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Collective Modes of a Two-Dimensional Wigner Crystal

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Calculations of the frequency spectrum of lattice vibrations of a finite two-dimensional electron crystal show a soundlike shear mode and a longitudinal mode that has an anomalous dispersion relation. At long wavelengths the frequency of the longitudinal mode varies as the square root of the wave vector. This arises because of the long-range nature of the Coulomb force between the electrons and is analogous to the longitudinal plasma vibration in a three-dimensional Wigner lattice.

I. INTRODUCTION

Wigner[1] pointed out that a system of electrons immersed in a compensating background of positive charge will, at sufficiently low density, assume a lattice configuration to minimize its energy. Since then there has been considerable theoretical interest in this model, even though the system was not expected to exist in the laboratory. Van Horn[2] suggested that the degenerate positive-ion core of a white dwarf star would be in a Wigner lattice configuration. Crandall and Williams[3] recently suggested that electrons on the surface of liquid helium might form a two-dimensional Wigner crystal. Therefore, it is appropriate to develop this idea and calculate the vibrational modes of a two-dimensional electron crystal to stimulate possible experiments. A new feature of this electron crystal is that the frequency of the longitudinal mode of vibration is proportional to the square root of the wave vector in the long-wavelength limit. This particular dispersion relation arises from the long-range nature of the Coulomb force. It is the two-dimensional analog of the longitudinal plasmalike mode in a three-dimensional Wigner crystal. The remaining mode is acoustic at long wavelengths.

The properties of a three-dimensional electron crystal have been investigated in detail by several authors. Wigner[1] predicted and Fuchs[4] confirmed that the bcc lattice is the minimum-energy configuration. Nevertheless, the other possible lattice configurations are only slightly higher in energy. Foldy[5] later showed that there is a phase transition to the fcc or some other lattice at higher temperature. Various estimates,[6] based on the harmonic approximation to the electron lattice, give 10–50 Å as the minimum distance between electrons required for crystallization. Kugler[7] demonstrated that an anharmonic electron lattice may not be stable for electron spacings less than about 350 Å. Clark,[8] Coldwell-Horsfall and Maradudin,[9] and Carr[10] calculated the vibrational spectrum in the harmonic approximation. Kugler[7] made corrections, due to anharmonicity, to the lattice spectrum.

In Sec. II we outline the model and calculate the dispersion curves. We calculate the electron distribution in Sec. III.

II. MODEL CALCULATION

It would be natural to model the two-dimensional Wigner crystal by analogy with its three-dimensional counterpart where the electrons are immersed in a uniform background of positive charge. However, since we are interested in calculating properties of a possible experimental system, we follow the suggestion of Crandall and Williams for obtaining electron crystallization. In this approach an electric field confines the electron system to a dielectric surface. This is fundamentally different from the three-dimensional case, where a uniform background of positive charge surrounds the electrons.

Figure 1 shows the proposed experimental arrangement for producing a two-dimensional electron crystal. The dielectric is liquid helium since it is known[11] that ^{4}He presents a potential barrier of about 1 eV to electron penetration. Since the helium surface is relatively free of charged impurities, electron-electron interactions should be the dominant electron localization mechanism in the x, y plane. The metal plates, A and B, biased in such a way as to force the electrons against the helium, confine the electrons, shown by a dotted line, to a finite region of the liquid surface. The electrostatic depression[12] of the liquid surface is exaggerated for clarity. Since the mean spacing between the electrons in the x, y plane is orders of magnitude larger than their average distance from the helium surface,[13] we may consider the system as two-dimensional. Thus, the function of the helium and the electric field between plates A and B is to confine the electrons in a two-dimensional configuration.

To minimize end effects, the distance between the plates is less than their diameter, and to ensure uniformity of the electric field between the plates, their separation is much greater than the mean spacing between electrons. Therefore, in the following calculation we neglect end effects at the edges of the electron sheet and furthermore assume that the electron density is uniform across the helium surface. These approximations affect only the very-long-wavelength vibrations. Also we assume that any interaction between the electron and the liquid that would modify the motion parallel to the surface can be accounted for in the framework of the effective-mass approximation.[14]

The above describes a system of electrons localized just outside the surface of liquid helium. However, ions inside liquid helium can similarly be localized near the surface for arbitrarily long times.[13,15,16] Inside the liquid an electron forms a bubble and the positive ion forms a solid sphere of helium about it. These highly correlated structures have a mass many times that of a helium atom.[16] Since this state is mobile and charged, the above considerations will apply. The following calculations will apply equally well to electrons on the surface and ions in the liquid.

For the system of electrons described by Fig. 1, the potential energy is

$$V(x,\ y,\ z) \approx V_p + V(z) + \tfrac{1}{2}\sum_{l,m} \frac{e^2}{|\vec{r}_l - \vec{r}_m|}\ , \quad l \neq m \tag{1}$$

where V_p represents the potential of interaction between the electrons and the plates A and B, and $V(z)$ is the image-potential interaction[14] between the electrons and the ^{4}He. The remaining term

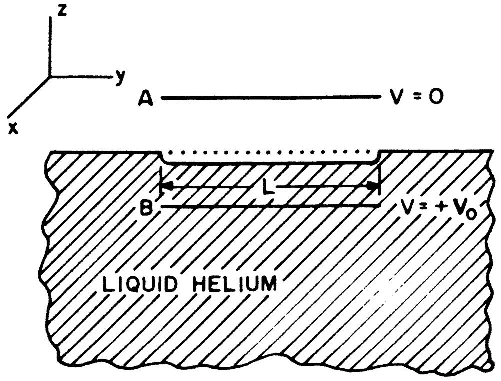

FIG. 1. Schematic of the electron system on liquid helium. The metal plate B is immersed in the helium, and the plate A held at ground potential is above the helium surface. The electrons are shown by dots on the helium surface. The length L, a rough measure of the size of the electron system, is given by the plate size. The z direction is perpendicular to the liquid surface.

is the electron-electron interaction where the $\vec{r}_l$ are the coordinates of the electrons.

The form of $V(x,\ y,\ z)$ differs from that for a three-dimensional Wigner crystal[8] by the absence of a term representing the interaction between the electrons and the uniform background of positive charge. In our case the positive charges reside on the plate B and their interaction is taken into account by the term V_p. Since the number of positive charges on the plate is equal to the number of electrons on the helium, the sytem is neutral. The image potential does not appear in the three-dimensional model. It depends on the z coordinate only and does not affect the x, y motion of the electron.

The combination of $V(z)$ and the electric field due to the metal plates localizes the electrons to within 70 Å of the surface.[14] A typical electron density is $10^8 \mathrm{cm}^{-2}$, giving a mean electron spacing a_0 of 10^4 Å. Therefore $a_0 \gg \langle z \rangle$. Essentially, the characteristic frequencies of the z motion are much higher than those of the x, y motion. Thus, the z motion can be ignored in calculating the x, y motion. This property permits the electron-electron interaction term to be separated into a term depending only on x and y and a remaining term depending on x, y, and z which is smaller than $V(z)$. Since we neglect end effects, V_p is independent of x and y. Of course the end effects are important in that they confine the charges to a finite region of space. Therefore, as long as the electrons remain within say 10^3 Å of the surface, $V(x,\ y,\ z)$ separates into a term depending on x and y, and one depending on z. Since the z-dependent term serves only to localize the electrons near the ^{4}He surface, we will not consider it further.

For a sufficiently dilute system, the energy of Coulomb repulsion between electrons will dominate the kinetic energy due to their localization. The Coulomb energy is minimized when the electrons are ordered in a crystalline array. For the very dilute system of electrons considered here, the Coulomb energy is certainly much larger than the localization energy. Thus we need only consider the Coulomb energy in determining the ground state of the system.

Using well-known methods[17] we can readily derive the classical equations of motion of an electron in the x, y plane. In the third term in Eq. (1), we replace $\vec{r}_l$ by $\vec{R}_l + \vec{u}_l$, where $\vec{R}_l$ is the coordinate of the l th lattice point and $\vec{u}_l$ is the displacement of the l th electron from its lattice point $\vec{R}_l = \vec{X}_l + \vec{Y}_l$. We then expand the third term in Eq. (1) about the lattice points. Because we are considering a dilute system of electrons, the harmonic approximation to the lattice expansion is reasonably well justified. The anharmonic terms contribute less than 1% to $V(x, y, z)$. This also makes it unnecessary to treat $V(x, y, z)$ self-consistently. Therefore, the potential energy that depends on x and y is

$$V = \tfrac{1}{2} \sum_{l, m} \vec{u}_l \cdot A^{lm} \cdot \vec{u}_m , \tag{2}$$

where the tensor A^{lm} has the property that for $l \neq m$

$$A^{lm} = \frac{-e^2 B^{lm}}{|\vec{R}_l - \vec{R}_m|^5} , \tag{3}$$

and for $l = m$

$$A^{ll} = +e^2 \sum_m \frac{B^{lm}}{|\vec{R}_l - \vec{R}_m|^5} , \quad m \neq l. \tag{4}$$

The tensor B^{lm} is

$$B^{lm} = \begin{bmatrix} 2(X_l - X_m)^2 - (Y_l - Y_m)^2 & 3(X_l - X_m)(Y_l - Y_m) \\ 3(X_l - X_m)(Y_l - Y_m) & 2(Y_l - Y_m)^2 - (X_l - X_m)^2 \end{bmatrix}, \tag{5}$$

where X_l and Y_l are the coordinates of the electron lattice points in the x, y plane. The normal modes of this coupled system, which are described by wave vector $\vec{q}$ and frequency $\omega(\vec{q})$, are those that diagonalize A^{lm}. Diagonalization proceeds in the normal way to give for the two eigenfrequencies

$$\omega^2(q) = \frac{1}{2m_{\parallel}} \left\{ D_{xx} + D_{yy} \pm \left[(D_{xx} - D_{yy})^2 + 4 |D_{xy}|^2 \right]^{1/2} \right\}, \tag{6}$$

where

$$D_{ij} = \sum_m A_{ij}^{lm} \, e^{i\vec{q} \cdot (\vec{R}_l - \vec{R}_m)} \tag{7}$$

and $m_{\parallel}$ is the effective mass of the electron in the x, y plane.

Because the Coulomb force is long range, the sums over lattice sites converge slowly. Therefore, we use the Ewald method[18] to sum the series in Eq. (7). However, the Ewald method applies to a lattice of infinite extent whereas, we are considering a finite lattice. This may be accounted for by writing a finite sum as $\sum_0^A = \sum_0^\infty - \sum_A^\infty$, where $\sum_0^A$ is the desired sum over a finite area and $\sum_0^\infty$ is the Ewald sum. The last sum may be converted into an integration for large area. When $q \gg L^{-1}$ this last sum can be neglected. Since we are not concerned with the region in which this condition is violated, we neglect the third sum in what follows.

Since the Ewald technique is well known,[18] we give only the results of its application to the sums in Eq. (7). This summation method replaces the sum over lattice sites by two sums: one over the reciprocal lattice and one over the real lattice. The advantage is that both sums are rapidly convergent. It is convenient to define the following dimensionless vectors: $\vec{\alpha} = \vec{g}/2G$, $\vec{\beta} = (\vec{g} + \vec{q})/2G$, and $\vec{\gamma} = \vec{R}_l G$, where $\vec{g}$ is a reciprocal-lattice vector and G is a parameter chosen to make both sums converge equally well.

After some algebra, the contributions to the tensor D_{ij} can be written

$$\begin{aligned} D_{xx} + D_{yy} = {} & 2ne^2 \sum_{\vec{g}} \{ 2\sqrt{\pi} G [e^{-\alpha^2} e^{-\beta^2}] + 2G\pi [\beta \operatorname{erfc}(\beta) - \alpha \operatorname{erfc}(\alpha)] \} \\ & + \frac{2e^2}{\pi} \sum_{\vec{R}} \frac{[1 - e^{i\vec{q} \cdot \vec{R}}]}{R^3} [\gamma e^{-\gamma^2} + \tfrac{1}{2}\pi \operatorname{erfc}(\gamma)], \end{aligned} \tag{8}$$

$$\begin{aligned} D_{xx} - D_{yy} = {} & -2e^2 n\pi \sum_{\vec{g}} \left[\frac{(g_x^2 - g_y^2)}{g} \operatorname{erfc}(\alpha) - \frac{(\beta_x^2 - \beta_y^2)}{2G\beta} \operatorname{erfc}(\beta) \right] \\ & + \frac{4e^2}{\sqrt{\pi}} \sum_{\vec{R}} \frac{[1 - e^{i\vec{q} \cdot \vec{R}}]}{R^5} (X^2 - Y^2)[0.75 \sqrt{\pi} \operatorname{erfc}(\gamma) + \gamma(1.5 + \gamma^2) e^{-\gamma^2}], \end{aligned} \tag{9}$$

and the off-diagonal element is

$$D_{xy} = -\frac{ne^2\pi}{G} \sum_{\vec{g}} \frac{\beta_x \beta_y}{\beta} \operatorname{erfc}(\beta) - 3e^2 \sum_{\vec{R}} \frac{e^{i\vec{q} \cdot \vec{R}}}{R^5} XY[\gamma(1.5 + \gamma^2) e^{-\gamma^2} + 0.75 \sqrt{\pi} \operatorname{erfc}(\gamma)], \tag{10}$$

where erfc is the complimentary error function and n the electron density. The subscripts x and y refer to the components along the coordinate axes. The vector $\vec{R} = \vec{X} + \vec{Y}$ is the distance from the lth to the mth lattice site. Because of the exponential factors in the sums, they converge rapidly and summation to next nearest neighbors gives sufficient accuracy.

In evaluating the dispersion relation (6) over the first Brillouin zone, we find that there are two important regions of $\vec{q}$ space. When $\vec{q}$ is smaller than the reciprocal of the length of the system, we have the hydrodynamic regime where, neglecting end effects, both ω_1 and ω_2 are proportional to q. However, this is not physically realizable since, for these long wavelengths, electron motion is a translation of the electron system against the charged plate B, which gives a restoring force. This restoring force produces a finite-frequency longitudinal vibration even at $q = 0$. This vibration is analogous to the plasma vibration in three dimensions. In the two-dimensional model the electric field due to the bottom plate is assumed uniform. Therefore, the plasma-like vibration arises from edge effect and not local charge imbalances.

To determine exactly the oscillation of the charge sheet against the fixed conducting plate is a difficult mathematical problem beyond the scope of this paper. Nevertheless, we can make an estimate of the frequency of this mode if we model our system by analogy with the three-dimensional Wigner crystal. To do this we replace the conducting plate by an insulating plate of the same configuration with a uniform fixed positive charge density equal to the electron density on the liquid surface. For the $q = 0$ mode, we consider the motion of the electron sheet to be a rigid displacement by a distance δ relative to the bottom plate. For small δ, it is straightforward to show that the restoring force leads to a harmonic vibration whose frequency ω_p is given by

$$\omega_p{}^2 = \frac{ne^2}{m_\parallel \sqrt{5}} L^{-1}. \tag{11}$$

In deriving this expression, we assumed that the distance between the plate and the electron sheet was small compared with the plate size. Similarly, the assumptions leading to Eqs. (6) and (7) are no longer valid for wavelengths the order of the size of the system. For wavelengths much shorter than the system size, end effects can be ignored and Eq. (6) is expected to give the vibrational frequencies.

In the region where the wavelength is longer than the lattice spacing, we find that only one of the frequencies is an acoustic-type vibration. The other vibration, which is longitudinal for any $\vec{q}$ direction, is given by taking the positive root of Eq. (6). This gives

$$\omega_l(q) = \left(\frac{2\pi n e^2}{m}\right)^{1/2} q^{1/2}. \tag{12}$$

This square-root dependence of frequency on wave vector holds for any lattice structure. This can be readily seen if the D_{ij} are expanded in powers of q. For a nonzero reciprocal-lattice vector, the leading term is proportional to q^2. However, the $g = 0$ terms, which are independent of lattice structure, are proportional to q. This leads, via Eq. (6), to one frequency being proportional to q and the other being proportional to $q^{1/2}$. Since the longitudinal dispersion relation comes from the $\vec{g} = 0$ term in the sum, we see that it is due to the long-range nature of the Coulomb force. At the end of this section we use the uniform-background model for the electron crystal to show this more clearly.

In Fig. 2 numerical solutions of the dispersion relation (6) are plotted for the principal directions of a square lattice. The frequency is in units of $(e^2/m_\parallel R_0^3)^{1/2}$, where R_0 is the lattice constant. The two directions (1, 0) and (1, 1) serve to illustrate the main features of the dependence of $\omega(q)$ on q. The values of the sound velocity S are given in the figure in units of $e(m_\parallel R_0)^{-1/2}$. For a lattice constant of 10^4 Å, the values of S are greater than 10^6 cm/sec. The high sound velocity is due to the light electron mass. Substantially the same results are found for the hexagonal lattice.

Of course ω_l does not go to zero as q tends to zero but approaches the limiting value ω_p. However, for $R_0 = 10^{-4}$ cm and $L \cong 1$ cm, ω_p is about four orders of magnitude smaller than the highest lattice frequency. Thus, the finite value of ω_l at $q = 0$ is not apparent in Fig. 2.

We can apply the above arguments to the ion lattice inside the liquid by simply scaling the expression for $\omega(q)$ by the square root of the mass ratio. This reduces the frequencies and sound velocity by a factor of about 500.

Because it might be inferred that the particular form of the longitudinal dispersion relation, Eq. (12), is due to the particular configuration of electrons on ^{4}He, we prove that this form of the dispersion relation is a general result of the uniform-background model of the electron lattice in two dimensions. It arises when the range of the Coulomb force exceeds the wavelength of the disturbance. We show the connection with the three-dimensional Wigner lattice and demonstrate that the longitudinal vibration in two dimensions is the counterpart of the longitudinal plasma

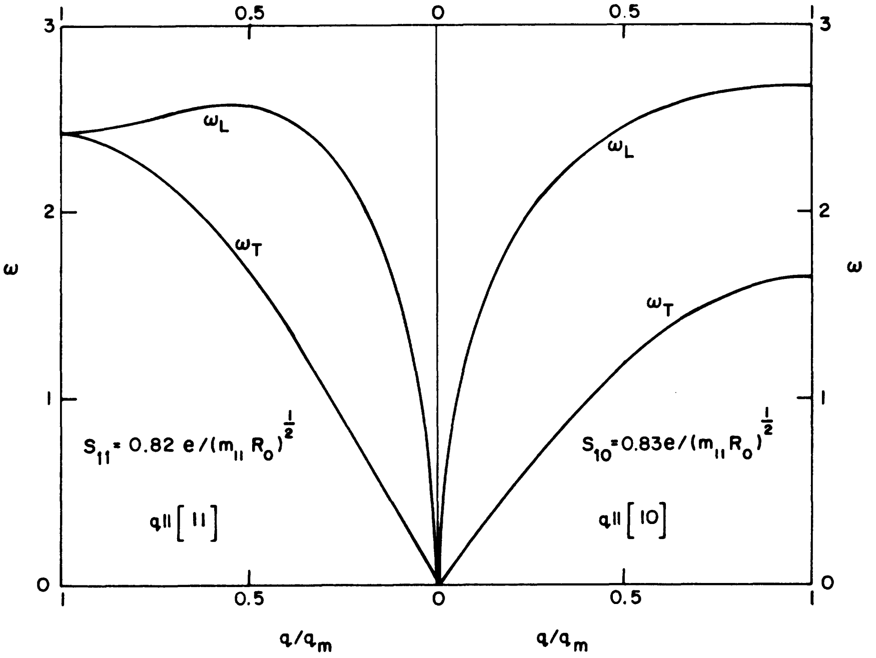

FIG. 2. Vibrational frequency $\omega(q)$ in units of $e/(m_{\parallel} R_0^3)^{1/2}$ is plotted vs normalized wave vector q for the two principal directions of a square lattice. The maximum wave vector q_m is the value of q at the zone boundary. The transverse and longitudinal branches are labeled at ω_T and ω_L, respectively.

vibration in three dimensions.

These results are best presented by expanding the potential energy that depends on only x and y in a Fourier series. This implies, then, that the lattice is of infinite extent. The dispersion relation can thus be written as

$$\omega(q)\vec{\epsilon} = \frac{N}{m_{\parallel}} V_{\vec{q}}\, \vec{q}\,\vec{q} \cdot \vec{\epsilon} + \frac{N}{m_{\parallel}} \sum_{g \neq 0} [V_{\vec{g}+\vec{q}} (\vec{q}+\vec{g})(\vec{q}+\vec{g}) \cdot \vec{\epsilon} - V_{\vec{g}}\, \vec{g}\,\vec{g} \cdot \vec{\epsilon}], \tag{13}$$

where $V_{\vec{q}}$ is the Fourier transform of the potential, $\vec{\epsilon}$ is a unit polarization vector, and N is the number of electrons. The first term is responsible for the longitudinal vibration. The second term is caused by the periodicity of the lattice and is thus responsible for the transverse vibration. In what follows we are only concerned with the long-wavelength behavior of the longitudinal vibration. Therefore, only the first term in Eq. (13) need be retained since it gives the longitudinal vibration at long wavelength.

Equation (13) shows that the $\vec{q}$ dependence of the dispersion relation is determined by the $\vec{q}$ dependence of $V_{\vec{q}}$. In three dimensions $V_{\vec{q}} \propto 1/q^2$ for a Coulomb potential. Therefore, the longitudinal vibration is independent of $\vec{q}$. This is the familiar result that the longitudinal frequency is equal to the plasma frequency.[19] In two dimensions, the Fourier transform of a Coulomb potential is $V_{\vec{q}} = 2\pi e^2 (qL)^{-1}$. Substituting this into Eq. (13) we derive a dispersion relation that is identical to that derived earlier for the system of electrons on liquid helium. In fact, this longitudinal vibration in two dimensions is the analog of the plasma vibration in three dimensions. However, the two-dimensional vibration is manifestly different because its frequency goes to zero as $\vec{q}$ goes to zero, whereas in three dimensions it remains finite as $\vec{q}$ goes to zero.

It may seem strange that there is no plasmalike vibration in two dimensions. However, one must remember that we are considering a specialized model where the positive background is uniform in space. Since plasma vibrations arise from polarizations, these can only be due to real charges at the edges of the system. In three dimensions these charges lead to finite forces when the size of the system becomes infinite. These forces give a plasma or finite frequency vibration at $q = 0$. In two dimensions, however, the force due to charges on the edges of the system goes to zero as the size of the system goes to infinity. Therefore, for this model, there is not a finite frequency vibration at $q = 0$. Of course, if one removes the restriction of a uniform background of positive charge and treats the positive charges as discrete and movable charges, then local charge imbalances lead to a plasma vibration of the same kind in both two and three dimensions.

The nonsoundlike longitudinal vibrations in two and three dimensions are a consequence of the long-range nature of the Coulomb force. This becomes apparent if a short-range force, such as a screened Coulomb force, is assumed. Then, for wavelengths longer than the range of the force (wavelengths larger than the screening radius), $V_{\vec{q}}$ is a constant. If $V_{\vec{q}}$ is constant then Eq. (13) gives a soundlike dispersion relation. This result is reminiscent of the argument given earlier, for the system of electrons on liquid helium, that for wavelengths larger then the size of the system (if end effects were neglected) the longitudinal vibrations were soundlike.

III. PARTICLE DISTRIBUTION IN THE LATTICE

Now that we have computed the vibrational modes, it is instructive to calculate the electron distribution in the electron lattice to justify the assumption that the electrons are localized on lattice points. The average value of the electron density distribution $\langle\rho(\vec{r})\rangle$ is a guide to the conditions under which we might expect the electron lattice to exist.

To find $\langle\rho(\vec{r})\rangle$ we need its Fourier transform, ρ_K, which is

$$\rho_K=\sum_l e^{-i\vec{K}\cdot\vec{r}_l}. \tag{14}$$

Recalling that $\vec{r}_l=\vec{R}_l+\vec{u}_l$, we take the transform of ρ_K and arrive at the average value of $\rho(r)$;

$$\langle\rho(\vec{r})\rangle=(2\pi\langle u^2\rangle)^{-1}\sum_{\vec{R}_l} e^{(-\vec{r}-\vec{R}_l)^2/2\langle u^2\rangle}. \tag{15}$$

The calculation of $\langle u^2\rangle$ is straightforward using the well-known result

$$\langle u^2\rangle=\sum_{\vec{q},p}\frac{\hbar}{Nm_\parallel}\frac{\coth(\hbar\omega_p(q)/2kT)}{2\omega_p(q)}, \tag{16}$$

where N is the number of particles, p is the polarization index, and the sum extends over a Brillouin zone. An exact determination of $\langle u^2\rangle$ requires a numerical integration using the calculated values of $\omega(q)$. Nevertheless, an upper limit to $\langle u^2\rangle$ can be obtained readily using a Debye approximation for $\omega(q)$. This is equivalent to replacing the longitudinal phonon branch by a soundlike branch. Then we find that

$$\langle u^2\rangle=\frac{kT}{\pi m_\parallel n\bar{S}^2}\ln\left(\frac{\sinh(\alpha q_m)}{\sinh(\alpha q_0)}\right), \tag{17}$$

where $\alpha=\hbar\bar{S}/2kT$, and q_m and q_0 are the maximum and minimum q values, respectively. $\bar{S}$ is an average sound velocity. The replacement of ω_l by ω_T gives an upper limit to $\langle u^2\rangle$ since $\omega_l>\omega_T$.

If the minimum q value is taken as zero, then $\langle u^2\rangle=\infty$ at finite temperature. This logarithmic divergence of $\langle u^2\rangle$ is the reason for the argument[20] that two-dimensional lattices are not stable. However, this argument is only true for infinite systems.[21] In a finite system it is not physically reasonable for vibrational modes of the system to have wavelengths longer than the size of the system. Thus q_0 is not zero in Eq. (7) since the system is finite. Therefore, it is possible that crystalline order might exist in this system of finite size.[21] It is reasonable to take the minimum value of q equal to the reciprocal of the length L of the system. Then

$$\langle u^2\rangle=\frac{hq_m}{nm_\parallel\bar{S}\pi}+\frac{kT}{\pi m_\parallel n\bar{S}^2}\ln\left[\frac{(1-e^{-2\hbar S\sqrt{\pi}/kTR_0})}{(1-e^{-\hbar S/kTL})}\right]. \tag{18}$$

The first term arises from the zero-point vibrations. In the high-temperature approximation, with a lattice constant of 10^4 Å and $L=1$ cm, we find that $\langle u^2\rangle\sim2\times10^{-11}+3\times10^{-10}T$ cm^2. The fluctuations due to thermal vibrations exceed the zero-point vibrations above 0.1 K. Nevertheless $\langle u^2\rangle\ll R_0^2$ at 1 K.

Returning to Eq. (15), we see that the smallness of $\langle u^2\rangle$ ensures that an electron is localized about at lattice point. Owing to zero-point fluctuations alone, an electron is confined to a volume of 10^{-3} of a unit cell. These arguments do not prove that long-range order exists. However, because of the localization of the electron to the vicinity of a lattice site, we might expect a crystalline state over a finite region.

IV. CONCLUSION

We conclude by mentioning some ways that one might observe electron crystallization in two dimensions and speculate as to whether crystallization might already have been observed. The most convincing evidence for crystallization would be the observation of the diffraction of laser light. This experiment would be best performed on the ion state inside the liquid because these states would give a larger scattering cross section for light than the bare electron outside the liquid. The propagation of a shear sound wave through the electron system would also demonstrate crystallization. The presence of the anomalous longitudinal mode could also be detected in this way. The lifetime of the electron in the image potential surface state outside the liquid is dependent on whether the electrons are crystallized or not. Similarly, one might expect the attenuation of surface waves by the electron system to depend on its state of crystallization.

At present, there is no direct evidence for a lattice of electrons on the outside surface of liquid helium. However, there is some indirect evidence or at least a hint of a lattice structure from some earlier measurements.[13,22] Sommer and Tanner[22] made puzzling observations during their mobility measurements of electrons on the surface of ^{4}He. They interpreted these observations as if there were regions of the surface charge that were immobile. This occurred only below a minimum electron density and at low temperature. The speculation of regions of immobile electrons is consistent with the idea of crystallization since an electron crystal is an insulator in the sense that a band gap must be overcome for single-particle excitations.[23] The entire crystal, however, can move in response to an electric field.[23] Crandall and Williams[13] measured a depression in liquid helium caused by the condenser force between the electrons and the bottom plate B (see Fig. 1). From a knowledge of the electric field on the electrons and an analysis of the equilibrium between electrons bound in the image potential and free in the gas, they concluded that the electron motion parallel to the surface must be localized.

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