Vacuum Polarization and Positronium-Ground-State Splitting

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The value of vacuum-polarization amplitude is investigated in the region of positronium bound-state poles, where the naive Feynman-graph expansion fails. The corresponding contribution to positronium-ground-state splitting is explicitly evaluated up to order α^6. The previously given result for the $\alpha^6 \ln\alpha^{-1}$ term coming from fourth-order vacuum-polarization corrections to the one-photon annihilation diagram is confirmed and put on a sounder basis. The α^6 term is found to receive, besides the obvious contribution from α^2 vacuum-polarization term, an additional α^6-contribution which can be considered as generated by an infinite series of terms of higher nominal order in α.

I. INTRODUCTION

As it is well known, the one-photon annihilation process leads to lowest order in α, to the triplet-singlet splitting of positronium ground state

$$\Delta\nu = \tfrac{1}{4} m\alpha^4 . \qquad (1)$$

The contribution from vacuum-polarization (vac.-pol.) corrections to the above process, within α^6 accuracy and without considering corrections to the nonrelativistic positronium wave function, as it can be shown in the perturbative treatment of the Bethe-Salpeter equation, is given by

$$\begin{aligned}\Delta\nu &= \tfrac{1}{4} m\alpha^4\{-[\Pi(\alpha, t) - \Pi_{\text{pole}}]\}_{t=t_1} \\ &\equiv \tfrac{1}{4} m\alpha^4[-\tilde{\Pi}(\alpha, t_1)] . \end{aligned} \qquad (2)$$

Here

$$\Pi_{\mu\nu}(\alpha, p) \equiv i(p_\mu p_\nu - p^2\delta_{\mu\nu})\Pi(\alpha, -p^2) \qquad (3)$$

is the vac.-pol. tensor and

$$t_1 = (2m - \tfrac{1}{4}\alpha^2 m)^2 \qquad (4)$$

is approximately the squared invariant mass of the positronium ground state. Note that, according to the usual rules of perturbation theory, the ground-state pole, which is present in the vac.-pol. amplitude, is to be subtracted when considering the energy level shift of the same state.

In the usual perturbative expansion in powers of the fine-structure constant one puts

$$\Pi(\alpha, t) = \sum_{n=1}^{\infty} \left(\frac{\alpha}{\pi}\right)^n \Pi^{(2n)}(t) . \qquad (5)$$

Only the first two terms of the expansion, at present, are known explicitly. In a previous paper[1] we noted that such explicit knowledge enables one to write

$$\begin{aligned}\tilde{\Pi}(\alpha, t_1) &\simeq \left(\frac{\alpha}{\pi}\right)\Pi^{(2)}(t_1) + \left(\frac{\alpha}{\pi}\right)^2 \Pi^{(4)}(t_1) \\ &= \left(\frac{\alpha}{\pi}\right)\frac{8}{9} + \tfrac{1}{2}\alpha^2 \ln\alpha^{-1} + O(\alpha^2) , \end{aligned} \qquad (6)$$

where the first and the second term on the right-hand side come, respectively, from second- and fourth-order vac. pol. $\Pi^{(2)}(t)$ and $\Pi^{(4)}(t)$. The first term had already been given by Karplus and Klein[2] together with all other α^5 contributions to the splitting resulting from interaction diagrams with one and two photons, getting the result

$$\Delta\nu = \tfrac{1}{2} m\alpha^4[\tfrac{7}{6} - (\tfrac{16}{9} + \ln 2)(\alpha/\pi)] . \qquad (7)$$

More recently, Fulton, Owen, and Repko[3] have considered all the $\alpha^6 \ln\alpha^{-1}$ recoil corrections to the previously computed α^5 terms.

Summing up their results with the second term of Eq. (6), which was not taken into account in Ref. 3, one obtains[1]

$$\begin{aligned}\Delta\nu &= \tfrac{1}{2} m\alpha^4[\tfrac{7}{6} - (\tfrac{16}{9} + \ln 2)(\alpha/\pi) + \tfrac{1}{2}\alpha^2 \ln\alpha^{-1}] + O(\alpha^6) \\ &= 203.404 \text{ GHz} + O(\alpha^6) . \end{aligned} \qquad (8)$$

It compares fairly well with the latest experimental value[4]

$$\Delta\nu_{\text{expt}} = 203.396 \pm 0.005 \text{ GHz} \quad (25 \text{ ppm}) .$$

Of course, since $\tfrac{1}{2} m\alpha^6 = 0.007$ GHz, calculation of the α^6 terms is now mandatory.[5]

The $\alpha^2 \ln\alpha^{-1}$ term in Eq. (6), instead of a naively expected α^2 term, comes from a logarithmic threshold behavior

$$-\tfrac{1}{4}\alpha^2 \ln|t - 4m^2|/m^2$$

of the fourth-order vac.-pol. amplitude. Such singularity, as already noted in the original paper by Källén and Sabry,[6] is connected to the vanishing of the photon mass; on the other hand, the vac.

pol. in perturbation theory is evaluated treating as free as the charged particles in the intermediate states, i.e., neglecting the Coulomb distortion. The problem of the consistency of the approach naturally arises. One has to prove, in particular, that Coulomb distortion does not alter the α and $\alpha^2 \ln\alpha^{-1}$ terms of Eq. (6), and to show how to avoid the trap of a "false expansion" in α. A "false expansion" would be an expansion in powers of some quantity, such as, for instance, $m\alpha/(t-4m^2)^{1/2}$, of the same nominal order of smallness as α, but which is converted into a constant when evaluated, say at $t=t_1 \simeq 4m^2 - \alpha^2 m^2$. We recall that the occurrence of false expansions is one of the main difficulties in bound state problems, as clearly shown, for instance, by Erickson and Yennie[7] in the Lamb-shift calculation. Here a series of terms of "nominal order" $\alpha(Z\alpha)^n$, $n \geq 4$, is converted in the so-called "Bethe log" of order $\alpha(Z\alpha)^4$.

The program of this paper is to investigate the effect of Coulomb distortions in vac. pol., evaluating explicitly $\tilde{\Pi}(\alpha, t_1)$ up to the α^2 term. The result is

$$\tilde{\Pi}(\alpha, t_1) = \left(\frac{\alpha}{\pi}\right)\frac{8}{9} + \frac{1}{2}\alpha^2 \ln\alpha^{-1} + \left(\frac{\alpha}{\pi}\right)^2 \left(-\frac{13}{324} + \frac{11}{32}\pi^2 - \frac{\pi^2}{4}\ln 2\right) + \frac{3}{4}\alpha^2, \tag{9}$$

confirming the previously given $\alpha^2 \ln\alpha^{-1}$ contribution of Eq. (6) and putting it on a sounder basis. Whereas the $\alpha^2 \ln\alpha^{-1}$ and the first α^2 term in large parentheses in Eq. (9) correspond to $\Pi^{(4)}(t_1)$, the usual fourth-order vac. pol. evaluated at $t=t_1$, the last term $\frac{3}{4}\alpha^2$ is the analog of the "Bethe log" in the present case and can be considered as generated by a series of terms of higher nominal order in α. Specifically, it receives the contribution $\frac{1}{2}\alpha^2$ from the continuum and $\frac{1}{4}\alpha^2$ from the discrete spectrum. Of course $\Delta\nu$ receives many other α^6 contributions besides that coming from the α^2 term of Eq. (9), and any speculation on its actual coefficient is meaningless at this level.

To obtain Eq. (9) we have to compute $\Pi(\alpha, t)$ in the region of positronium bound states. They manifest themselves as poles in t of the form $g_n(\alpha)/(t-4m^2+m^2\alpha^2/n^2)$, which, in turn, can be regarded as singularities in the coupling constant α and make $\Pi(\alpha, t)$ not expandible in a perturbation-theory series for $|t-4m^2| \leq \alpha^2 m^2$.

Besides bound states, other singularities in α are due to the continuum of e^-e^+ scattering states for $\alpha m/(t-4m^2)^{1/2} > 1$.

In Secs. II and III we explicitly compute the contributions to the part $\Pi_s(\alpha, t)$ of the vac.-pol. amplitude, singular in α, corresponding to continuum and bound states, respectively. The result agrees with the corresponding expression given by Braun[8] in a remarkable paper on the same subject.

The connection of $\Pi_s(\alpha, t)$ and of the usual Feynman-graph expansion with the vac.-pol. amplitude $\Pi(\alpha, t)$ is given in Sec. IV and the complete result, Eq. (9), is therefore justified. Some final remarks are made in Sec. V, concerning the threshold properties ($t \to 4m^2$) of functions $\Pi^{(2n)}(t)$ appearing in the perturbative expansion of the vac.-pol. amplitude equation (5) and the related dispersive representations.

II. CONTINUUM CONTRIBUTION

We take as our starting point the spectral representation for the vac.-pol. amplitude

$$\Pi(\alpha, t) = \frac{t}{\pi}\mathcal{P}\int_{t_0}^{\infty} \frac{dt'}{t'(t'-t)} \operatorname{Im}\Pi(\alpha, t') \tag{10}$$

where $\mathcal{P}$ means "principal part of" and with one subtraction accounting for renormalization and

$$\operatorname{Im}\Pi(\alpha, t) = \frac{\pi}{3t}\sum_n \langle 0|J_\mu(0)|n\rangle\langle n|J_\mu(0)|0\rangle\,\delta(\sqrt{t} - E_n), \tag{11}$$

where E_n is the energy of the intermediate state $|n\rangle$ in the center-of-mass system.

As we have to compute $\Pi(\alpha, t)$ at $t=t_1 \simeq 4m^2 - m^2\alpha^2$, it is crucial to have an accurate expression for $\operatorname{Im}\Pi(\alpha, t)$ near this point. In particular, we are not allowed to neglect Coulomb interaction between charged particles in the intermediate states, as done in usual lowest-order perturbative treatment, in which such particles are treated as free. The intermediate states contributing to Eq. (11) consist of electron-positron states; states with an electron-positron pair and one photon; states with more pairs and/or more photons, etc.

Let us start by discussing the continuum of $e^+ - e^-$ states, labeled by the asymptotic relative momentum $\vec{k}$ and a spin index s. Their contribution to $\operatorname{Im}\Pi(\alpha, t)$ is

$$\begin{aligned}\operatorname{Im}\Pi^{e^+e^-}_{\text{cont}}(\alpha, t) &= \frac{\pi}{3t}\sum_s \int \frac{1}{8}\frac{d^3k}{(2\pi)^3}\langle 0|J_\mu(0)|\vec{k}, s\rangle\langle\vec{k}, s|J_\mu(0)|0\rangle\,\delta(\sqrt{t} - (k^2+4m^2)^{1/2}) \\ &= \frac{\pi}{3t}4\pi\alpha\sum_s \int \frac{1}{8}\frac{d^3k}{(2\pi)^3}\operatorname{Tr}[\bar{\phi}(s,\vec{k};0)\gamma_\mu]\operatorname{Tr}[\gamma_\mu\phi(s,\vec{k};0)]\delta(\sqrt{t} - (k^2+4m^2)^{1/2}),\end{aligned} \tag{12}$$

where $\phi(s,\vec{k};0)$ is the Bethe-Salpeter 4×4 matrix wave function at zero relative coordinate. For t approaching the threshold $4m^2$, use of nonrelativistic wave functions becomes correct giving

$$\mathrm{Im}\Pi_{\mathrm{cont}}^{e^+e^-}(\alpha,t)\simeq\frac{1}{4}\frac{\alpha^2}{\nu}\,|\psi_\nu(0)|^2, \qquad (13)$$

where

$$|\psi_\nu(0)|^2=\frac{2\pi\nu}{1-e^{-2\pi\nu}}, \qquad \nu=\frac{\alpha m}{(t-4m^2)^{1/2}}. \qquad (14)$$

Equation (14) displays an essential singularity for $\nu\to\infty$ and shows that for $|\nu|>1$ an expansion in powers of α would fail. It is easily seen that $|\psi_\nu(0)|^2$ admits the representation

$$|\psi_\nu(0)|^2=1+\pi\nu+2\nu^2\sum_{n=1}^{\infty}\frac{1}{n^2+\nu^2} \qquad (15)$$

and for $|\nu|<1$ it can be expanded as

$$|\psi_\nu(0)|^2=1+\pi\nu+2\sum_{k=1}^{\infty}(-1)^{k-1}\zeta(2k)\nu^{2k}. \qquad (16)$$

where $\zeta(n)$ is the Riemann ζ function of argument n. By using Eq. (16), Eq. (13) becomes, for $|m\alpha/(t-4m^2)^{1/2}|<1$,

$$\mathrm{Im}\Pi_{\mathrm{cont}}^{e^+e^-}(\alpha,t)\simeq\tfrac{1}{4}\alpha\,\frac{(t-4m^2)^{1/2}}{m}+\frac{\pi}{4}\alpha^2+\frac{\alpha^2}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\zeta(2k)\left(\frac{\alpha m}{(t-4m^2)^{1/2}}\right)^{2k-1}. \qquad (17)$$

Of course to lowest order in α, i.e., if $|\psi_\nu(0)|^2$ is replaced by 1 and therefore Coulomb interaction is neglected, Eq. (17) reduces, as $t\to 4m^2$, to the usual second-order vac.-pol. discontinuity

$$\left(\frac{\alpha}{\pi}\right)\mathrm{Im}\Pi^{(2)}(t)=\left(\frac{\alpha}{\pi}\right)\pi\,\frac{t+2m^2}{3t}\left(1-\frac{4m^2}{t}\right)^{1/2}. \qquad (18)$$

The right-hand side of Eq. (13) is obviously an approximation of the whole spectral function for vac. pol., because it gives only the nonrelativistic contribution of e^+-e^- continuum intermediate states and neglects anything else. It accounts, however, for the exchange of infinite uncrossed Coulomb photons, and we assert that its expansion in α [Eq. (17)] contains the leading singularities for $t\to 4m^2$ of the usual Feynman-graph expansion.

This is easy to check in perturbation theory. According to the known expression of the vac.-pol. amplitude up to fourth order, as $t\to 4m^2$, the e^+e^- and $e^+e^-\gamma$ cuts give, respectively,[6,9]

$$\mathrm{Im}\Pi^{e^+e^-}(t)\simeq\pi\left(\frac{\alpha}{\pi}\right)\frac{(t-4m^2)^{1/2}}{4m}+\pi\left(\frac{\alpha}{\pi}\right)^2\frac{\pi^2}{4}, \qquad (19)$$

$$\mathrm{Im}\Pi^{e^+e^-\gamma}(t)\simeq\pi\left(\frac{\alpha}{\pi}\right)^2 O(t-4m^2). \qquad (20)$$

We note, in particular, that at threshold the terms in Eq. (19) are correctly given by the corresponding ones in the α expansion of Eq. (13). From a general point of view the above-given statement could be proved by inspection of the discontinuities of Feynman graphs at any order in perturbation theory along the lines of Ref. 10.

This discussion makes, therefore, clear the role played by the e^+e^- Coulomb interaction for the threshold behavior of the spectral function.

We define the part $\Pi_s^{\mathrm{cont}}(\alpha,t)$ of $\Pi(\alpha,t)$, containing the leading singularities in α coming from the continuum intermediate states as

$$\begin{aligned}\Pi_s^{\mathrm{cont}}(\alpha,t)&\equiv\frac{\alpha}{4\pi m}\int_{4m^2}^{\infty}\frac{dt'}{t'-t}(t'-4m^2)^{1/2}\times\left[|\psi_{\nu'}(0)|^2-1-\pi\nu'\right]_{\nu'=\alpha m/(t'-4m^2)^{1/2}}\\&=\frac{\alpha^2}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+i\nu}\right),\end{aligned} \qquad (21)$$

where the integration is performed by means of Eq. (15). At variance with $\Pi(\alpha,t)$ [Eq. (10)], the above $\Pi_s^{\mathrm{cont}}(\alpha,t)$ is not subtracted at $t=0$. Note also that the term $1+\pi\nu'$ in the integrand has been subtracted out. Such changes are however irrelevant for later use (see Sec. IV), since they only modify $\Pi_s^{\mathrm{cont}}(\alpha,t)$ by terms regular in α.[11]

At $t=t_1$, $\nu=-i$ and, therefore,

$$\Pi_s^{\mathrm{cont}}(\alpha,t_1)=\frac{\alpha^2}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=\frac{\alpha^2}{2}. \qquad (22)$$

III. BOUND-STATE CONTRIBUTION

In close analogy with the discussion of Sec. II, we consider here the α singularities connected to the existence of positronium bound states. Their contribution to the spectral function is

$$\begin{aligned}\mathrm{Im}\Pi_s^{\mathrm{bound}}(\alpha,t)&=\frac{4\pi^2\alpha}{m}\sum_{n=1}^{\infty}|\varphi_n(0)|^2\delta(t-E_n^2)\\&=\frac{\pi m^2\alpha^4}{2}\sum_{n=1}^{\infty}\frac{1}{n^3}\delta(t-E_n^2),\end{aligned} \qquad (23)$$

where $\varphi_n(0)$ is the nonrelativistic wave function for the state of principal quantum number n and of zero angular momentum at zero relative coordinate, and $E_n=2m-m\alpha^2/4n^2$. Correspondingly, we define the part $\Pi_s^{\mathrm{bound}}(\alpha,t)$ containing the leading singularities in α coming from bound intermediate states as

$$\Pi_s^{\text{bound}}(\alpha,t)=\frac{1}{\pi}\int\frac{dt'}{t'-t}\,\mathrm{Im}\Pi_s^{\text{bound}}(\alpha,t')$$
$$=-\frac{\alpha^2}{2}\sum_{n=1}^{\infty}\left(\frac{2}{n}-\frac{1}{n+i\nu}-\frac{1}{n-i\nu}\right). \tag{24}$$

This expression is obviously not expandable in powers of α for $|\nu|>1$. Its value at $t=t_1$, after subtraction of the ground-state pole, is $\frac{1}{4}\alpha^2$.

IV. SINGULAR AND REGULAR PART OF VAC.-POL. AMPLITUDE

We can now give the full singular part $\Pi_s(\alpha,t)$ of the vac.-pol. amplitude:

$$\Pi_s(\alpha,t)=\Pi_s^{\text{cont}}(\alpha,t)+\Pi_s^{\text{bound}}(\alpha,t)$$
$$=-\tfrac{1}{2}\alpha^2\sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+1-i\nu}\right). \tag{25}$$

Recalling the formula

$$\psi(z)=-C+\sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+z}\right), \tag{26}$$

where C is the Bernouilli constant and

$$\psi(z)\equiv\frac{d}{dz}\ln\Gamma(z), \tag{27}$$

we can rewrite $\Pi_s(\alpha,t)$ as

$$\Pi_s(\alpha,t)=-\tfrac{1}{2}\alpha^2[\psi(1-i\nu)+C]. \tag{28}$$

This result agrees with the one quoted by Braun[8] and obtained here in a different context.

If $|\nu|<1$, i.e., $|t-4m^2|>\alpha^2m^2$, $\Pi_s(\alpha,t)$ can be expanded as

$$\Pi_s(\alpha,t)=\tfrac{1}{2}\alpha^2\sum_{k=2}^{\infty}\zeta(k)(i\nu)^{k-1}. \tag{29}$$

For $|\nu|\geqslant 1$, Eq. (29) would actually be a "false expansion." For the very definition of $\Pi_s(\alpha,t)$, $\Pi(\alpha,t)-\Pi_s(\alpha,t)$ is regular in α. From Eqs. (5) and (29), up to order α^n, one has

$$\Pi(\alpha,t)-\Pi_s(\alpha,t)\simeq\sum_{k=1}^{n}\left(\frac{\alpha}{\pi}\right)^k\Pi^{(2k)}(t)$$
$$-\frac{\alpha^2}{2}\sum_{k=2}^{n-1}\zeta(k)(i\nu)^{k-1} \tag{30}$$

(the last term on the right-hand side is to be replaced by zero for $n\leqslant 2$). Whereas Eqs. (5) and (29) are valid only for $|t-4m^2|>\alpha^2m^2$, Eq. (30) holds also for $t-4m^2=m^2O(\alpha^2)$.

Equation (30) can be rewritten[8]

$$\Pi(\alpha,t)\simeq\sum_{k=1}^{n}\left(\frac{\alpha}{\pi}\right)^k\Pi^{(2k)}(t)$$
$$+\left(\Pi_s(\alpha,t)-\frac{\alpha^2}{2}\sum_{k=2}^{n-1}\zeta(k)(i\nu)^{k-1}\right), \tag{31}$$

which can be regarded as an improved approximation to $\Pi(\alpha,t)$ up to order n.

If $|t-4m^2|>\alpha^2m^2$, $\Pi_s(\alpha,t)$ can be expanded in series of α, so that the term in the large parentheses in Eq. (31) gives a negligible correction of order $\alpha^{(n+1)}$ to $\Pi(\alpha,t)$, and Eq. (31) coincides up to α^n with the usual Feynman-graph expansion. If, on the contrary $|t-4m^2|\leqslant\alpha^2m^2$, the term in the large parentheses is only of nominal order $\alpha^{(n+1)}$, whereas its actual value can be higher.

This discussion makes it therefore clear that, for $\tilde{\Pi}(\alpha,t_1)$ as defined by Eq. (2), one can write, up to order α^2,

$$\tilde{\Pi}(\alpha,t_1)=\left(\frac{\alpha}{\pi}\right)\Pi^{(2)}(t_1)+\left(\frac{\alpha}{\pi}\right)^2\Pi^{(4)}(t_1)+\tilde{\Pi}_s(\alpha,t_1), \tag{32}$$

where, from the known analytic expressions,[9,12] one has

$$(\alpha/\pi)\Pi^{(2)}(t_1)=(\alpha/\pi)\,\tfrac{8}{9}+O(\alpha^3), \tag{33}$$

$$\left(\frac{\alpha}{\pi}\right)^2\Pi^{(4)}(t_1)=\tfrac{1}{2}\alpha^2\ln\alpha^{-1}+\left(\frac{\alpha}{\pi}\right)^2$$
$$\times\left(-\frac{13}{324}+\frac{11}{32}\pi^2-\frac{\pi^2}{4}\ln 2\right)$$
$$+(\text{higher-order terms}), \tag{34}$$

and, from Eq. (25),

$$\tilde{\Pi}_s(\alpha,t_1)=\left(\Pi_s(\alpha,t)+\frac{\alpha^4m^2}{t-t_1}\right)_{t=t_1}=\tfrac{3}{4}\alpha^2. \tag{35}$$

From Eq. (31) it is seen that the next neglected term, of nominal order α^3, would give to $\tilde{\Pi}(\alpha,t_1)$, the contribution

$$\left(\frac{\alpha}{\pi}\right)^3\left(\Pi^{(6)}(t_1)-\frac{\pi^3}{2}\zeta(2)\frac{im}{(t_1-4m^2)^{1/2}}\right),$$

which goes to zero more rapidly than α^2 since the second term in large parentheses, coming from the expansion of the nonrelativistic contribution to $\Pi(\alpha,t)$, subtracts out from $\Pi^{(6)}(t)$, its leading threshold singularity.

The result of Eq. (8) is therefore proved.

V. THRESHOLD BEHAVIOR OF VAC.-POL. IN PERTURBATION THEORY

We want to comment here a little more on the threshold properties ($t\to 4m^2$) of the functions $\Pi^{(2n)}(t)$ occurring in the perturbative expansion equation (5) of the vac.-pol. amplitude. This is mathematically a meaningful problem. The Feynman graphs $\Pi^{(2n)}(t)$ are indeed defined for every t, even if, as discussed in the preceding sections, the sum of a finite number of them does not provide any adequate approximation to $\Pi(\alpha, t)$ for $t \to 4m^2$.

The threshold behavior is well described by the nonrelativistic contribution of e^-e^+ intermediate states, Eq. (13), or, in a Feynman-graph language, by the sum of the one-loop graphs with the exchange of any number of uncrossed Coulomb photons. In particular, we note from Eq. (17) that the imaginary parts of the terms of Eq. (29) of nominal order k in α are as singular as $(t-4m^2)^{(2-k)/2}$ for odd k, whereas, for even k, with the exception of the α^2 term, the singularity is weaker.

Therefore, while it remains true that at any order in perturbation theory the discontinuities $\mathrm{Im}\Pi^{(2n)}(t)$ are given by the Cutkosky-Veltman rule,[13] the corresponding dispersion relations

$$\Pi^{(2n)}(t) = t\,\frac{1}{\pi}\,\mathcal{P}\int \frac{dt'}{t'(t'-t)}\,\mathrm{Im}\Pi^{(2n)}(t') \tag{36}$$

hold only for small n, since the right-hand side diverges at threshold for $n>4$.

Along the lines of a previous work on the vertex function,[14] an appropriate dispersive representation could be written, for instance, for functions such as

$$\left[\frac{(t-4m^2)}{t}\right]^k \Pi^{(2n)}(t),$$

which have the same asymptotic behavior as $\Pi^{(2n)}(t)$. The power k is to be chosen in such a way as to make $(t-4m^2)^k\,\mathrm{Im}\Pi^{(2n)}(t)$ integrable at $t=4m^2$.

If $k=1$, for instance, one has

$$\Pi^{(2n)}(t) = -\frac{4m^2 t}{t-4m^2}\,\frac{d}{dt}\,\Pi^{(2n)}(0) + \frac{t^2}{t-4m^2}\,\frac{1}{\pi}\,\mathcal{P}\int \frac{dt'(t'-4m^2)}{t'^2(t'-t)}\,\mathrm{Im}\Pi^{(2n)}(t'), \tag{37}$$

and the integral converges, for the above discussion, up to $n=6$. Note that, in this case, the knowledge of the discontinuity is not sufficient for the evaluation of the whole perturbative amplitude by means of a Hilbert transform, but it is necessary to supply, independently, the value of its derivative at $t=0$.

As in the case of the electron form factors, the threshold singularities of $\Pi^{(2n)}(t)$ are a consequence of the vanishing photon mass. If the photon is given a "small" mass λ, the corresponding functions $\Pi^{(2n)}(t, \lambda)$ would display a smooth threshold behavior, so that the relations

$$\Pi^{(2n)}(t, \lambda) = t\,\frac{1}{\pi}\,\mathcal{P}\int \frac{dt'}{t'(t'-t)}\,\mathrm{Im}\Pi^{(2n)}(t', \lambda) \tag{38}$$

remain valid for any n as far as λ is different than zero. Exchange of the $\lambda \to 0$ limit with the integration is, of course, not allowed.

Note, on the other hand, that no indication emerges against the validity of the spectral representation Eq. (10) for the whole amplitude $\Pi(\alpha, t)$. Indeed, at $t \simeq 4m^2$ one has, from Eqs. (12) and (13),

$$\mathrm{Im}\Pi(\alpha, t) \simeq \mathrm{Im}\Pi_{\mathrm{cont}}^{e^+e^-}(\alpha, t) \simeq \tfrac{1}{2}\pi\alpha^2,$$

so that the integration over the continuum spectrum is convergent at threshold.

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[11] Besides $\Pi_s^{\mathrm{cont}}(\alpha,t)$, Eq. (21), the continuum can have of course other singularities in α, corresponding, for instance, to the exchange of a finite number of transverse photons together with infinite Coulomb photons. According to the general arguments of Sec. III, these further singularities are expected to be smaller than $\Pi_s^{\mathrm{cont}}(\alpha,t)$ by some over-all power of α, and will be ignored in our discussion. Any irreducible part, including $\Pi_s(\alpha,t)$, appears then in $\Pi(\alpha,t)$ with all its higher-order iterations.

[12] Equation (34) receives a contribution $-(8/9)^2(\alpha/\Pi)^2$ from the Feynman graph representing the iteration of the second-order vacuum-polarization bubble. This

contribution is not contained in Eq. (49) of Ref. 9.

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