z* dispersion curves for $k_1 = 0.6$ and $\lambda' = \frac{1}{5}\tau$. There are four bound-state curves ($s = 1, 2, 3, 4$) with the $s=4$ curve degenerate with the *X-Y*-like free states. To compare the bound states with the free states (a two-parameter curve), we have set $q_1 = q_2 = q$. ΔE is in arbitrary units in that we have factored out $-J_z\,\mathrm{sn}(2\zeta, l)K_1(K_l')^{-1}$ from all curves.

The bound-state dispersion curves reduce to the known results[11] in the ferromagnetic, Heisenberg-Ising limit:

$1 < \Delta < \infty$:

$$J_x = 1,\ J_y = -1,\ J_z = -\Delta,$$
$$l = 1,\ K_l = \infty,\ K_l' = \tfrac{1}{2}\pi,$$
$$\tau \to \infty,\ \lambda \to \infty,\ \tau - \lambda = \lambda',$$
$$\Delta = \cosh\lambda', \quad (7.13)$$
$$\Delta E = (2\sinh\lambda'/\sinh s\lambda') \times [\cosh s\lambda' - \cos\Delta Q].$$

In our formalism, the free states correspond to the two largest strings. These are of length $\gamma' - 1$ and γ'. In the limit $l \to 1$, we see $\gamma' \to \infty$. This suggests there are no free states in the ferromagnetic Heisenberg-Ising limit.

In the ferromagnetic Ising limit $\Delta \to \infty$, (7.13) yields $\Delta E/\Delta \to 2$. The s-string corresponds to turning over s adjacent spins in the ferromagnetic ground state. The $s = 1$ string corresponds to what is usually called the "free state" in the ferromagnetic Ising model.

ACKNOWLEDGMENT

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APPENDIX A: BAXTER'S PARAMETRIZATION

Following Baxter, we denote the pairwise equal vertex weights by a, b, c, d, and we define

$$w_1 = \tfrac{1}{2}(c+d),\quad w_2 = \tfrac{1}{2}(c-d)$$
$$w_3 = \tfrac{1}{2}(a-b),\quad w_4 = \tfrac{1}{2}(a+b).$$

The partition function has the symmetry property

$$Z(w_1, w_2, w_3, w_4) = Z(\pm w_i, \pm w_j, \pm w_k, \pm w_l), \quad (A1)$$

where i, j, k, l is any permutation of $1, 2, 3, 4$. Hence, it is sufficient to compute the partition function in the fundamental region

$$w_1 > w_2 > w_3 > w_4 \ge 0, \quad (A2)$$

in order to know it everywhere.

Parametrize the w's in terms of elliptic functions of modulus l in the following manner:

$$w_1 : w_2 : w_3 : w_4 = \frac{\mathrm{cn}(V, l)}{\mathrm{cn}(\zeta, l)} : \frac{\mathrm{dn}(V, l)}{\mathrm{dn}(\zeta, l)} : 1 : \frac{\mathrm{sn}(V, l)}{\mathrm{sn}(\zeta, l)}, \quad (A3)$$

where

$$l^2 = \frac{(w_1^2 - w_4^2)(w_2^2 - w_3^2)}{(w_1^2 - w_3^2)(w_2^2 - w_4^2)}, \quad (A4)$$

$$\mathrm{sn}(\zeta, l) = \left[\frac{w_1^2 - w_3^2}{w_1^2 - w_4^2}\right]^{1/2}. \quad \text{(A5)}$$

Then the FR is

$$0 \leq V < \zeta < K_l, \quad 0 < l < 1. \quad \text{(A6)}$$

The principal motivation for the parametrization (A3) was that if ζ, l are considered fixed, and the transfer matrix $\underline{T}(V)$ is regarded as a function of V, then

$$[\underline{T}(V), \underline{T}(V')] = 0.$$

Baxter now considers elliptic functions of the new modulus

$$k = (1 - l)/(1 + l). \quad \text{(A7)}$$

Then

$$K_l = \tfrac{1}{2}(1 + k)K_k', \quad K_l' = (1 + k)K_k. \quad \text{(A8)}$$

Defining

$$v = iV/(1 + k), \quad \eta = i\zeta/(1 + k), \quad \text{(A9)}$$

we see that

$$a: b: c: d = \mathrm{sn}(v + \eta, k): \mathrm{sn}(v - \eta, k): \mathrm{sn}(2\eta, k): k\,\mathrm{sn}(v + \eta, k)\,\mathrm{sn}(v - \eta, k)\,\mathrm{sn}(2\eta, k). \quad \text{(A10)}$$

Equivalently, one can write

$$a = \rho\,\theta(2\eta)\theta(v - \eta)H(v + \eta), \quad \text{(A11)}$$

$$b = \rho\,\theta(2\eta)H(v - \eta)\theta(v + \eta), \quad \text{(A12)}$$

$$c = \rho H(2\eta)\theta(v - \eta)\theta(v + \eta), \quad \text{(A13)}$$

$$d = \rho H(2\eta)H(v - \eta)H(v + \eta). \quad \text{(A14)}$$

It is useful to define the scaled variables

$$\tau = \pi K_k'/2K_k = \pi K_l/K_l', \quad \text{(A15)}$$

$$\lambda = -i\pi\eta/K_k = \pi\zeta/K_l', \quad \mu = \pi\zeta/K_l, \quad \text{(A16)}$$

$$\alpha = -i\pi v/K_k = \pi V/K_l', \quad U = \pi V/K_l. \quad \text{(A17)}$$

In terms of these quantities, the FR can be expressed as either

$$0 < k < 1, \quad 0 \leq \alpha < \lambda < \tau, \quad \text{(A18)}$$

or

$$0 < k < 1, \quad 0 \leq U < \mu < \pi. \quad \text{(A19)}$$

Baxter has shown that in the FR, the free energy has a singularity when $w_2 \to w_3$. In this limit, $l \to 0$, $K_l \to \frac{1}{2}\pi$ and $K_l' \to \ln(4/l) \to \infty$. Hence, τ, λ, α all approach zero.

The "decoupling limit" is specified by the equivalent forms $4\eta = iK_k'$, $\tau = 2\lambda$, or $\mu = \frac{1}{2}\pi$. In both our work and in Baxter's[1,14] it is useful to define the new moduli k_1 and k_2 by

$$\pi K_1'/K_1 = \lambda \text{ and } \pi K_2'/K_2 = 2\lambda. \quad \text{(A20)}$$

In the decoupling limit, note that $k_2 = l'$.

In the decoupling limit, the eight-vertex model becomes equivalent to two independent Ising lattices. Let $H_i = \beta J_i$, where J_1 and J_2 are the horizontal and vertical exchange constants of the corresponding Ising model. We shall tabulate the relation between the Ising-model variables and Baxter's parametrization of the vertex weights:

$T > T_c$:

$$w_1 = \cosh(H_1 + H_2), \quad w_3 = \sinh(H_1 + H_2),$$

$$w_2 = \cosh(H_1 - H_2), \quad w_4 = \sinh(H_1 - H_2),$$

$$k_2 = l' = \sinh 2H_1 \sinh 2H_2,$$

$$\zeta = \tfrac{1}{2}K_l$$

$$\mathrm{sn}(V, l) = \mathrm{sn}(\zeta, l)\sinh(H_1 - H_2)/\sinh(H_1 + H_2),$$

$$\sinh 2H_1 = -ik_2\,\mathrm{sn}(\tfrac{1}{2}iK_2' + iV, k_2),$$

$$\cosh 2H_1 = \mathrm{dn}(\tfrac{1}{2}iK_2' + iV, k_2),$$

$$\sinh 2H_1^* = -i\,\mathrm{sn}(\tfrac{1}{2}iK_2' - iV, k_2),$$

$$\cosh 2H_1^* = \mathrm{cn}(\tfrac{1}{2}iK_2' - iV, k_2).$$

Here, $\sinh 2H^* \sinh 2H = 1$. The expressions for $H_1 \to H_2$ are obtained by letting $V \to -V$.

$T < T_c$:

$$w_1 = \cosh(H_1 + H_2), \quad w_3 = \cosh(H_1 - H_2),$$

$$w_2 = \sinh(H_1 + H_2), \quad w_4 = \sinh(H_1 - H_2),$$

$$k_2 = l' = (\sinh 2H_1 \sinh 2H_2)^{-1}$$

$$\zeta = \tfrac{1}{2}K_l,$$

$$\mathrm{sn}(V, l) = \mathrm{sn}(\zeta, l)\tanh(H_1 - H_2),$$

$$\sinh 2H_2^* = -ik_2\,\mathrm{sn}(\tfrac{1}{2}iK_2' + iV, k_2),$$

$$\cosh 2H_2^* = \mathrm{dn}(\tfrac{1}{2}iK_2' + iV, k_2),$$

$$\sinh 2H_2 = -i\,\mathrm{sn}(\tfrac{1}{2}iK_2' - iV, k_2),$$

$$\cosh 2H_2 = \mathrm{cn}(\tfrac{1}{2}iK_2' - iV, k_2).$$

The corresponding expressions for $H_2 \to H_1$ are obtained by letting $V \to -V$.

Let us point out that these relations are precisely the uniformization substitutions introduced by Onsager,[6] in his original paper on the two-dimensional Ising model. Equation (2.2) of the Appendix of that paper is obtained by noting that Onsager's parameter a is related to Baxter's parameters by

$$a = \tfrac{1}{2}K_2' - V.$$

APPENDIX B: ELLIPTIC θ FUNCTIONS

Let K be the complete elliptic integral of the first kind of modulus k, and K' the same integral of the complementary modulus $k' = (1 - k^2)^{1/2}$. We

define

$$q = e^{-\pi K'/K}. \tag{B1}$$

Corresponding to the variables x, y, it is useful to introduce the scaled variables

$$X = \pi x/K, \quad Y = -i\pi y/K, \quad \tau_1 = \pi K'/K. \tag{B2}$$

The elliptic θ functions satisfy the quasiperiodic conditions

$$H(x+2K) = -H(x), \tag{B3}$$

$$H(x+iK') = iq^{-1/4} e^{-(1/2)iX}\theta(x), \tag{B4}$$

$$\theta(x+2K) = \theta(x), \tag{B5}$$

$$\theta(x+iK') = iq^{-1/4} e^{-(1/2)iX}H(x). \tag{B6}$$

The following Fourier expansions are of great use:

$$\ln\frac{\theta(x+y)}{\theta(x-y)} = 4\sum_{m=1}^{\infty}\frac{1}{m}\frac{q^m}{1-q^{2m}} \times \sin mX \sin miY, \tag{B7}$$

$$\ln\frac{H(x+y)}{H(x-y)} = \ln\left((-)e^{-ix}\frac{\theta(x+y-iK')}{\theta(x-y+iK')}\right) = i(-\pi - X) + 4\sum_{m=1}^{\infty}\frac{1}{m}\frac{q^m}{1-q^{2m}} \times \sin mX \sin mi(Y-\tau_1). \tag{B8}$$

The expansion (B7) is valid for

$$|\mathrm{Im}X| + |\mathrm{Im}iY| < \tau_1. \tag{B9}$$

Hence (B8) is valid for

$$|\mathrm{Im}X| + |\mathrm{Im}(iY - i\tau_1)| < \tau_1. \tag{B10}$$

Note that the function in (B8) is not periodic, but is continuous as a function of x.

Let $H_{(1/2)}(x)$ and $\theta_{(1/2)}(x)$ denote the θ functions corresponding to

$$q_{(1/2)} = e^{-\pi K'/2K} = e^{-\tau}.$$

Then $\tau = \frac{1}{2}\tau_1$, and we define the modulus j corresponding to τ by

$$\tau = \pi K_j'/K_j = \pi K'/2K.$$

From Landen's transformation, we find

$$k = (1-j')/(1+j').$$

Explicitly

$$H_{(1/2)}(x) = (K_j/K)H(K_j x/K, j),$$

$$\theta_{(1/2)}(x) = (K_j/K)\theta(K_j x/K, j).$$

Liouvilles theorem implies

$$h(x) \equiv H(x)\,\theta(x) = (\mathrm{const})\times H_{(1/2)}(x).$$

Since it is $h(x)$ which enters Baxter's equations, the natural τ to consider is that of (A15), $\tau = \frac{1}{2}\tau_1$.

In the text, we have used the functions (with $y = \eta$):

$$F_p(X) = \ln\frac{H_{(1/2)}(x+py)}{H_{(1/2)}(x-py)},$$

$$G_p(X) = \ln\frac{\theta_{(1/2)}(x+py)}{\theta_{(1/2)}(x-py)}.$$

We specify the branches of the logarithms by demanding that no cuts of F_p cross the real X axis, and $F_p(0) = -i\pi$, except $F_0(0) = 0$. Also, G_p has no branch cuts crossing the real axis and $G_p(0) = 0$ except $G_\gamma(0) = G_{\gamma+1}(0) = -2\pi i$, where $\gamma - 1$ is the greatest integer less than τ/Y.

When Y is real, we find

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} dX\, e^{-imX} G_p(X - X_0) = 0, \quad (m=0) = \frac{\sinh mpy}{m\sinh m\tau} e^{-imX_0} \tag{B11}$$

for $|pY| + |\mathrm{Im}X_0| < \tau$;

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} dX\, e^{-imX} G_p(X - X_0) = -2\pi i + 2iX_0, \quad (m=0) = \frac{2}{m}e^{-im\pi} + \frac{\sinh m(pY - 2\tau)}{m\sinh m\tau} e^{-imX_0} \tag{B12}$$

for $|pY - 2\tau| + |\mathrm{Im}X_0| < \tau$;

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} dX\, e^{-imX} F_p(X - X_0) = -i\pi + iX_0, \quad (m=0) = \frac{1}{m}e^{-im\pi} + \frac{\sinh m(pY - \tau)}{m\sinh m\tau} e^{-imX_0} \tag{B13}$$

for $|pY - \tau| + |\mathrm{Im}X_0| < \tau$;

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} dX\, e^{-imX}\frac{d}{dX}F_p(X - X_0) = -i, \quad (m=0) = i\frac{\sinh m(pY - \tau)}{\sinh m\tau} e^{-imX_0} \tag{B14}$$

for $|pY - \tau| + |\mathrm{Im}X_0| < \tau$.

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