Theory of a Ring Laser*

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A systematic formulation of the theory of a ring laser is given, based on first principles and using a well-known model for laser operation. The discussion begins with a simple physical derivation of the electromagnetic field equations to first order in Ω for a noninertial reference frame in uniform rotation, and a qualitative analysis of the traveling-wave Fox-Li modes for a polygonal cavity. The polarization of the active medium is obtained by using a Fourier-series method which permits the formulation of a strong-signal theory. The formalism can also be applied to another problem of current interest: the absorption line shape for a weak wave in the presence of a strong one traveling in the opposite direction. In the last few sections, the small-signal ring-laser theory is recovered as a special case, and a systematic calculation of the various equations of this theory is included. The limitations of laser gyroscopes arise mainly because of effects of backscattering of radiation and nonreciprocities of the optical path, leading, for example, to frequency-locking phenomena. Nonreciprocal losses have been used to shift the locking threshold, but for rotation rates above this threshold the observed beat note departs from the desired rotation rate in a typical manner shown in previous articles. However, the theory indicates that if more-detailed measurements are made, they should provide sufficient information for determining the rotation rate (apart from noise fluctuations).

I. INTRODUCTION

In the past seven years, a large number of experimental and theoretical articles have been published dealing with important aspects of ring-laser operation at small signals. These features include the effects of rotation, stability of the different modes of operation, behavior of the beat note, influence of backscattering of radiation in frequency-locking phenomena and hysteresis effects, effects of collisions and admixture of isotopes, and noise fluctuations. These are all of practical importance in understanding the limitations of laser gyroscopes. Some articles dealing with these problems are those by Aronowitz,[1,2] Aronowitz and Collins,[3] Lee and Atwood,[4] Hutchings *et al.*,[5] Klimontovich *et al.*,[6-8] Landa and Lariontsev,[9,10] Zhelnov et al.,[11] Andronova and Bershtein,[12,13] Bidikhov *et al.*,[14] and Rybokov *et al.*[15] An extensive list of others can be found in the review article by Privalow and Fridrikhov.[16]

What appears to be lacking in this extensive literature is a sufficiently systematic calculation, based on first principles and more closely related to a well-known model for laser operation,[17] and a formulation of a strong-signal theory for a ring laser.

There is considerable disagreement concerning the form of the electromagnetic field equations, in terms of $\vec{E}$ and $\vec{B}$, for a noninertial reference frame in uniform rotation. A simple physical derivation of these equations, to first order in Ω, has been included in Sec. II. Section III A represents an attempt to clarify the nature of the Fox-Li modes for an open polygonal resonator. To derive convenient amplitude- and phase-determining equations, one has to choose an appropriate representation for $\vec{E}(\vec{r}, t)$. Because of the effects of backscattering of radiation, it is not clear *a priori* that the running-wave representation is in every case the most convenient one. This problem is discussed in Sec. III and IV.

The atomic system and its interaction with the optical field is described in Sec. V. In Sec. VI A the polarization of the active medium is obtained by using a Fourier-series method which permits the formulation of a strong-signal theory, and solutions are given in terms of continued fractions. Analytical solutions are easily obtained in special cases of current interest. One of these has immediate application to the problem of the absorption line shape of a weak wave in the presence of a strong one traveling in the opposite direction.[18,19] When the continued fraction is expanded to third order in the fields, as in Sec. VI B, one obtains the familiar small-signal ring-laser theory; and the remainder of the article is devoted to a systematic calculation of the various equations for this theory. The effects of collisions, velocity flow, and admixture of isotopes are included in Sec. VII.[20] Owing to effects of backscattering of radiation and nonreciprocities, the time average of the beat note, $\langle \dot{\phi}(t)\rangle$, departs from the desired value of the rotation rate in the typical way shown by different authors (see, for example, Refs. 6, 9, 20). However, the theory indicates that more detailed measurements should provide sufficient information for determining the rotation rate. This possibility is discussed in Sec. VIII.

II. ELECTROMAGNETIC FIELD EQUATIONS

Maxwell's equations have been the starting point for the semiclassical theory of gas lasers. To follow the same general procedure with ring lasers, one has to use a proper generalization of the Maxwell equations for a noninertial frame of reference in uniform rotation with angular velocity Ω. A rigorous approach requires general relativity theory,[21] and many papers have been devoted to this purpose.[22-26] Although there is no disagreement regarding the covariant form of the equations written in terms of the electromagnetic field tensor $F_{\mu\nu}$, there is, however, considerable disagreement concerning the form of the equations expressed in terms of the field intensities E and B.

In dealing with the theory of the ring laser, a less formal approach may be used to derive a sufficiently accurate set of equations. This is possible because in actual experiments, perimetral speeds $|\vec{v}(\vec{r})|=|\vec{\Omega}\times\vec{r}|\ll c$ occur and therefore only first-order effects in $|\vec{\Omega}|$ are important. Under these circumstances, the electromagnetic field equations up to first order in (v/c) can be obtained by using the following three assumptions.

(a) For an inertial observer characterized by the superscript 0, the electromagnetic equations are given by the well-known Maxwell equations:

$$\begin{aligned}
&\vec{\nabla}^0\cdot\vec{B}^0=0,\\
&\vec{\nabla}^0\times\vec{E}^0+\partial\vec{B}^0/\partial t^0=0,\\
&\vec{\nabla}^0\cdot\vec{E}^0=\rho^0/\epsilon_0,\\
&\vec{\nabla}^0\times\vec{B}^0-\mu_0\epsilon_0\partial\vec{E}^0/\partial t^0=\mu_0\vec{J}^0,
\end{aligned}\tag{2.1}$$

where polarized media will be described by assigning suitable charge and current distributions.

(b) Since we are interested only in terms of the first order in Ω, the equations for coordinate transformation can be taken to be

$$\begin{aligned}
&x=x^0\cos\Omega t^0+y^0\sin\Omega t^0,\\
&y=-x^0\sin\Omega t^0+y^0\cos\Omega t^0,\\
&z=z^0,\\
&t=t^0.
\end{aligned}\tag{2.2}$$

A careful definition of simultaneity in a rotating frame is not necessary because the "local" dependence of the Lorentz time dilation

$$dt=[1-(\Omega r/c)^2]^{1/2}dt^0$$

is of the order Ω^2, and consequently has been neglected (dt is the time interval between events taking place at the same point for a rotating observer).

(c) It is assumed that an observer on the rotating frame, after neglecting the effects of inertial forces, can relate his "physical" field quantities $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$ to the inertial ones $\vec{E}^0(\vec{r}^0,t^0)$ and $\vec{B}^0(\vec{r}^0,t^0)$ by the well-known formulas of the special relativity (order v/c),

$$\begin{aligned}
&\vec{E}^0_\parallel=\vec{E}_\parallel,\quad \vec{B}^0_\parallel=\vec{B}_\parallel,\\
&\vec{E}^0_\perp=(\vec{E}_\perp-\vec{v}\times\vec{B}),\quad \vec{B}^0_\perp=(\vec{B}_\perp+\mu_0\epsilon_0\vec{v}\times\vec{E})
\end{aligned}\tag{2.3}$$

($\parallel,\perp$ implies parallel and perpendicular to $\vec{v}$), except that here

$$\vec{v}(\vec{r})=\vec{\Omega}\times\vec{r}\tag{2.4}$$

varies from point to point. By the word "physical" we mean that $\vec{E}$ and $\vec{B}$ are related to force on a test charge and torque on a current loop in the same way as they are in inertial systems. Assumptions (2.3) has been tested for quasistatic fields and slowly rotating bodies ($v/c\ll 1$) in experiments on unipolar induction and electrically polarized bodies.[27,28] For current and charges $(\vec{J},\rho)$ we assume the same transformation equations as for the coordinates. To first order in Ω,

$$\vec{J}^0=\vec{J}+\rho\vec{v},\ \rho^0=\rho.\tag{2.5}$$

The electromagnetic field equations for a rotating observer can now be easily obtained from the Maxwell equations in the inertial frame. Using (2.2) for coordinate transformation we have

$$(\partial/\partial x_i^0)_{t^0}=\sum_j(\partial x_j/\partial x_i^0)_{t^0}\,\partial/\partial x_j,\tag{2.6}$$

which means $\vec{\nabla}^0=\vec{\nabla}$. Similarly, we find

$$(\partial/\partial t^0)=(\partial/\partial t)-(\vec{v}\cdot\vec{\nabla})+\vec{\Omega}\times,\tag{2.7}$$

where the last term of (2.7) has been added so that the operation on vectors will take into account the rate of change of the vector due to rotation. Using the above operators in the Maxwell equations (2.1) and using some auxiliary formulas, we obtain the electromagnetic field equations for a rotating system to first order in $|(\vec{\Omega}\times\vec{r})/c|$:

$$\begin{aligned}
&\vec{\nabla}\cdot[\vec{B}+\mu_0\epsilon_0(\vec{\Omega}\times\vec{r})\times\vec{E}]=0,\\
&\vec{\nabla}\times\vec{E}+(\partial/\partial t)[\vec{B}+\mu_0\epsilon_0(\vec{\Omega}\times\vec{r})\times\vec{E}]=0,\\
&\vec{\nabla}\cdot[\vec{E}-(\vec{\Omega}\times\vec{r})\times\vec{B}]=\rho/\epsilon_0,\\
&\vec{\nabla}\times\vec{B}-\mu_0\epsilon_0(\partial/\partial t)[\vec{E}-(\vec{\Omega}\times\vec{r})\times\vec{B}]=\mu_0\vec{J}.
\end{aligned}\tag{2.8}$$

The discrepancy between the above set of equations and the one proposed by Schiff and by Heer (to first order in Ω), i.e.,[22,25]

$$\begin{aligned}
&\vec{\nabla}\cdot\vec{B}=0,\\
&\vec{\nabla}\times\vec{E}+\partial\vec{B}/\partial t=0,\\
&\vec{\nabla}\cdot[\vec{E}-(\vec{\Omega}\times\vec{r})\times\vec{B}]=\rho/\epsilon_0,\\
&\vec{\nabla}\times[\vec{B}-\mu_0\epsilon_0(\vec{\Omega}\times\vec{r})\times\vec{E}]-\mu_0\epsilon_0(\partial/\partial t)\\
&\qquad\qquad\times[\vec{E}-(\vec{\Omega}\times\vec{r})\times\vec{B}]=\mu_0\vec{J},
\end{aligned}\tag{2.9}$$

can be attributed to a different definition of the tensor $F_{\mu\nu}$ in terms of the fields $\vec{E}$ and $\vec{B}$ in a rotating frame of reference. In fact, using the covariant equations

$$\begin{aligned}
&(\partial F_{\mu\nu}/\partial x^\sigma)+(\partial F_{\nu\sigma}/\partial x^\mu)+(\partial F_{\sigma\mu}/\partial x^\nu)=0,\\
&(\partial/\partial x^\nu)[(-g)^{1/2}F^{\mu\nu}]=(-g)^{1/2}J^\mu,
\end{aligned}\tag{2.10}$$

Schiff and Heer have obtained set (2.9) by assuming that

$$F_{\mu\nu}=\begin{vmatrix} 0 & B_3 & -B_2 & E_1\\ -B_3 & 0 & B_1 & E_2\\ B_2 & -B_1 & 0 & E_3\\ -E_1 & -E_2 & -E_3 & 0\end{vmatrix}. \tag{2.11}$$

On the other hand the set of Eqs. (2.8) suggest that a proper definition of $F_{\mu\nu}$ in terms of physical fields $\vec{E}$ and $\vec{B}$ for a noninertial rotating frame should be (first order in Ω)

$$\begin{vmatrix} 0 & [\vec{B}+(\vec{\Omega}\times\vec{r})\times\vec{E}]_3 & -[\vec{B}+(\vec{\Omega}\times\vec{r})\times\vec{E}]_2 & E_1\\ -[\vec{B}+(\vec{\Omega}\times\vec{r})\times\vec{E}]_3 & 0 & [\vec{B}+(\vec{\Omega}\times\vec{r})\times\vec{E}]_1 & E_2\\ [\vec{B}+(\vec{\Omega}\times\vec{r})\times\vec{E}]_2 & -[\vec{B}+(\vec{\Omega}\times\vec{r})\times\vec{E}]_1 & 0 & E_3\\ -E_1 & -E_2 & -E_3 & 0\end{vmatrix}, \tag{2.12}$$

which for inertial systems ($\Omega=0$) reduces to the more familiar definition (2.11).

Our results are in agreement to first order in v/c with those obtained by Irvine[24] using a covariant method with the requirement that "the fields $\vec{E}$ and $\vec{B}$ in a rotating reference system are given by the same physical measurements as determine those quantities in an inertial system, once the effects of the inertial forces has been subtracted out."

In deriving Eqs. (2.8), we have considered only charge and current sources, deliberately avoiding any discussion of constitutive relations for a macroscopic medium. Such a medium will be considered here simply as an array of atoms with an electrical state described by a polarization $\vec{P}(\vec{r},t)$ (electric dipole moment density). In the usual way, we may regard the macroscopic charge and current densities as partially due to polarization charge and polarization current, assuming the validity of the familiar replacement for nonmagnetic materials[29]

$$\begin{aligned}
&\rho\rightarrow\rho-\operatorname{div}\vec{P},\\
&\vec{J}\rightarrow\vec{J}+\partial\vec{P}/\partial t,
\end{aligned}\tag{2.13}$$

where, after the substitution, $\vec{J}$ and ρ stands for free current and charge densities. Thus, according to our assumptions, replacing (2.13) on the right side of Eqs. (2.8) gives us the "macroscopic" Maxwell equations for a rotating system. Combining Eqs. (2.8) with the specifications (2.13), we may obtain the following wave equation for the electric field (first order in Ω):

$$\begin{aligned}
&\operatorname{curl}\operatorname{curl}\vec{E}+\mu_0\epsilon_0\,\partial^2\vec{E}/\partial t^2+\mu_0\epsilon_0(\partial/\partial t)\\
&\qquad\times\{\operatorname{curl}[(\vec{\Omega}\times\vec{r})\times\vec{E}]+(\vec{\Omega}\times\vec{r})\times\operatorname{curl}\vec{E}\}\\
&\qquad\qquad=-\mu_0(\partial/\partial t)[\vec{J}+\partial\vec{P}/\partial t].
\end{aligned}\tag{2.14}$$

Equation (2.14) is a generalization of Eq. (3) of Ref. 17, and will be used as a starting point for our discussion of the ring laser. The same wave equation may also be obtained from (2.9) with specifications (2.13).

III. RING LASER–WAVE EQUATION

A. Fox and Li Modes for a Polygonal Cavity

Before trying to reduce the wave equation (2.14) to a form more appropriate for ring lasers, it is useful to discuss the normal-modes problem for a cavity of polygonal type. An open cavity, with finite mirrors, has a continuum of modes because it is not enclosed by reflecting walls. However, for a cavity of the Fabry-Perot type shown in Fig. 1(a), Fox and Li[30] have shown that there are discrete sets of quasistationary states for which the losses from the tube are small. Their iterative method of calculation was analogous to the physical process of launching an initial wave configuration in the interferometer and letting it bounce back and forth between the mirrors. It was found that after many reflections a state is

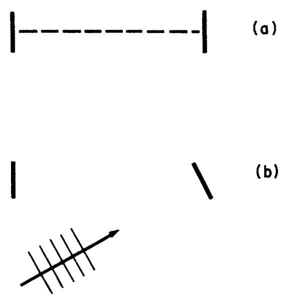

FIG. 1. (a) shows schematically the mirror arrangement for a Fabry-Perot resonator. (b), the same as (a), but the mirrors are badly misaligned. A plane optical wave is incident on the mirror system.

reached in which the relative field distribution does not vary from transit to transit and the amplitude decays at an exponential rate owing to leakage from the cavity. This field distribution is regarded as a normal mode of the interferometer. The analysis of Fox and Li shows that the cavity modes of lowest diffraction loss have even symmetry across the tube diameter, and frequencies determined by the familiar Fabry-Perot condition that the plate separation d be a half-integer multiple of the wavelength,

$$\Omega_n = (\pi n c/d),$$

typically $d \approx 100$ cm, $n = 2\times 10^6$.

For these longitudinal modes of highest Q, the field has its largest intensity at the middle of the tube diameter, diminishing toward the edges with a slow variation over distances of the order of a wavelength. It is also shown that the diffraction losses diminish very rapidly with increase of the number of Fresnel zones seen in one mirror from the center of the other. For example, for a cavity of the first type used by Javan, Bennett, and Herriott,[31] this number was about 200 and the diffractive spillover per transit, being much less than 0.1%, was negligible compared to the 1% mirror reflectance loss.[32]

Similar calculations for the polygonal geometry of a ring laser are necessarily more complicated than for a Fabry-Perot resonator. For our purposes, it is more instructive to take a qualitative approach based on well-known consequences of scattering theory. Consider the arrangement shown in Fig. 1(b), where the perfectly conducting mirrors of Fig. 1(a) are badly misaligned. Let a plane optical (scalar) wave be incident on the mirror system. The wave function has the form

$$\psi(\vec{K}, \vec{r}) = e^{i\vec{K}\cdot\vec{r}} + S(\vec{K}, \vec{r})$$

of an incident plane of unit amplitude and wave vector $\vec{K}$ and a scattered wave $S(\vec{K}, \vec{r})$. Each of the eigenfunctions $\psi(\vec{K}, \vec{r})$ describes a normal mode of the open system, and the spectrum of allowed K values is continuous. These normal modes supply us with a complete set of complex-orthogonal functions which can be used to expand the space dependence of any physical wave field satisfying the boundary conditions at the mirrors.

At large distances from the mirrors, $|\psi(\vec{K}, \vec{r})| \to 1$, and for the geometry of Fig. 1(b) one would expect that, away from the edges, $|\psi(\vec{K}, \vec{r})|$ would not be much larger than unity anywhere. For the direction of the incident wave shown in Fig. 1(b) there would also be a reflection at one of the mirrors and the wave amplitude would have a standing-wave character for a distance of the order a^2/λ in front of the mirror, where a is the mirror width and λ is the wavelength.

If the mirrors are nearly aligned as in Fig. 1(a), and the wave vector is adjusted so that $|\vec{K}|d \approx n\pi$ (where n is an integer), with the direction of $\vec{K}$ nearly parallel to the laser axis, a very sharp resonance phenomenon occurs.[33] The amplitude of $\psi(\vec{K}, \vec{r})$ in the space *between* the mirrors will be very dependent on the detuning $||\vec{K}|d - n\pi|$. The average energy density *there* will be larger than for the incident wave by a factor $Q \gg 1$, and its dependence on detuning will be approximately Lorentzian with a range of K given by $|(\Delta K/K)| \approx (1/Q)$. In the region between the mirrors, the spatial dependence of $\psi(\vec{K}, \vec{r})$ will be nearly that of the standing wave obtained in the Fox and Li treatment. Although the eigenfunctions $\psi(\vec{K}, \vec{r})$ for a range $\Delta\vec{K}$ of $\vec{K}$ values near resonance have nearly the same spatial dependence in the region of the Fox-Li mode, all of the eigenfunctions are orthogonal in the unbounded domain. The transverse Fox-Li modes of lower Q can be obtained by forming the derivatives of the functions $\psi(\vec{K}, \vec{r})$ with respect to $\vec{K}$. It should be noted that even at resonance the function $\psi(\vec{K}, \vec{r})$ would have, besides the dominant standing wave, a small traveling-wave component. This could either be neglected or canceled out by sending in a plane wave in the direction $-\vec{K}$.

We now consider the scattering of a plane wave on the array of three mirrors shown in Fig. 2(a). As before, the eigenfunctions $\psi(\vec{K}, \vec{r})$ have a continuous spectrum in K space, and provide us with a complete set of complex, orthogonal basis functions. In general, the amplitude of the distorted-plane-wave field will not differ much from unity. However, when the three mirrors are oriented as shown in Fig. 2(b), a triangular optical path can be found which satisfies the condition for reflection at all three mirrors. Under these conditions, a plane wave incident on the mirrors can form a resonant field in the vicinity of the geo-

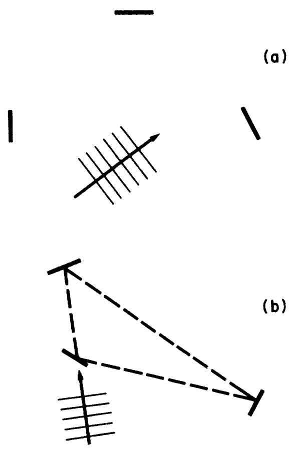

FIG. 2. Scattering of a plane wave of unit amplitude incident on the array of three mirrors is shown in (a). In general, the amplitude of the distorted-plane-wave field will not differ much from unity. However, when the mirrors are oriented as shown in (b), a triangular optical path can be found which satisfies the condition for reflection at all three mirrors. If the wavelength of the plane wave is adjusted to fit an integral number of times into the optical path, a sharp resonance occurs and the field will have a region of strongly augmented amplitude in the vicinity of the geometrical optical path.

metrical optical path. For this to occur, several conditions must be satisfied. The wavelength of the plane wave must fit an integral number of times into the optical path, with due allowance for phase shifts at each mirror reflection. In general, these depend on the polarization of the electric vector. Also, the direction of propagation should be parallel to one of the triangular arms. Under these conditions, the field $\psi(\vec{K}, \vec{r})$ will have a region of strongly augmented amplitude in the vicinity of the geometrical optical path and will *there* have very nearly the spatial dependence of the corresponding traveling-wave Fox-Li mode. The opposite traveling-wave solution can be obtained by reversing the direction of the wave vector of the incident plane wave. Each longitudinal mode has a twofold degeneracy. In general, the traveling-wave Fox-Li modes will have a (very) small admixture of the opposite traveling wave because of back diffraction at the mirrors.

B. Backscattering of Radiation, Nonreciprocal and Localized Losses

The wave equation (2.14) will now be reduced to a form more appropriate for a theory of a ring laser. Following the discussion of Sec. III A, it is assumed that a polygonal cavity of the ring laser type possesses normal modes of a similar nature to the Fox and Li modes for a parallel-plate resonator, and that the modes of highest Q vary little in directions which are transverse to the optical path $\hat{s}$. Then, the variation of the field only along the polygonal path will be taken in account; i.e., $\vec{\nabla} = \hat{s}(\partial/\partial s)$. Furthermore, an electric field polarized along a direction perpendicular to the plane of the structure ($\vec{E} = \hat{z}E$) will be assumed (Fig. 3). With these simplications, Eq. (2.14) reduces to

$$c^2 \frac{\partial^2 E}{\partial s^2} - \frac{\partial^2 E}{\partial t^2} + 2[\hat{s} \cdot (\vec{\Omega} \times \vec{r})] \frac{\partial^2 E}{\partial s \partial t} = \frac{1}{\epsilon_0} \frac{\partial}{\partial t}\left(J + \frac{\partial P}{\partial t}\right), \quad (3.1)$$

where $(\mu_0 \epsilon_0)^{-1} = c^2$.

As in an earlier paper,[17] an Ohmic current $J = \sigma E$ will be introduced as one simple way to deal with energy losses. In addition, it is convenient to take into account the possibility of having different losses for clockwise and counterclockwise senses of propagation. This may be done by simply adding to Eq. (3.1) a differential loss term of the type $(c/n)(\sigma_d/\epsilon_0)\,\partial E/\partial s$, where n is the index of refraction of any nonactive linear medium. In some experiments, controllable nonreciprocal losses have been produced by using magneto-optic Faraday rotators in conjunction with Brewster-angle surfaces.[4,15] The fictional constants σ and σ_d are finally adjusted to give the desired losses.

The electric polarization acting as a source in Eq. (3.1) will be divided into contributions of the active nonlinear medium

$$P(s, t),$$

and the contributions of any linear, nonresonant media

$$P_l = \epsilon^0 \chi E.$$

Experiments with ring lasers[3-15] suggest that under some working conditions it is necessary to consider the effect of backscattering of radiation

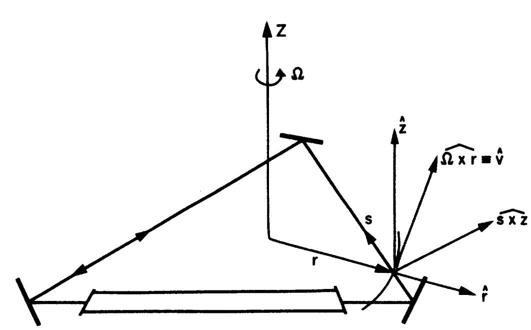

FIG. 3. This shows the laser geometry and defines some of the notation used in the derivation of Eq. (3.1) from Eq. (2.14). Carets denote unit vectors.

at the finite mirrors of the cavity or by dust particles, as well as reflections on dielectric windows. These effects involve a transfer of radiation from one sense of propagation to the other, which most authors have described by simply adding phenomenological terms to the amplitude and phase equations. The approach assumes that a fraction (R_+, R_-) of fields (E_+, E_-) is scattered into the opposite direction with phase changes δ_+, δ_-, respectively. If one assumes that $(R_+, \delta_+) \neq (R_-, \delta_-)$ it becomes possible to give a schematic treatment of cases cited in Refs. 5, 6, and 15 in which a portion of the energy is transferred from mode $E_\pm$ to mode $E_\mp$ via additional mirrors, external to the ring.

Dealing with a nonlinear problem we prefer to bring these effects into Eq. (3.1) by permitting steep space variations (bumps) in the electric susceptibility of the linear, nonresonant media, i.e.,

$$P_l = \epsilon_0 \chi(s) E.$$

Possible differences in reflection coefficients and phases will be allowed later in the final equations. However, they also may be brought into Eq. (3.1) by considering, for instance, a localized, resonant medium [an extra $P(s,t)$ term] with a velocity flow in one of the propagation senses. The resonant atoms will then provide for different dispersive properties for E_+ and E_-, by Doppler effect.

Aronowitz has shown[2,20] that losses which may be associated with $\chi(s)$, e.g., finite conductivity of dielectric bumps, dust particle and localized mirror losses, etc., also contribute to the coupling between oppositely directed running waves. This will be taken into account by simply allowing

$$\chi(s) = \chi_r(s) + i\chi_i(s). \tag{3.2}$$

Of course, the complex function $\chi(s)$ may not be very well known, and consequently will be treated here in the lowest approximation by using a minimum number of related parameters.

Finally, with the substitution $P \to P(s,t) + \epsilon_0 \chi(s) E$, our wave equation (3.1) for the rotating ring laser reads

$$c^2 \frac{\partial^2 E}{\partial s^2} - [1+\chi(s)] \frac{\partial^2 E}{\partial t^2} + 2[\hat{s}\cdot(\vec{\Omega}\times\vec{r})] \frac{\partial^2 E}{\partial s \partial t} - (\sigma/\epsilon_0)\frac{\partial E}{\partial t} + \frac{c}{n}\frac{\sigma_d}{\epsilon_0}\frac{\partial E}{\partial s} = \frac{1}{\epsilon_0}\frac{\partial^2 P(s,t)}{\partial t^2}, \tag{3.3}$$

where $\chi(s)$ is given by (3.2). Any physical solution of the above equation must satisfy

$$E(s,t) = E(s+L, t), \tag{3.4}$$

where L is the perimeter of the polygon.

C. Free Oscillations of the Ring-Laser Wave Equation

The main task of a ring-laser theory is the study of the quasistationary self-sustained oscillations of $E(s,t)$ as described by (3.3). To derive convenient amplitude- and phase-determining equations from (3.3), one has to choose an appropriate representation for $E(s,t)$. It will help us to select this representation, to give first a discussion of the free oscillations of Eq. (3.3). Thus, removing $P(s,t)$ and the damping terms, we seek solutions of the type

$$E(s,t) = U(s) e^{-i\nu t} + \text{c.c.} \tag{3.5}$$

Substituting this form into the wave equation without losses or active medium, we have

$$\frac{d^2 U(s)}{ds^2} + \left(\frac{\nu}{c}\right)^2 n(s)^2 U(s) - 2i\left(\frac{\nu}{c}\right)[c^{-1}(\vec{\Omega}\times\vec{r})\cdot\hat{s}] \times \frac{dU}{ds} = 0, \tag{3.6}$$

where

$$n(s)^2 = [1+\chi_r(s)] \tag{3.7}$$

is the index of refraction of the linear transparent media. We assume that $n(s)$ does not vary too violently, except for discontinuities or small regions of very steep behavior, which will provide for reflections. For simplicity, discontinuities will be replaced by "equivalent" bumps adjusted to give the same amount of reflection (see Fig. 4).

The rotation term $(v/c \ll 1)$, and the bumps $\delta n(s)$ in the index of refraction, $n(s) = n_0(s) + \delta n(s)$, will be considered as small perturbations of Eq.

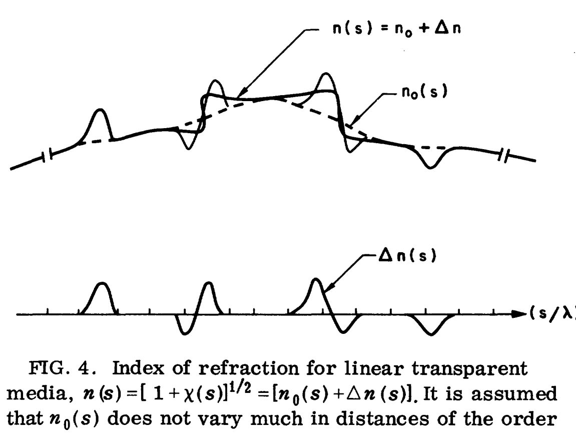

FIG. 4. Index of refraction for linear transparent media, $n(s) = [1+\chi(s)]^{1/2} = [n_0(s) + \Delta n(s)]$. It is assumed that $n_0(s)$ does not vary much in distances of the order of a wavelength. Discontinuities or small regions of very steep behavior of $n(s)$ are replaced by equivalent bumps $\Delta n(s) \neq 0$, adjusted to give the same amount of reflection.

(3.6). Solutions of the unperturbed equation away from bumps may be given in WKB approximations,[34]

$$U_\pm(s)=[n_0(s)]^{-1/2}\exp[\pm i(\nu_0/c)\int^s n_0(s')ds'].\qquad(3.8)$$

They should be valid in regions where

$$\left(\frac{\nu}{c}\right)^2 n(s)^2 \gg \left(\frac{\nu}{c}\right)\frac{dn(s)}{ds},\qquad(3.9)$$

that is to say, regions where $n(s)[=n_0(s)]$ does not change by an appreciable fraction of itself in a distance of a wavelength. The rotation term could have been included in the WKB solution. In that case, $n_0(s)$ should be replaced by $[n_0(s)\pm c^{-1}(\vec{\Omega}\times\vec{r})\cdot\hat{s}]$, from which it is apparent that rotation introduces nonreciprocity in the index of refraction [see Ref. 25, especially Eq. (32)]. The electric field will be single-value if

$$[(\nu_0/c)\int_0^L n_0(s')ds']+\Phi=2\pi m\quad(m=1,2,3,\dots),$$

where Φ represents possible phase shifts at mirrors. This equation determines the value of the possible unperturbed frequencies

$$(\nu_0/c)_m=(2\pi m-\Phi)/(\langle n_0\rangle L),\qquad(3.10)$$

where $\langle n_0\rangle$ is the average index of refraction defined by

$$\langle n_0\rangle=(1/L)\int_0^L n_0(s)ds\approx\langle n\rangle.\qquad(3.11)$$

The unnormalized functions $U_{m,\pm}$ satisfy the orthogonality relation

$$\frac{1}{\langle n_0\rangle L}\int_0^L n_0(s)^2 U_{m,\mu}U_{m',\mu'}\,ds=\delta_{mm'}\delta_{\mu\mu'}.\qquad(3.12)$$

A solution of the entire Eq. (3.6), with bumps and rotation term included, may now be obtained by degenerate perturbation theory by writing $U(s)$ as a superposition of the unperturbed $U_{m,\pm}$. Dealing with the theory of the single-mode operation of the ring laser, it will be sufficient to consider only one m value in the expansion of the field, that one which has $(\nu_0)_m$ closest to atomic resonance $\omega=(E_a-E_b)/h$. Consequently, we shall expand

$$U(s)=a_+U_+(s)+a_-U_-(s),\qquad(3.13)$$

where the index m has been dropped from $U_{m,+}$ and $U_{m,-}$. Substituting (3.13) into (3.6) by using (3.9), and taking projections onto $U_\pm$ with the help of (3.12),

$$\{\nu_0^2-\nu^2\}a_\pm\mp2\nu\nu_0\left\{\frac{1}{\langle n\rangle L}\int_0^L[c^{-1}(\vec{\Omega}\times\vec{r})\cdot\hat{s}]ds\right\}a_\pm$$
$$-\nu^2\left\{\frac{1}{\langle n\rangle L}\int_0^L n(s)^2U_\mp U_\pm^* ds\right\}a_\mp=0.\qquad(3.14)$$

A closer examination of the integrals indicate that the second curly bracket is the average value of the tangential velocity (in c units) divided by the average value of the index of refraction, i.e., $[\langle c^{-1}v_s\rangle/\langle n\rangle]$. The third curly bracket is the matrix element of $(n/n_0)^2$ between $U_\mp$ and $U_\pm$. Comparing this integral with the orthogonality relation (3.12), one can see that its nonzero value arises from the presence of bumps in the index of refraction, $n(s)=n_0(s)+\delta n(s)$ ($|\delta n|\ll n_0$).

At this point, it is convenient to define the frequencies κ and Δ characterising the effects of rotation and bumps on laser operation:

$$\kappa=[\nu_0/\langle n\rangle L]\int_0^L[c^{-1}(\vec{\Omega}\times\vec{r})\cdot\hat{s}]ds$$
$$=\nu_0[c^{-1}\vec{\Omega}\cdot(2\vec{A}/L)]/\langle n\rangle,\qquad(3.15)$$

where $\vec{A}=\frac{1}{2}\int_0^L\vec{r}\times d\vec{s}$ is the area of the polygon,[35,36] and

$$2\Delta e^{\pm i\delta_r}=\nu_0\left\{\frac{1}{\langle n\rangle L}\int_0^L n(s)^2U_\mp U_\pm^* ds\right\},\qquad(3.16)$$

where $n(s)^2=[1+\chi_r(s)]$. It is easily shown that Δ and the amplitude reflection coefficient R, associated with the dielectric bumps, are related as follows:

$$\Delta\approx\frac{1}{2\pi}\left(\frac{c}{L}\right)R\ \text{(Hz)},\qquad(3.17)$$

assuming $R\ll1$. Neglecting higher-order terms in κ, $\Delta\ll\nu_0$, Eqs. (3.14) may now be written as

$$[\nu_0^2-\nu^2\mp2\nu_0\kappa]a_\pm-2\nu_0\Delta e^{\pm i\delta_r}a_\mp=0.\qquad(3.18)$$

The compatibility condition for (3.18) gives us two positive frequencies,

$$\nu_1=\nu_0-(\kappa^2+\Delta^2)^{1/2},$$
$$\nu_2=\nu_0+(\kappa^2+\Delta^2)^{1/2}.\qquad(3.19)$$

These frequencies together with the corresponding two values for the ratio (a_-/a_+), which can be obtained from (3.18), enable us to write explicit expressions for the normal modes

$$E_j(s,t)=[U_j(s)e^{-i\nu_jt}+\text{c.c.}]\quad(j=1,2),\qquad(3.20)$$

where the eigenfunctions U_j are given by

$$U_1(s)=[1+M^2]^{-1/2}[U_+(s)+Me^{-i\delta_r}U_-(s)],$$
$$U_2(s)=[1+M^2]^{-1/2}[U_-(s)-Me^{+i\delta_r}U_+(s)],\qquad(3.21)$$

where

$$M=\frac{\Delta}{\kappa+(\kappa^2+\Delta^2)^{1/2}}\qquad(3.22)$$

and

$$\frac{1}{\langle n\rangle L}\int_0^L n_0(s)^2U_iU_j^*\,ds=\delta_{ij}.\qquad(3.23)$$

Depending on the values of κ and Δ we distinguish four cases:

(i). $\kappa=0$, $\Delta=0$. Then M is indeterminate and the normal modes may be running or standing waves. We have degeneracy, and $\nu_{1,2}=\nu_0$.

(ii). $\kappa\neq 0$, $\Delta=0$. Then $M=0$ and

$$U_1=U_+=[n_0(s)]^{-1/2}\exp[+i(\nu_0/c)\int^s n_0(s')ds'],$$
$$U_2=U_-=[n_0(s)]^{-1/2}\exp[-i(\nu_0/c)\int^s n_0(s')ds']. \tag{3.24}$$

The normal modes are counterrotating running waves with frequencies

$$\nu_1=\Omega_+=(\nu_0-\kappa),$$
$$\nu_2=\Omega_-=(\nu_0+\kappa). \tag{3.25}$$

Rotation of the platform removes the degeneracy, giving rise to the Sagnac frequency splitting 2κ. The expression (3.15) for κ in conjunction with (3.10) for ν_0 means that Sagnac's frequency in the presence of matter is reduced from its vacuum value by a factor which is the square of the average index of refraction, $\langle n(s)\rangle=\langle(1+\chi_r)^{1/2}\rangle$, for the optical path. The influence of matter, as expressed by (3.10) and (3.15), agrees with the results of Anderson and Ryon.[37]

(iii). $\kappa=0$, $\Delta\neq 0$. Then $M=1$. Furthermore, the phase factors $e^{\pm i\delta_r}$ may be dropped from the matrix elements (3.16), and consequently from $U_{1,2}$. In fact, the origin of the s coordinate has until now been arbitrary and, from (3.8) and (3.16), it is easy to see that a translation of the origin by a distance $\bar{s}$ determined by

$$2(\nu_0/c)\int_0^{\bar{s}} n_0(s)\,ds=\delta_r \tag{3.26}$$

will change the phase factor to unity. The eigenfunctions may now be written as

$$U_1=U_c=[2/n_0(s)]^{1/2}\cos[(\nu_0/c)\int_0^s n_0(s')ds'],$$
$$U_2=U_s=-[2/n_0(s)]^{1/2}\sin[(\nu_0/c)\int_0^s n_0(s')ds'], \tag{3.27}$$

showing that the normal modes are standing waves with frequencies

$$\nu_1=\Omega_c=\nu_0-\Delta,$$
$$\nu_2=\Omega_s=\nu_0+\Delta. \tag{3.28}$$

These standing waves have loops and nodes correlated with the origin determined by (3.26).

(iv). $\kappa\neq 0$, $\Delta\neq 0$. Then $0<M<1$, and the normal modes are some combinations of opposite running (or standing) waves depending on the value of M.

IV. SELF-CONSISTENCY EQUATIONS

In the presence of a given polarization $P(s,t)$, the forced oscillations described by (3.3) can be expanded in any complete set of functions satisfying the periodicity condition (3.4). In dealing with quasistationary self-sustained oscillations around a single frequency ν_0, we may choose to express the field in any of the bases that have been described in Sec. III, namely, $\{U_+(s),U_-(s)\}$, $\{U_c(s),U_s(s)\}$, $\{U_1(s),U_2(s)\}$.

In linear problems, where the principle of superposition holds, the field is more conveniently expanded in terms of the normal-mode eigenfunctions U_j's of Eq. (3.6). In that case, the expansion coefficients have their simplest form $\langle E(s,t)|U_j(s)\rangle=\mathcal{E}_j e^{-i\nu_j t}$, with constant $\mathcal{E}_j$. However, when the polarization of the active medium is included in Eq. (3.3), the problem becomes nonlinear and the normal-mode eigenfunctions' basis in general will not be the most convenient one for expanding the field. In principle, the sets of functions written above may be considered as equally appropriate for single-frequency operation around ν_0. This is so because they are eigenfunctions of operators which differ very little from each other and from $[\partial^2/\partial s^2-(\nu_0/c)^2 n_0(s)^2]$ $(\kappa,\Delta\ll\nu_0)$. Thus, our choice will simply depend on mathematical simplicity, and we shall describe the quasistationary autonomous oscillations of $E(s,t)$ in both $\{U_+,U_-\}$ and $\{U_c,U_s\}$ representations. They will be called "running-wave representation" (RWR), and "standing-wave representation" (SWR), respectively.

A. Self-Consistency Equations in the Running-Wave Representation

We adopt the following form for the electric field:

$$E(s,t)=\tfrac{1}{2}[\mathcal{E}_+(t)U_+(s)+\mathcal{E}_-(t)U_-(s)]+\text{c.c.} \tag{4.1}$$

As in Ref. 17, in order to determine the amplitudes, frequencies, and phases of the possible quasistationary oscillations, we look for solutions of the type

$$\mathcal{E}_\pm(t)=E_\pm(t)\,e^{-i[\nu t+\phi_\pm(t)]}, \tag{4.2}$$

where the amplitudes $E_\pm(t)$ and phases are real functions of t slowly varying compared with $\cos\nu t$, and ν is not yet determined, but will be a frequency around ν_0.

The polarization $P(s,t)$ is split up into a part in phase with the field, and a part with a phase difference of $\pi/2$,

$$P(s,t)=\tfrac{1}{2}\sum_{\mu=\pm}\{[C_\mu(t)+iS_\mu(t)]e^{-i[\nu t+\phi_\mu(t)]}U_\mu(s)\}+\text{c.c.}, \tag{4.3}$$

where, in agreement with (3.12), the (coefficients are defined by

$$[C_\mu(t)+iS_\mu(t)]=\frac{1}{\langle n\rangle L}\int_0^L n_0(s)^2P(s,t)U_\mu(s)^*ds. \tag{4.4}$$

The amplitudes $C_\mu(t)$ and $S_\mu(t)$ are also slowly varying functions of t.

The expressions (4.1)–(4.3) are substituted in Eq. (3.3). In evaluating the space derivatives we use (3.9). Only first time derivatives of $E_\mu(t)$ and $\phi_\mu(t)$ are retained. Small terms like $\ddot{E}_\mu$, $\dot{\phi}_\mu\dot{E}_\mu$, $\ddot{\phi}E_\mu$, $(\dot{\phi}_\mu)^2E_\mu$, and $c^{-1}(\vec{\Omega}\times\vec{r})\nu\dot{E}$, $c^{-1}(\vec{\Omega}\times\vec{r})\nu\dot{\phi}_\mu E_\mu$, $(\sigma/\epsilon_0)\dot{E}_\mu$, etc., are neglected compared to terms of the order ν^2E_μ. In the source term we replace $\partial^2P/\partial t^2$ by $-\nu^2P$ as in Ref. 17, because terms proportional to $\dot{C}_\mu$, $\dot{S}_\mu$, etc., are negligible.

After the above simplifications, we take projections onto $U_\mu(s)$ by using (3.12), (4.4) and the expressions (3.11), (3.16) for the average value and matrix elements of $n(s)^2=[1+\chi_r(s)]$. Similar expressions are also used in dealing with localized losses $[i\chi_i(s)]$, i.e.,

$$\langle\chi_i(s)\rangle=\frac{1}{L}\int_0^L\chi_i(s)ds,$$
$$e^{\pm i\delta_i}\Theta=\frac{\nu_0}{\langle n\rangle L}\int_0^L\chi_i(s)U_\mp U_\pm^*ds. \tag{4.5}$$

Finally, two equations are obtained which may be conveniently written in matrix form:

$$i\frac{d}{dt}\begin{vmatrix}E_+e^{-i(\nu t+\phi_+)}\\E_-e^{-i(\nu t+\phi_-)}\end{vmatrix}-\left\{\begin{vmatrix}(\nu_0-\kappa) & -\Delta e^{i\delta_r}\\-\Delta e^{-i\delta_r} & (\nu_0+\kappa)\end{vmatrix}-\tfrac{1}{2}i\begin{vmatrix}(\nu/Q_+) & \Theta e^{i\delta_i}\\ \Theta e^{-i\delta_i} & (\nu/Q_-)\end{vmatrix}\right\}\begin{vmatrix}E_+e^{-i(\nu t+\phi_+)}\\E_-e^{-i(\nu t+\phi_-)}\end{vmatrix}$$
$$=-\frac{\nu}{2\epsilon_0\langle n\rangle^2}\begin{vmatrix}(C_++iS_+)e^{-i(\nu t+\phi_+)}\\(C_-+iS_-)e^{-i(\nu t+\phi_-)}\end{vmatrix}, \tag{4.6}$$

where (3.15) and (3.16) are used for κ and Δ, localized losses associated with dielectric bumps are given by (4.5), and nonreciprocal losses $(\nu/Q_\pm)$ and differential losses (Λ) are defined by

$$(\nu/Q_\pm)=\{[(\sigma/\epsilon_0)+\nu_0\langle\chi_i\rangle]\pm(\sigma_d/\epsilon_0)\}/\langle n\rangle^2$$
$$=(\nu/Q)\pm\Lambda. \tag{4.7}$$

The right-hand-side members of Eqs. (4.6) and (4.7) suggest the convenience of defining

$$\epsilon=\epsilon_0\langle n\rangle^2=\epsilon_0(1+\langle\chi_r\rangle). \tag{4.8}$$

Multiplying Eqs. (4.6) by $e^{+i(\nu t+\phi_\pm)}$, and taking the real and imaginary parts, we obtain the amplitude- and phase-determining equations

$$\dot{E}_\pm+\tfrac{1}{2}(\nu/Q_\pm)E_\pm\pm\Delta E_\mp\sin(\phi+\delta_r)+\tfrac{1}{2}\Theta E_\mp\cos(\phi+\delta_i)=-(\nu/2\epsilon)S_\pm,$$
$$[(\nu+\dot{\phi}_\pm)-(\nu_0\mp\kappa)]E_\pm+\Delta E_\mp\cos(\phi+\delta_r)\mp\tfrac{1}{2}\Theta E_\mp\sin(\phi+\delta_i)=-(\nu/2\epsilon)C_\pm \tag{4.9}$$

where

$$\phi=(\nu t+\phi_+)-(\nu t+\phi_-)\equiv(\phi_+-\phi_-). \tag{4.10}$$

In some experiments,[5,6,15] a portion of the energy is transferred from one running wave to the other via an auxiliary mirror. This extra coupling implies different reflection coefficients and phases for the opposite beams. Possible differences, $(R_+,\delta_+)\neq(R_-,\delta_-)$, may be incorporated into Eqs. (4.9) by the following substitutions:

$$\Delta E_\mp\sin(\phi+\delta_r)\rightarrow\Delta_\mp E_\mp\sin(\phi+\delta_\mp),$$
$$\Delta E_\mp\cos(\phi+\delta_r)\rightarrow\Delta_\mp E_\mp\cos(\phi+\delta_\mp). \tag{4.11}$$

The corresponding formulas for the standing-wave representation are given in Appendix B.

V. THE ACTIVE MEDIUM–POPULATION MATRIX

The next task is to evaluate the macroscopic polarization as a statistical summation over the atomic dipole moments. The atomic system and its interaction with the optical field was described in Ref. 17. Briefly, the model for the active medium is an ensemble of independent atoms with two excited energy levels a and b, between which the laser transition takes place in the presence of the field. The frequency of the transition $a\rightarrow b$ is designated by $\omega=(W_a-W_b)$, and both levels are allowed to decay to lower states at rates indicated by γ_a and γ_b. The rate at which atoms are excited to the state α $(=a$ or $b)$ at the space-time point (s_0,t_0) with velocity v (s component) is described by

$$\lambda_\alpha(s_0,t_0,v)=W(v)\Lambda_\alpha(s_0,t_0), \tag{5.1}$$

where $W(v)$ is taken to be the normalized Maxwellian distribution function

$$W(v)=(\pi^{1/2}u)^{-1}e^{-(v/u)^2}, \tag{5.2}$$

with u related to an effective temperature T by $u^2=2\langle v^2\rangle_{av}=(2k_BT/M)$, and $\Lambda_\alpha(s_0,t_0)$ the number of atoms excited to state α per unit volume and unit time. $\Lambda_\alpha(s_0,t_0)$ will be assumed to be slowly varying over distances of several wavelengths and over times comparable to the lifetime of the atomic states. If collisions are neglected, an atom introduced at position s_0 at time t_0 with velocity v is at a later time $t>t_0$ situated at

$$s=s_0+v(t-t_0). \tag{5.3}$$

Consider what happens to an atom initially specified by the labels $[\alpha, s_0, t_0, v]$. Owing to the presence of the optical-field-induced transitions $a \leftrightarrow b$ the atomic wave function, which is ψ_α at time t_0, becomes a time-dependent linear combination $\psi(t)=a(t)\psi_a+b(t)\psi_b$. Correspondingly, the atom acquires an induced electric dipole moment given by the expectation value of the dipole operator,

$$\langle\psi(t)|e\sum_i \vec{r}_i\cdot\hat{z}|\psi(t)\rangle=\wp(ab^*+a^*b),$$

where $\wp$ (assumed real) is the matrix element of the dipole operator between a and b. Instead of describing the atomic state by its wave function, it has proved advantageous to use the equivalent description which is provided by its ("pure case") density matrix (in the subspace of ψ_a and ψ_b)

$$\rho(\alpha,s_0,t_0,v;t)=\begin{vmatrix}|a|^2 & ab^*\\ a^*b & |b|^2\end{vmatrix}. \tag{5.4}$$

The equation of motion of the density matrix is

$$i\hbar\frac{\partial}{\partial t}\rho(\alpha,s_0,t_0,v;t)=[\mathcal{H}(s,t),\rho]-\tfrac{1}{2}i\hbar(\Gamma\rho+\rho\Gamma), \tag{5.5}$$

where

$$\mathcal{H}=\hbar\begin{vmatrix}W_a & V(s,t)\\ V(s,t) & W_b\end{vmatrix},\quad \Gamma=\begin{vmatrix}\gamma_a & 0\\ 0 & \gamma_b\end{vmatrix}, \tag{5.6}$$

and $\hbar V(s,t)=-\wp E(s,t)$ is the matrix element between states a and b of the interaction energy operator.

A. Population Matrix—Formal Solution

All the macroscopic quantities that we shall deal with, such as polarization $P(s,v,t)$ and population inversion density $N(s,v,t)$ for atoms of a given velocity v, can be expressed in terms of a population matrix $\rho(s,v,t;\hat{t})$.[38,39] This is obtained by summing the microscopic density matrix contributions, up to a time $\hat{t}\ (\le t)$, of all atoms which, regardless of s_0, t_0, and α, are characterized by the three labels: velocity v and time t of arrival at a given place s, i.e.,

$$\rho(s,v,t;\hat{t})=\sum_{\alpha=a,b}\int_{-\infty}^{\hat{t}}dt_0 \times\int ds_0\,\lambda_\alpha(s_0,t_0,v)\,\rho(\alpha,s_0,t_0,v;\hat{t}) \times\delta(s-s_0-v(t-t_0)). \tag{5.7}$$

For $\hat{t}=t$, we have $[\rho(s,v,t;\hat{t})]_{\hat{t}=t}=\rho(s,v,t)$, where the right-hand-side member is the matrix defined in expression (24) of Ref. 17. It is easy to see that $\rho(s,v,t;\hat{t})$ and $\rho(s,v,t)$ satisfy

$$\rho(s,v,t;\hat{t})=\rho(s-v(t-\hat{t}),v,\hat{t}),$$

which shows the relation between the "substantial" and "local" time derivatives of the two matrices $[d/d\hat{t}\leftrightarrow\partial/\partial\hat{t}+v(\partial/\partial s)]$.

Using (5.7), the macroscopic polarization and population-inversion density at a given place s, and time t, are given by

$$P(s,t)=\int_{-\infty}^{+\infty}P(s,v,t)\,dv =\wp\int_{-\infty}^{+\infty}[\rho_{ab}(s,v,t;\hat{t})+\rho_{ba}]_{\hat{t}=t}\,dv, \tag{5.8}$$

$$N(s,t)=\int_{-\infty}^{+\infty}N(s,v,t)\,dv =\int_{-\infty}^{+\infty}[\rho_{aa}(s,v,t;\hat{t})-\rho_{bb}(s,v,t;\hat{t})]_{\hat{t}=t}\,dv. \tag{5.9}$$

The population matrix $\rho(s,v,t;\hat{t})$ obeys an equation of motion which can be obtained with the help of the equation of motion (5.5) for the atomic density matrix (Refs. 38 and 39). In component form,

$$\frac{d}{d\hat{t}}\rho_{aa}(s,v,t;\hat{t}) =-\gamma_a\rho_{aa}+iV(\hat{s},\hat{t})[\rho_{ab}-\rho_{ba}]+\lambda_a(\hat{s},\hat{t},v),$$

$$\frac{d}{d\hat{t}}\rho_{bb}(s,v,t;\hat{t}) =-\gamma_b\rho_{bb}-iV(\hat{s},\hat{t})[\rho_{ab}-\rho_{ba}]+\lambda_b(\hat{s},\hat{t},v),$$

$$\frac{d}{d\hat{t}}\rho_{ab}(s,v,t;\hat{t})=-i(\omega-i\gamma_{ab})\rho_{ab}+iV(\hat{s},\hat{t})[\rho_{aa}-\rho_{bb}],$$

$$\rho_{ba}=\rho_{ab}^*, \tag{5.10}$$

where $\gamma_{ab}=\frac{1}{2}(\gamma_a+\gamma_b)$. When the contribution to γ_a, γ_b, γ_{ab} of damping mechanisms of nonradiative type, such as atomic collisions in gases, are taken into account, the relationship between the three decay constants is destroyed. We shall therefore regard γ_a, γ_b, and γ_{ab} as independent. In Eq. (5.10), $V(\hat{s},\hat{t})$ is given by

$$V(\hat{s},\hat{t})=-\frac{\wp}{2\hbar}\sum_\mu[E_\mu(\hat{t})e^{-i[\nu\hat{t}+\phi_\mu(\hat{t})]}U_\mu(\hat{s})+\text{c.c.}], \tag{5.11}$$

with

$$\hat{s} = s - v(t - \hat{t}), \tag{5.12}$$

and the index μ takes on the values $(+, -)$ or (c, s) depending on the representation adopted for the ring-laser field. We note that in the region where the active medium is confined, the normal-mode eigenfunctions (3.24) and (3.27) take on the simple form $[n_0(s) \approx 1]$

$$U_\pm(s) = e^{\pm iKs}, \tag{5.13}$$

or

$$U_c(s) = \sqrt{2} \cos Ks, \quad U_c(s) = -i\sqrt{2} \sin Ks, \tag{5.14}$$

with K given by (3.10).

The pumping terms λ_α in the equation of motion for $\rho(s, v, t; \hat{t})$ make possible a quasisteady-state solution. Leaving aside transients, from Eqs. (5.10) it is easy to obtain the formal expressions given below which couple only $\rho_{ab}(s, v, t; \hat{t})$ and $N(s, v, t; \hat{t}) = [\rho_{aa}(s, v, t; \hat{t}) - \rho_{bb}]$:

$$N(s,v,t;\hat{t}) = \left[\frac{\lambda_a(\hat{s},\hat{t},v)}{\gamma_a} - \frac{\lambda_b(\hat{s},\hat{t},v)}{\gamma_b}\right] + \int_{-\infty}^{\hat{t}} d\hat{t}' \{V(\hat{s}',\hat{t}')(e^{-\gamma_a(\hat{t}-\hat{t}')} + e^{-\gamma_b(\hat{t}-\hat{t}')})[\rho_{ab}(s,v,t;\hat{t}') - \rho_{ab}^*]\}, \tag{5.15}$$

$$\rho_{ab}(s,v,t;\hat{t}) = i\int_{-\infty}^{\hat{t}} d\hat{t}'[V(\hat{s}',\hat{t}')e^{-i(\omega - i\gamma_{ab})(\hat{t}-\hat{t}')} N(s,v,t;\hat{t}')], \tag{5.16}$$

where $\lambda_\alpha(\hat{s}, \hat{t}, v)$ has been assumed to be slowly varying in the sense already indicated.

The coupled Equations (5.15) and (5.16) with the form (5.11) for $V(\hat{s}, \hat{t})$ are simple to solve in the case where $N(s, v, t; \hat{t}')$ can be treated as a slowly varying function of $\hat{t}'$ over periods comparable to the lifetime of the atomic states $(1/\gamma_a)$. In that case an iterative procedure can be applied in which one begins with expression (5.16) for $\rho_{ab}(s, v, t; \hat{t})$ with $N(s, v, t; \hat{t}')$ taken out of the integral and evaluated at $\hat{t}' = \hat{t}$. After integrating, the result obtained is inserted into (5.15), which gives $N(s, v, t; \hat{t})$ in terms of the parameters of the system. The expression for N is then put back into (5.16) to obtain an improved expression for $\rho_{ab}(s, v, t; \hat{t})$, etc. In the absence of atomic motion $(v = 0)$ this method is essentially exact since the two $(+, -)$ or (c, s) modes have very close frequencies $\nu + \dot{\phi}_\mu \approx \nu_0$ and the iterative procedure outlined is similar to the "rate-equation approach" for a single mode as described in Sec. 16 of Ref. 17.

For moving atoms $(v \neq 0)$, the temporal variations introduced through $U_\pm(\hat{s}') = e^{\pm iK[s - v(t - t')]}$, or $U_{c,s}(\hat{s}')$, make it difficult to judge the accuracy of treating $N(s, v, t; \hat{t}')$ as a slowly varying function in the $\hat{t}'$ integrals (5.15) and (5.16), and a closer examination is necessary to justify the rate-equation approximation. A more rigorous treatment of the problem with moving atoms is possible by using a Fourier-series method applied by Stenholm and Lamb,[39] by Holt,[19] and by Feldman and Feld[40] for a high-intensity single-mode linear laser. We shall see this in Sec. VI, where the abovementioned method, with slight modifications, is used to solve the coupled equations (5.15) and (5.16) with the form (5.11) for the perturbation $V(\hat{s}, \hat{t})$.

VI. POPULATION INVERSION AND POLARIZATION (RWR)

A. Strong-Signal Ring-Laser Theory

An examination of the coupled equations (5.15) and (5.16) shows that the population difference has a dc term, and both N and ρ_{ab} are proportional to

$$N_0(\hat{s}, \hat{t}) W(v) = \left[\frac{\lambda_a(\hat{s},\hat{t},v)}{\gamma_a} - \frac{\lambda_b(\hat{s},\hat{t},v)}{\gamma_b}\right]. \tag{6.1}$$

Furthermore, considering that $V(\hat{s}, \hat{t})$ given by (5.11) contains the factors

$$U_\pm(\hat{s}) = e^{\pm iK\hat{s}} \equiv e^{\pm iK[s - v(t - \hat{t})]}$$

suggests that solutions may be found in the form of Fourier series of the type

$$N(s, v, t; t) = N_0 W \sum_{\substack{n=-\infty \\ n \text{ even}}}^{\infty} d_n e^{-in\phi(\hat{t})/2} e^{inK\hat{s}} \quad (d_{-n} = d_n^*), \tag{6.2}$$

$$\rho_{ab}(s, v, t; t) = N_0 W e^{-i\nu\hat{t}} \times \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} p_n e^{-i[(n+1)\phi_+ - (n-1)\phi_-]/2} e^{inK\hat{s}}, \tag{6.3}$$

with the relative phase angle

$$\phi(\hat{t}) = (\nu\hat{t} + \phi_+) - (\nu\hat{t} + \phi_-) \equiv \phi_+ - \phi_-. \tag{6.4}$$

The form adopted for the phase factors in (6.2) and (6.3) is taken over from the case where the field is weak enough that successive iterations beginning with $N_0 W$ can be applied to solve the coupled equations. The possible dependence of both d_n and p_n on $\{s, v, t; \hat{t}\}$ is not shown explicitly. We assume that they are slowly varying functions, e.g.,

$$\frac{\partial d_n}{\partial \hat{t}} \ll \gamma_{ab} d_n, \quad \frac{\partial p_n}{\partial \hat{t}} \ll \gamma_{ab} p_n .$$

The Fourier series are now inserted into the Eqs. (5.15) and (5.16), and the expressions are evaluated with the following considerations (a), (b) taken into account:

(a) Only resonant terms $\sim e^{\pm i(\nu-\omega)\hat{t}'}$ are retained, and terms with the time dependence $\sim e^{\pm i(\nu+\omega)\hat{t}'}$, $\sim e^{\pm 2i\nu\hat{t}'}$ are neglected (rotating-wave approximation).

(b) Because of the competition between cavity losses and the gain provided by the active medium, a typical time for appreciable net growth of the field is much greater than $(\nu/Q)^{-1}$ ($\approx 10^{-6}$ sec). Thus for times $\hat{t}-\hat{t}' \leq 1/\gamma_{ab}$ ($\approx 10^{-8}$ sec), the replacements

$$E_\pm(\hat{t}') \approx E_\pm(\hat{t}),$$

$$\phi_\pm(\hat{t}') \approx \phi_\pm(\hat{t}) - \dot{\phi}_\pm(\hat{t})(\hat{t}-\hat{t}'),$$

$$\phi_\pm(\hat{t}') = \phi_+ - \phi_- \approx \phi(\hat{t}) - \dot{\phi}(\hat{t})(\hat{t}-\hat{t}')$$

are reasonable in the $\hat{t}'$ integrals (5.15) and (5.16). Correspondingly, it will prove convenient to introduce the notation for the "local" frequencies

$$\nu_\pm = \nu + \dot{\phi}_\pm \tag{6.5}$$

and the "beat" frequency

$$\dot{\phi} = (\nu_+ - \nu_-) \equiv (\dot{\phi}_+ - \dot{\phi}_-). \tag{6.6}$$

The coefficients of equal powers of e^{inKs} are then equated on the two sides of the equations, leading to a system of coupled algebraic equations for the Fourier amplitudes d_n and p_n. For $n=0$, ± 2, $\pm 4, \ldots,$

$$d_n = \delta_{n,0} + \left[\sum_{\alpha(a,b)} \frac{1}{\frac{1}{2}n[(\nu_+ - Kv) - (\nu_- + Kv)] + i\gamma_\alpha} \right] \times \{(\wp E_+/2\hbar)[p_{n+1} - p^*_{-n+1}] + (\wp E_-/2\hbar)[p_{n-1} - p^*_{-n-1}]\}, \tag{6.7}$$

and for $n = \pm 1, \pm 3, \pm 5, \ldots,$

$$p_n = \frac{1}{(\nu_\pm \mp Kv) - \frac{1}{2}(n\mp 1)[2Kv - (\nu_+ - \nu_-)] - \omega + i\gamma_{ab}} \times \left[\left(\frac{\wp E_+}{2\hbar}\right) d_{n-1} + \left(\frac{\wp E_-}{2\hbar}\right) d_{n+1}\right], \tag{6.8}$$

where the upper signs in (6.8) are convenient for $n \geq 1$ and the lower signs for $n \leq -1$. The quantities $(\nu_+ - Kv)$ and $(\nu_- + Kv)$ are the Doppler-shifted frequencies of the $\pm$ waves as seen by an atom of velocity v.

It will be convenient to write the above equations in a more compact form by using

$$\Gamma_s = [\sum_\alpha (1/\gamma_\alpha)]^{-1} = \frac{\gamma_a\gamma_b}{\gamma_a + \gamma_b} \tag{6.9}$$

with dimensions of a rate, and the dimensionless quantities

n even:

$$\mathfrak{D}_2(n) = \sum_\alpha \frac{i\Gamma_s}{-nKv + \frac{1}{2}n(\nu_+ - \nu_-) + i\gamma_\alpha},$$

$$\mathfrak{D}_2(0) = 1, \quad \mathfrak{D}_2(-n) = \mathfrak{D}_2(n)^*; \tag{6.10}$$

n odd:

$$\mathfrak{D}_1(n) = \frac{i\gamma_{ab}}{\nu_\pm - nKv + \frac{1}{2}(n\mp 1)(\nu_+ - \nu_-) - \omega + i\gamma_{ab}},$$

$n \geq 1 \rightarrow$ upper signs, $n \leq -1 \rightarrow$ lower signs. (6.11)

Equations (6.7) and (6.8) will now have the form

n even:

$$d_n = \delta_{n,0} - i(\gamma_{ab}/\Gamma_s)^{1/2}\mathfrak{D}_2(n) \times \{I_+^{1/2}[p_{n+1} - p^*_{-n+1}] + I_-^{1/2}[p_{n-1} - p^*_{-n-1}]\}; \tag{6.12}$$

n odd:

$$p_n = -i(\Gamma_s/\gamma_{ab})^{1/2}\mathfrak{D}_1(n)(I_+^{1/2} d_{n-1} + I_-^{1/2} d_{n+1}) \tag{6.13}$$

Here

$$I_\mu = \frac{\wp^2 E_\mu^2}{4\hbar^2\Gamma_s\gamma_{ab}} = \frac{\wp^2 E_\mu^2(\gamma_a + \gamma_b)}{4\hbar^2\gamma_{ab}\gamma_a\gamma_b} \tag{6.14}$$

is introduced as a dimensionless measure of the intensity of mode μ.

It is not difficult to see that the above equations are simple generalizations of Eqs. (60) of Ref. 39 for a single-mode linear laser. Also, as in Ref. 39, Eqs. (6.12) with arbitrary values for the field amplitudes can be rewritten for computer solution by using properly truncated continued fractions. In fact, inserting (6.13) into (6.12), and taking into account that $d_{-n} = d_n^*$, we obtain

$$a_2(n)d_{n+2} + c(n)d_n + a_{-2}(n)d_{n-2} = \delta_{n,0} \tag{6.15}$$

where

$$a_2(n) = (I_+I_-)^{1/2}\mathfrak{D}_2(n)[\mathfrak{D}_1(n+1) + \mathfrak{D}_1^*(-n-1)],$$

$$a_{-2}(n) = (I_+I_-)^{1/2}\mathfrak{D}_2(n)[\mathfrak{D}_1(n-1) + \mathfrak{D}_1^*(-n+1)] = a_2(-n)^*, \tag{6.16}$$

$$c(n) = 1 + I_+\mathfrak{D}_2(n)[\mathfrak{D}_1(n+1) + \mathfrak{D}_1^*(-n+1)] + I_-\mathfrak{D}_2(n)[\mathfrak{D}_1(n-1) + \mathfrak{D}_1^*(-n-1)].$$

Dividing by d_n, Eq. (6.15) can be written, for $n=0$, as

$$d_0 = \frac{1}{c(0) + 2\mathrm{Re}[a_2(0)(d_2/d_0)]}, \tag{6.17}$$

while for $n \geq 2$,

$$\left(\frac{d_n}{d_{n-2}}\right) = \frac{-a_{-2}(n)}{c(n) + a_2(n)(d_{n+2}/d_n)}\,, \tag{6.18}$$

with the complex conjugate for $n \leq -2$.

Setting $n = 2$ and iterating Eq. (6.18), an expression for (d_2/d_0) in the form of a continued fraction is obtained:

$$\left(\frac{d_2}{d_0}\right) = \cfrac{-a_{-2}(2)}{c(2) + a_2(2)\cfrac{-a_{-2}(4)}{c(4) + a_2(4)\cfrac{-a_{-2}(6)}{c(6) + a_2(6)\cdots}}}\,. \tag{6.19}$$

After a number of iterations, the factor

$$a_2(n) = f_n((I_+ I_-)^{1/2},\ Kv,\ \nu - \omega) \tag{6.20}$$

becomes small compared with $c(n) - 1$, and the continued fraction can be truncated. An examination of the coefficients shows that for small intensities or large velocities only small values of n are needed to make (6.20) small. For high intensities or small velocities many terms have to be included in the continued fraction. Valuable theorems dealing with "convergents" of truncated continued fractions show that more accurate convergents (larger n) can be obtained from less accurate ones by means of simple linear relations.

Numerical analysis and a general computer program which contains this theory as a special case have been given by Sargent.[41] For our present purposes it is sufficient to see that the value obtained for $(d_{\pm 2}/d_0)$, once inserted into (6.17), gives an explicit expression for d_0. Expressions for $d_{\pm 2}$ are next obtained by multiplication $d_0 \times (d_{\pm 2}/d_0)$. The remaining Fourier amplitudes d_n and p_n are obtained by using (6.15) and (6.13).

The method outlined above gives the basis for a strong-signal ring-laser theory. In the particular case where one of the signals E_- (or E_+) is sufficiently weak, solutions in analytic form can be derived from (6.15) and (6.13). This is easily accomplished by taking a perturbation expansion in terms of the weak field E_-,

$$d_n = d_n^{(0)} + d_n^{(1)} + \cdots,\quad p_n^{(0)} + p_n^{(1)} + \cdots,$$

which gives

$$d_n^{(0)} = \frac{\delta_{n,0}}{c^{(0)}(n)}\,,\quad p_n^{(0)} = -i(\Gamma_s/\gamma_{ab})^{1/2}\mathfrak{D}_1(n) I_+^{1/2} d_{n-1}^{(0)}\,, \tag{6.21}$$

and for $l \geq 1$,

$$d_n^{(l)} = -\{[c^{(2)}(n) d_n^{(l-2)} + a_2(n) d_{n+2}^{(l-1)} + a_{-2}(n) d_{n-2}^{(l-1)}]/c^{(0)}(n)\},$$

$$p_n^{(l)} = -i(\Gamma_s/\gamma_{ab})^{1/2}\,\mathfrak{D}_1(n)\{I_+^{1/2} d_{n-1}^{(l)} + I_-^{1/2} d_{n+1}^{(l-1)}\}, \tag{6.22}$$

where $c^{(2)}(n) d_n^{(l-2)}$ appears only for $l \geq 2$. Recently, much attention has been paid to questions such as the absorption line shape of a weak in the presence of a strong one traveling in the opposite direction.[18,19] Equations (6.21) and (6.22) can also be used to discuss this problem, and our results up to first order in E_- agree with those independently derived by Baklanov and Chebotaev. [See Ref. 18 especially Eqs. (18)–(22).]

B. Applications to a Third-Order Ring-Laser Theory

In the remainder of the present article, the general formalism of Sec. VI A will be applied only to the case of a small-signal ring laser (I_+ and $I_- \ll 1$). Most of the results have appeared in the extensive literature of the past six years, especially in papers by Aronowitz,[1-3,20] Klimontovich *et al.*,[6-8] Landa and Lariontsev.[9,10] Thus, no attempt will be made to give a complete treatment of all experimentally important aspects of ring-laser operation at small signals. However, it is desirable to have a systematic calculation, based on first principles and the model described in Ref. 17.

For low excitations such that $I_\pm \ll 1$, only lower-order terms are needed in Eqs. (6.17) and (6.18). In restricted situations fifth-order terms seem to be necessary [see Ref. 7, especially (1.7)–(1.10)], but most features can be described by a third-order expansion. Then,

$$d_0 \approx c(0)^{-1} \approx \{1 - [2I_+\mathfrak{L}(\nu_+ - Kv - \omega) + 2I_-\mathfrak{L}(\nu_- + Kv - \omega)]\}, \tag{6.23}$$

where

$$\mathfrak{L}(\nu_\pm \mp Kv - \omega) = \tfrac{1}{2}[\mathfrak{D}_1(\pm 1) + \mathfrak{D}_1^*(\pm 1)] = \frac{\gamma_{ab}^2}{(\nu_\pm \mp Kv - \omega)^2 + \gamma_{ab}^2} \tag{6.24}$$

and

$$d_2 \approx -a_{-2}(2) = -(I_+ I_-)^{1/2} \times \left[\sum_\alpha \frac{i\Gamma_s}{(\nu_+ - Kv) - (\nu_- + Kv) + i\gamma_\alpha}\right] \times \left[\frac{i\gamma_{ab}}{\nu_+ - Kv - \omega + i\gamma_{ab}} + \frac{-i\gamma_{ab}}{\nu_- + Kv - \omega - i\gamma_{ab}}\right], \tag{6.25}$$

$d_{-2}=d_2^*$, and d_n for $|n|>2$ are of higher order. Using the above expressions for d_0 and $d_{\pm 2}$ and (6.13) for p_n we obtain

$$p_1=\frac{(\wp E_+/2\hbar)}{\nu_+-Kv-\omega+i\gamma_{ab}}\Bigg\{1-\Bigg[2I_+\mathcal{L}(\nu_+-Kv-\omega)+2I_-\mathcal{L}(\nu_-+Kv-\omega)+I_-\left(\sum_\alpha\frac{i\Gamma_s}{(\nu_+-Kv)-(\nu_-+Kv)+i\gamma_\alpha}\right)\left(\frac{i\gamma_{ab}}{\nu_+-Kv-\omega+i\gamma_{ab}}+\frac{-i\gamma_{ab}}{\nu_-+Kv-\omega-i\gamma_{ab}}\right)\Bigg]\Bigg\}. \tag{6.26}$$

p_{-1} can be obtained from (6.26) by interchanging subscripts $\pm$ and changing the sign of the Kv terms. The amplitudes $p_{\pm 3}$ have terms which are also of third order in the field amplitudes; however, their contribution to the total polarization will be zero after projection onto the $\{U_+, U_-\}$ modes.

The expressions for $N(s,v,t)$ and $P(s,v,t)$ to be used in (5.8) and (5.9) are obtained by substituting d_0, $d_{\pm 2}$, $d_{\pm 1}$ in the Fourier series (6.2), (6.3) evaluated at $\hat{t}=t$:

$$N(s,v,t)=N_0(s,t)\,W(v)\Bigg\{1-2\Bigg[I_+\mathcal{L}(\nu_+-Kv-\omega)+I_-\mathcal{L}(\nu_-+Kv-\omega)+(I_+I_-)^{1/2}\mathrm{Re}\,e^{2iKs}e^{-i\phi}\times\left(\sum_\alpha\frac{i\Gamma_s}{(\nu_+-Kv)-(\nu_-+Kv)+i\gamma_\alpha}\right)\left(\frac{i\gamma_{ab}}{\nu_+-Kv-\omega+i\gamma_{ab}}+\frac{-i\gamma_{ab}}{\nu_-+Kv-\omega-i\gamma_{ab}}\right)\Bigg]\Bigg\}, \tag{6.27}$$

$$\begin{aligned}P(s,v,t)=\wp[\rho_{ab}(s,v,t;\hat{t})+\text{c.c.}]_{\hat{t}=t}&=\wp N_0(s,t)W(v)\frac{(\wp E_+/2\hbar)e^{-i(\nu t+\phi_+)}e^{+iKs}}{\nu_+-Kv-\omega+i\gamma_{ab}}\\&\times\Bigg\{1-\Bigg[2I_+\mathcal{L}(\nu_+-Kv-\omega)+2I_-\mathcal{L}(\nu_-+Kv-\omega)+I_-\left(\sum_\alpha\frac{i\Gamma_s}{(\nu_+-Kv)-(\nu_-+Kv)+i\gamma_\alpha}\right)\\&\times\left(\frac{i\gamma_{ab}}{\nu_+-Kv-\omega+i\gamma_{ab}}+\frac{-i\gamma_{ab}}{\nu_--Kv-\omega-i\gamma_{ab}}\right)\Bigg]\Bigg\}\\&+\text{same changing }\{(+\longleftrightarrow-),\ (e^{+iKs}\to e^{-iKs}),\ \text{sign of }Kv\}+\text{c.c.}\end{aligned} \tag{6.28}$$

Expression (6.27) is a generalization of one obtained by Aronowitz,[1] who neglected the space-dependent terms. These terms arise from interference between both ($\pm$) waves interacting with atoms of velocity $K|v|\lesssim\gamma_{ab}$. The expression for $P(s,v,t)$ shows that the interference term will be small in the Doppler limit, but in general should not be neglected. The steady-state inverted population in the absence of optical oscillations, $N_0(s,t)W(v)$, is modified by the presence of the field-induced transitions $a\longleftrightarrow b$. The terms enclosed between the square brackets are proportional to the spectral energy density giving rise to stimulated emission or absorption. Expression (6.27) shows that for a given detuning $(\nu_0-\omega)$ of the cavity frequency $\nu_0(\approx\nu)$ from the atomic transition frequency ω, there are two Lorentzian holes "burned" into the Gaussian velocity distribution $W(v)$. For $|(\nu_0-\omega)|>\gamma_{ab}$, the holes are centered at opposite sides of the $W(v)$ curve at

$$Kv_+=+(\nu_+-\omega)\quad\text{and}\quad Kv_-=-(\nu_--\omega)$$

and do not overlap, since we assume that the beat note $\dot{\phi}=(\nu_+-\nu_-)$ does not exceed γ_{ab}. In this case the opposite running waves primarily interact with different sets of atoms. For $|(\nu_0-\omega)|<\gamma_{ab}$ the two holes overlap and mode competition is to be expected because the $(+,-)$ waves interact with the same set of atoms. In this case, the interference term gives an additional kind of s-dependent hole burning, because the Doppler-shifted frequencies of the two waves (ν_+-Kv) and (ν_-+Kv), as seen by an atom of velocity v ($|Kv|<\gamma_{ab}$), do not differ by much compared to γ_α. The space striations caused by the term e^{2iKs} in the inverted population can lead to observable effects through Bragg-like reflections. For solid-state lasers, $Kv=0$, the two waves are driven by the same atoms and strong mode competition arises. The criterion

$$I_\mu=\frac{\wp^2E_\mu^2(\gamma_a+\gamma_b)}{4\hbar^2\gamma_{ab}\gamma_a\gamma_b}\ll 1 \tag{6.29}$$

for validity of the perturbation expansion in powers of the I_μ's means a small population depletion.

From now on, in dealing with $P(s,v,t)$, the difference between ν_+ and ν_- will be neglected in the small third-order terms, and $\nu_\pm$ will be replaced by ν_0. To obtain the total polarization $P(s,t)$ from expression (6.28) we have to integrate over velocities as required by (5.8). From the first-order terms we have

$$Z\left(\frac{\nu-\omega}{Ku}, \frac{\gamma_{ab}}{Ku}\right) = -Ku \int_{-\infty}^{+\infty} \frac{W(v)}{(\nu-\omega+i\gamma_{ab}) \mp Kv}\, dv, \tag{6.30}$$

which is a well-known representation of the complex "plasma dispersion function" used in the theory of Doppler broadening.[42] Since $W(v)$ is an even function of v, the result (6.30) is independent of the sign of Kv. The third-order contribution $P^{(3)}(s,t)$ will appear in terms of the functions

$$J_{\mu,\lambda}\left(\frac{\nu-\omega}{Ku}, \frac{\gamma_{ab}}{Ku}\right) = -2Ku \times \int_{-\infty}^{+\infty} \frac{W(v)\mathcal{L}(\nu-\omega-(\lambda)Kv)}{(\nu-\omega+i\gamma_{ab})-(\mu)Kv}\, dv, \tag{6.31}$$

with indices μ, $\lambda=\pm$. The symbol (μ) is defined to be ±1 when $\mu=\pm$ and similarly for (λ). Third-order contributions corresponding to interferences [see Eqs. (6.27) and (6.28)] will be given in terms of

$$Y\left(\frac{\nu-\omega}{Ku}, \frac{\gamma_{ab}}{Ku}, \frac{\gamma_\alpha}{Ku}\right) = -Ku \int_{-\infty}^{+\infty} \frac{W(v)}{(\nu-\omega+i\gamma_{ab}) \mp Kv}\left[\sum_\alpha \frac{i\Gamma_s}{i\gamma_\alpha \mp 2Kv}\right]\left[\frac{i\gamma_{ab}}{(\nu-\omega+i\gamma_{ab})\mp Kv} + \frac{-i\gamma_{ab}}{(\nu-\omega-i\gamma_{ab})\pm Kv}\right] dv. \tag{6.32}$$

It will be convenient to deal with Z, $J_{+,+}=J_{-,-}$, $J_{+,-}=J_{-,+}$, and Y as functions of a single complex dimensionless quantity

$$\zeta = \xi + i\eta = \frac{(\nu-\omega)}{Ku} + i\frac{\gamma_{ab}}{Ku} \tag{6.33}$$

which permits also a more compact form for the integral representations (6.30)–(6.32). Expressions of $J_{\mu,\lambda}$ and Y in terms of the plasma dispersion function $Z=Z_r(\xi,\eta)+iZ_i(\xi,\eta)$ are given in Appendix A.

After projection onto the $\{U_+, U_-\}$ modes, the total polarization, $P(s,t)=\int_{-\infty}^{+\infty}P(s,v,t)dv$, can be written in the form

$$P(s,t) = -(\wp^2\overline{N}_0/2\hbar Ku)\big(E_+ e^{-i[\nu t+\phi_+(t)]}U_+(s) \times [Z(\zeta_+) - \{\mathcal{S}(\zeta)I_+ + \mathcal{C}(\zeta)I_-\}] + \text{same with } \pm \rightarrow \mp\big) + \text{c.c.}, \tag{6.34}$$

where

$$\overline{N}_0(t) = (1/L)\int_0^L N_0(s,t)ds, \qquad \zeta_\pm = \xi_\pm + i\eta \equiv \frac{(\nu_\pm-\omega)}{Ku} + \frac{i\gamma_{ab}}{Ku}. \tag{6.35}$$

In the third-order terms of (6.34) the approximation $\nu_\pm = \nu + \dot\phi_\pm \approx \nu$ has been taken, and the "saturation" ($\mathcal{S}$) and "coupling" ($\mathcal{C}$) functions stand for

$$\mathcal{S} = (\mathcal{S}_r + i\mathcal{S}_i) \equiv J_{\pm,\pm}, \qquad \mathcal{C} = (\mathcal{C}_r + i\mathcal{C}_i) \equiv (J_{\pm,\mp} + Y). \tag{6.36}$$

Functions $(\mathcal{S}_r + i\mathcal{S}_i)$ and $(\mathcal{C}_r + i\mathcal{C}_i)$ will have a role similar to $(\rho + i\beta)$ and $(\tau + i\theta)$ of Eqs. (130) and (136) of Ref. 17.

A simple comparison of (6.34) and (4.3) shows the explicit form of the in-phase and quadrature coefficients $C_\pm(t)$ and $S_\pm(t)$. Correspondingly, the amplitude- and phase-determining equations in the RWR are given by (see Sec. IV)

$$\dot E_\pm + \tfrac{1}{2}(\nu/Q_\pm)E_\pm \pm \Delta_\mp E_\mp \sin(\phi+\delta_\mp) + \tfrac{1}{2}\Theta E_\mp \cos(\phi+\delta_i) = \left(\frac{\nu\wp^2\overline{N}_0}{2\epsilon\hbar Ku}\right) E_\pm\{Z_i(\xi_\pm) - [\mathcal{S}_i(\xi)I_\pm + \mathcal{C}_i(\xi)I_\mp]\}, \tag{6.37}$$

$$[(\nu+\dot\phi_\pm) - (\nu_0 \mp \kappa)]E_\pm + \Delta_\mp E_\mp \cos(\phi+\delta_\mp) \mp \tfrac{1}{2}\Theta E_\mp \sin(\phi+\delta_i) = \left(\frac{\nu\wp^2\overline{N}_0}{2\epsilon\hbar Ku}\right) E_\pm\{Z_r(\xi_\pm) - [\mathcal{S}_r(\xi)I_\pm + \mathcal{C}_r(\xi)I_\mp]\}. \tag{6.38}$$

The dimensionless intensities $I_\pm$ have been defined in (6.14); nonreciprocal losses $(\nu/Q_\pm)$ and localized losses Θ are defined in (4.7) and (4.5). The parameters κ, $\Delta \ll \nu_0$ which take into account the rotation of the platform and the effect of backscattering of radiation are given by (3.15) and (3.16), (3.17). A translation of the origin of the s coordinate, allow us to eliminate one of the phases $\delta_\pm$, δ_i [see (3.16), (3.26), and (4.5)].

Approximate analytical solutions of the nonlinear, coupled equations (6.37) and (6.38) can be obtained in a few simple cases.[2,6,10] However, in most cases a computer treatment is necessary,[3,9] and for this purpose it is convenient to rewrite the equations in a more compact form. This is done by the following steps (a)–(c).

(a) The equations are multiplied by $(\wp^2E_\pm/4\hbar^2\Gamma_s\gamma_{ab})$ so as to express them in terms of the dimensionless intensities $I_\pm = [\wp^2E_\pm^2(\gamma_a+\gamma_b)/$

$4\hbar^2\gamma_a\gamma_b\gamma_{ab}]$.

(b) The unspecified frequency ν has been introduced in (4.2) with the sole restriction that it must be close to the cavity eigenfrequency ν_0, and without loss of generality can be taken as

$$\nu \equiv \nu_0 . \tag{6.39}$$

(c) Usually, the degree of inversion $\overline{N}_0$ of the atomic medium is described in terms of the relative excitation

$$\mathfrak{N} = (\overline{N}_0/\overline{N}_T), \tag{6.40}$$

where $\overline{N}_T$ is the excitation required for threshold when the cavity frequency ν_0 is tuned to the peak ω of the atomic transition curve. Neglecting the complications arising from the possible presence of bumps Δ or nonreciprocal and localized losses Λ, $\Theta \ll (\nu/Q)$, the value of $\overline{N}_T$ it is determined by setting the linear gain equal to cavity losses:

$$(\nu\wp^2\overline{N}_T/2\epsilon\hbar Ku)Z_i(0) = \tfrac{1}{2}(\nu/Q). \tag{6.41}$$

Using (6.40) and (6.41), the parameter $(\nu\wp^2\overline{N}_0/2\epsilon\hbar Ku)$ which appears in Eqs. (6.37) and (6.38) can be expressed in terms of the cavity losses and relative excitation as

$$(\nu\wp^2\overline{N}_0/2\epsilon\hbar Ku) = \frac{1}{2}\frac{(\nu/Q)\mathfrak{N}}{Z_i(0)} \equiv \tfrac{1}{2}G. \tag{6.42}$$

Writing $(\nu/Q_\pm) = (\nu/Q) \pm \Lambda$, the net linear gain of the $(\pm)$ modes becomes

$$GA_\pm \equiv G[Z_i(\xi_\pm,\eta) - (\nu/Q_\pm)G^{-1}] = G[Z_i(\xi_\pm,\eta) - Z_i(0)\mathfrak{N}_\pm^{-1}], \tag{6.43}$$

where

$$\mathfrak{N}_\pm = \mathfrak{N}[(\nu/Q)/(\nu/Q_\pm)] \approx \mathfrak{N}[1 \mp \Lambda/(\nu/Q)]. \tag{6.44}$$

Finally, Eqs. (6.37) and (6.38) for the running-wave representation take the form

$$\dot{I}_\pm = \mp 2\,\Delta_\mp(I_+I_-)^{1/2}\sin(\phi+\delta_\mp) - \Theta(I_+I_-)^{1/2}\cos\phi + G[A_\pm - \mathcal{S}_i(\xi)I_\pm - \mathcal{C}_i(\xi)I_\mp]I_\pm, \tag{6.45}$$

$$\dot{\phi}_\pm I_\pm = \mp\kappa I_\pm - \Delta_\mp(I_+I_-)^{1/2}\cos(\phi+\delta_\mp) \pm \tfrac{1}{2}\Theta(I_+I_-)^{1/2}\sin\phi + \tfrac{1}{2}G[Z_r(\xi_\pm) - \mathcal{S}_r(\xi)I_\pm - \mathcal{C}_r(\xi)I_\mp]I_\mp, \tag{6.46}$$

where δ_i has been eliminated by a translation of the origin of the s coordinate, and $(\delta_\pm - \delta_i)$ has been written as $\delta_\pm$. Frequency parameters $\Delta_\pm$ and corresponding amplitude reflection coefficients $R_\pm$ are related by $\Delta_\pm = (c/L)R_\pm$. Functions Z, $\mathcal{S}$, $\mathcal{C}$ are given by (6.36) and corresponding expressions of Appendix A. In the case of two-mode operation (I_+ and $I_- \neq 0$), by subtracting one of Eqs. (6.46) from the other, we obtain a differential equation for the phase difference $\phi = (\nu t + \phi_+) - (\nu t + \phi_-) \equiv (\phi_+ - \phi_-)$, which gives the behavior of the beat frequency

$$\dot{\phi} \equiv (\dot{\phi}_+ - \dot{\phi}_-) = -2\kappa + \tfrac{1}{2}G[Z_r(\xi_+) - Z_r(\xi_-)] - \tfrac{1}{2}G(C_r - S_r)(I_- - I_+) - (I_+I_-)^{-1/2}[I_-\,\Delta_-\cos(\phi+\delta_-) - I_+\,\Delta_+\cos(\phi+\delta_+) - \tfrac{1}{2}\Theta(I_+ + I_-)\sin\phi]. \tag{6.47}$$

Equations (6.45)–(6.47) agree with those previously used by a number of authors. To deal with them, a computer treatment is generally necessary. Approximate analytical solutions can be obtained in some cases of practical interest, especially when $(\dot{\phi}/GA)$, (Λ/GA), and $(2\Delta/GA)$ are much less in magnitude than $[(S_i - C_i)/(S_i + C_i)]$, which is usually called the "weak-coupling" case in the Russian literature.[7,10] In this case the equations are more easily handled in terms of the quantities $Y = (I_- + I_+)$ and $X = (I_- - I_+)$.

VII. COLLISIONS, VELOCITY FLOW, ADMIXTURE OF ISOTOPES

The effects of collisions have been studied both theoretically and experimentally for standing-wave gas lasers,[43,44] and are easily brought into Eqs. (6.45) and (6.46). With the definition $I_\pm = [\wp^2E_\pm^2(\gamma_a+\gamma_b)/4\hbar^2\gamma_a\gamma_b\gamma_{ab}]$ for the dimensionless intensities, it can be shown that allowance for collisions at moderate pressures reduces to a redetermination of the parameters, i.e., a shift of the transition frequency, and a linear pressure (p) dependence of the damping constants

$$\gamma_\alpha = \gamma_\alpha^{(0)} + K_\alpha p, \qquad \gamma_{ab} = \tfrac{1}{2}(\gamma_a^{(0)} + \gamma_b^{(0)}) + K_{ab}p. \tag{7.1}$$

In this approximation, Eqs. (6.45) and (6.46) are left formally unchanged.

In the case where a dc discharge is used to obtain population inversion, there is a net flow of the gas components toward one of the electrodes.[20,45] An obvious way to deal with this effect is to replace the Maxwellian velocity distribution by a suitably modified one containing adjustable parameters. A sufficiently good approximation would seem to be a Maxwellian distribution having some nonzero average $\overline{v}$,

$$W(v) = (\pi^{1/2}u)^{-1}e^{-[(v-\overline{v})/u]^2}. \tag{7.2}$$

It is apparent that this will imply an extra Doppler shift. With a shift of $\overline{v}$ in $W(v)$, the two Lorentzian holes in $N(s,v,t)$ are equally shifted by $K\bar{v}$. Formulas (6.27) and (6.28) for $N(s,v,t)$ and $P(s,v,t)$, in conjunction with the integral representations (6.30)–(6.32) for Z, J, and Y, indicate that this is equivalent to the substitution of $\nu_\pm = \nu + \dot{\phi}_\pm$ by

$$\nu_\pm \to \nu_\pm \mp K\bar{v} \quad \text{or} \quad \zeta_\pm \to \zeta_\pm \mp (\bar{v}/u). \tag{7.3}$$

The effects in the net gain $GA_\pm$ and frequency splitting can now be obtained by using the following first-order Taylor-series expansions [see (A6)]:

$$Z_i(\xi + \Delta\xi, \eta) = Z_i(\xi, \eta) - 2[\xi Z_i(\xi,\eta) + \eta Z_r(\xi,\eta)]\,\Delta\xi + \cdots,$$

$$Z_r(\xi + \Delta\xi, \eta) = Z_r(\xi, \eta) - 2[1 + \xi Z_r(\xi,\eta) - \eta Z_i(\xi,\eta)]\,\Delta\xi + \cdots. \tag{7.4}$$

Then, letting $\Delta\xi_\pm = [(\dot{\phi}_\pm \mp K\bar{v})/Ku] \approx \mp[(\kappa + K\bar{v})/Ku]$ in expressions (6.43) and (6.47), one obtains

$$GA_\pm = G\{Z_i(\xi,\eta)[1 \pm 2\xi(\kappa + K\bar{v})/Ku] - Z_i(0,\eta)\mathfrak{N}_\pm^{-1}\}, \tag{7.5}$$

$$\dot{\phi} = -2\kappa + 2G[(\kappa + K\bar{v})/Ku] + (\text{rest of the terms}). \tag{7.6}$$

Expression (7.5) indicates that the beam traveling in the direction of the velocity flow will have a greater gain (for $\xi > 0$). Expression (7.6) in conjunction with (6.47) indicates that because of the dispersive properties of the active medium [shape of $Z_r(\xi,\eta)$], for a velocity flow in the direction of the rotation, the frequency-splitting contributions of the rotation and the velocity flow tend to cancel ($G \approx 0.6$ MHz for typical values of the parameters).[20]

Equations (6.45) and (6.46) are valid for gas lasers containing a single isotope (for example ^{20}Ne in a Ne-He laser). The presence of even a trace of another isotope (^{22}Ne) can greatly change mode-competition phenomena. The influence of admixture of a second isotope has been theoretically described by Aronowitz[1] and Klimontovich *et al.*[7] Thus, we simply write down the general equations, which can be derived by an appropriate weighted average of the excitation inversion densities $N_j(s,v,t)$, and corresponding polarization $P_j(s,v,t)$, for each isotopic component j:

$$f_j = (\text{relative concentration of isotope } j), \quad f_1 + f_2 = 1$$

$$\omega_j = (\text{transition frequency of isotope } j), \tag{7.7}$$

$$W_j(v) = (\pi^{1/2}u_j)^{-1}\,e^{-(v/u_j)^2}, \quad u_j = (2k_BT/M_j)^{1/2}.$$

The population-inversion density will now be given by [see (6.27)]

$$N(s,v,t) = N_0 \sum_{j=1,2} f_j W_j(v) \times\{1 - 2[I_+\mathcal{L}(\nu_+ - \omega_j - Kv) + I_-\mathcal{L}(\nu_- - \omega_j + Kv) + \mathcal{G}]\}, \tag{7.8}$$

where $\mathcal{G}$ are interference terms, and a similar average for $P(s,v,t)$ [see (6.28)]. After integration over velocities, the amplitude and phase equations can be written with the same format as (6.45) and (6.46); and

$$A_\pm = \{[f_1 Z_i(\xi_\pm^{(1)}) + f_2 Z_i(\xi_\pm^{(2)})] - Z_i^{\max}\mathfrak{N}_\pm^{-1}\},$$

$$Z_r = \{f_1 Z_r(\xi_\pm^{(1)}) + f_2 Z_r(\xi_\pm^{(2)})\},$$

$$\mathcal{S} = \{f_1 \mathcal{S}(\xi^{(1)}) + f_2 \mathcal{S}(\xi^{(2)})\},$$

$$\mathcal{C} = \{f_1 \mathcal{C}(\xi^{(1)}) + f_2 \mathcal{C}(\xi^{(2)})\},$$

$$\xi^{(j)} = (\nu - \omega_j)/Ku, \quad G = (\nu/Q)\mathfrak{N}/Z_i^{\max},$$

$$Z_i^{\max} = \max[f_1 Z_i(\xi^{(1)}) + f_2 Z_i(\xi^{(2)})]. \tag{7.9}$$

Approximations $Ku_1 \approx Ku_2 = Ku$ and $\wp_1 \approx \wp_2 = \wp$ have been used in expressions (7.9).

From expression (7.8) one can see that besides the two Lorentzian holes of width γ_{ab} burned into the Gaussian velocity distribution $W_1(v)$ of the main isotope (1),

$$Kv_+^{(1)} = +(\nu_+ - \omega_1) \quad \text{and} \quad Kv_-^{(1)} = -(\nu_- - \omega_1),$$

there are also two Lorentzian holes

$$Kv_+^{(2)} = +(\nu_+ - \omega_2) \quad \text{and} \quad Kv_-^{(2)} = -(\nu_- - \omega_2)$$

burned into $W_2(v)$. For a typical case (0.633-μ transition in He-Ne laser at 2.5 Torr), $Ku \approx 1000$ MHz, $|\nu_+ - \nu_-| \ll \gamma_{ab} \approx 200$ MHz, and $(\omega_2 - \omega_1) \approx 850$ MHz.

In the detuning region of strong mode competition (in terms of the main isotope), where $(\nu - \omega_1) < \gamma_{ab}$, $Kv_\pm^{(1)} \approx 0$, the mode with lower losses would be able to quench the oscillations of the other mode. However, the presence of the two holes, mode (+) interacting with atoms with $Kv_+^{(2)} \approx -(\omega_2 - \omega_1)$, and mode (−) interacting with atoms with $Kv_-^{(2)} \approx +(\omega_2 - \omega_1)$, where $|\omega_2 - \omega_1| > \gamma_{ab}$, indicates that if f_2 is such that $f_2 \bar{N}_0 W(v^{(2)})$ exceeds a minimum value, the losses may be overcome by an extra increase of the "effective" population. In this case both modes will continue to oscillate.

VIII. RING LASER AS A ROTATION SENSOR

Depending on the values of the different parameters, i.e., κ, Δ, etc., interference of the two opposed beams emerging from a ring laser may produce a beat-frequency signal. The lower solid line of Fig. 5 shows qualitatively a typical diagram of the observed beat frequency as a function of the rotation rate (-2κ).[6,9,15,20] As the rotation rate increases, interference of the two beams results in beats whose frequency asymptotically approaches -2κ. When the magnitude of the parameter 2κ is decreased to a value $0<2|\kappa|\leq 2\kappa_L$, the observed beat note disappears, and both fields E_+ and E_- are present but have the same frequency in spite of a nonzero rotation rate. At rotation rates much greater than the lock-in threshold $2\kappa_L$, the beat signal is nearly sinusoidal and for rotation rates approaching the locking-in threshold, the signal has periodic time pulsations with a distorted waveform.[3,20] It is generally observed that the solid curve for the observed beat frequency $\dot{\phi}_{\text{obs}}$ agrees very well with the relationship[46]

$$\dot{\phi}_{\text{obs}} = [(2\kappa)^2 - (2\kappa_L)^2]^{1/2}. \tag{8.1}$$

The experimental behavior of the beat note, and theoretical descriptions using equations of the type (6.45)–(6.47)[3,6,9] appear to be in close correspondence when the observed beat note is associated with the time average value of $\dot{\phi}(t)$ [see Eq. (8.15) and Refs. 9 and 20]. The vertical lines in Fig. 5 schematically indicate temporal pulsations of $\dot{\phi}(t)$, and the lower solid line, the time average value of $\dot{\phi}(t)$ as obtained by a computer treatment of Eqs. (6.45) and (6.47). As the rotation rate approaches the locking threshold $2\kappa_L = f(\Lambda, \Delta, \Theta)$, the pulsating $\dot{\phi}(t)$ shows a distorted waveform, spending more time in the region near the threshold. The time average $\langle\dot{\phi}\rangle$, then departs from the desired value of the rotation rate (-2κ). However, the theory indicates that a measurement of a few features of the time behavior of $I_\pm(t)$ and $\dot{\phi}(t)$ may provide sufficient information for determining the value of the rotation rate above the locking zone. This possibility will be illustrated here by considering a simple case which can be handled analytically. In more complicated cases, a fitting of experimental curves with theoretical expressions will be necessary.

Above the locking zone, it is generally observed that both I_+ and I_- have pulsations at the beat frequency rate, i.e., with a fundamental frequency $\langle\dot{\phi}\rangle$, and with I_+ and I_- nearly 180° out of phase. Then, it will be sufficient for us to consider Eqs. (6.45) and (6.47) with $\Delta_\pm = \Delta$, $\delta_\pm = \delta$, and $\Theta = 0$. If rotation rates are not too high, and effects such as a net flow of the active gas atoms are small enough, the term $\frac{1}{2}G[Zr(\xi_+) - Zr(\xi_-)]$ can be dropped from Eq. (6.47). The remaining phase angle δ can also be ignored.

Introducing the new variables

$$Y = (I_- + I_+), \quad X = (I_- - I_+), \tag{8.2}$$

Equations (6.45) and (6.47) become

$$\dot{Y} = GA\{Y[1 - Y(\mathcal{S}_i + \mathcal{C}_i)/2A] + (\delta A/A)X - X^2(\mathcal{S}_i - \mathcal{C}_i)/2A\}, \tag{8.3}$$

$$\dot{X} = GA\{X[1 - Y(\mathcal{S}_i/A)] + (\delta A/A)Y + (2\Delta/GA)(Y^2 - X^2)^{1/2}\sin\phi\}, \tag{8.4}$$

$$\dot{\phi} = -\{2\kappa + 2\Delta X(Y^2 - X^2)^{-1/2}\cos\phi + \tfrac{1}{2}G(\mathcal{C}_r - \mathcal{S}_r)X\}, \tag{8.5}$$

where the net linear gain of $I_\pm$ has been written as [see (6.43)]

$$A_\pm = A \mp \delta A, \quad |\delta A| \ll A. \tag{8.6}$$

In the ideal case of zero coupling due to backscattering ($\Delta = 0$), the steady-state two-mode solution is given by

$$Y^0 = 2A(\mathcal{S}_i + \mathcal{C}_i), \quad X^0 = 2\delta A/(\mathcal{S}_i - \mathcal{C}_i), \tag{8.7}$$

or

$$I^0_\mp = A/(\mathcal{S}_i + \mathcal{C}_i) \pm \delta A/(\mathcal{S}_i - \mathcal{C}_i), \tag{8.7'}$$

which for excitations above threshold ($A>0$) is stable only if $|\delta A| < A(\mathcal{S}_i - \mathcal{C}_i)/(\mathcal{S}_i + \mathcal{C}_i)$, or equivalently, if $|X^0| < Y^0$ (i.e., away from the region of strong mode competition).

For $\Delta \neq 0$, and rotations rates above the locking zone, the intensities I_+ and I_- undergo time pulsations. We will restrict ouselves to those solutions oscillating around $\{Y^0, X^0\}$, i.e.,

$$Y(t) = Y^0 + Y^1(t), \quad |Y^1| \ll Y^0$$
$$X(t) = X^0 + X^1(t), \quad X^2 \ll (Y^0)^2. \tag{8.8}$$

Of course, a solution of this type will be valid only for restricted values of Δ, which will be determined later on, (weak-coupling case so-called in JETP, etc.). Substituting (8.8) into Eq.

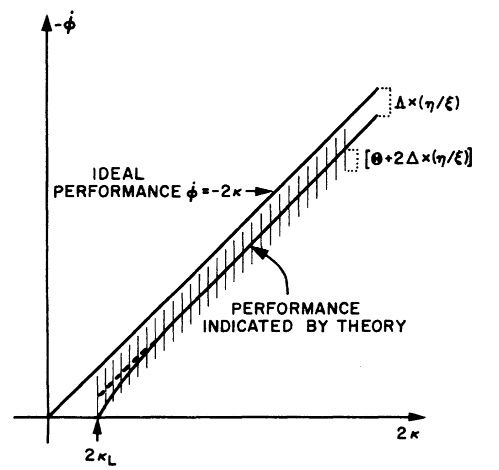

FIG. 5. Beat frequency $\phi(t)$ vs the Sagnac difference frequency (3.15). The vertical lines indicate the range of the temporal pulsations. The lower solid line indicates the time-average value of $\dot{\phi}(t)$. As κ approaches the locking value κ_L, the pulsating $\dot{\phi}$ spends more time in the region near $2/\kappa/-2\kappa_L$.

(8.4) with the restrictions imposed, one obtains

$$Y^1 = 0,$$

$$X^1 = 2\Delta Y^0 \int_0^t \sin\phi(t') \exp\{-[GA(\mathcal{S}_i - \mathcal{C}_i)/(\mathcal{S}_i + \mathcal{C}_i)] \times (t - t')\}\, dt', \tag{8.9}$$

after transients have decayed. Away from the region ($\mathcal{S}_i \approx \mathcal{C}_i$) of strong-mode competition, and for rotation rates not too far from the locking zone, the integral is extended over times $(t - t')$ short enough that $\phi(t') = \phi(t) - \dot\phi(t)(t - t')$ with $\dot\phi(t)(t - t') < 1$ should be a good approximation. Then

$$X^1(t) = 2\Delta Y^0 [GA(\mathcal{S}_i - \mathcal{C}_i)/(\mathcal{S}_i + \mathcal{C}_i)]^{-1} \times \{\sin\phi(t) - \dot\phi(t)[GA(\mathcal{S}_i - \mathcal{C}_i)/(\mathcal{S}_i + \mathcal{C}_i)]^{-1} \times \cos\phi(t)\}. \tag{8.10}$$

The initial assumption, $X^2 << Y^2$ is then valid for $(\delta A/A) = [\nu/Q_+) - (\nu/Q_-)]/GA$, $(\dot\phi/GA)$, and $(2\Delta/GA)$ much less in magnitude than $(\mathcal{S}_i - \mathcal{C}_i)/(\mathcal{S}_i + \mathcal{C}_i)$.[7,10]

The results to first order in the small parameters can be written in the form

$$Y = (I_- + I_+) = 2A/(\mathcal{S}_i + \mathcal{C}_i), \tag{8.11}$$

$$X = (I_- - I_+) = Y[P(\delta Q) + P(\Delta)\sin\phi(t)], \tag{8.12}$$

$$\dot\phi(t) = 2\kappa - 2\Delta[P(\delta Q) + P(\Delta)\sin\phi]\cos\phi(t) - \tfrac{1}{2}G(\mathcal{C}_r - \mathcal{S}_r)Y[P(\delta Q) + P(\Delta)\sin\phi(t)], \tag{8.13}$$

where

$$P(\delta Q) = (\delta A/GA)[(\mathcal{S}_i - \mathcal{C}_i)/(\mathcal{S}_i + \mathcal{C}_i)]^{-1},\quad P(\Delta) = (2\Delta/GA)[(\mathcal{S}_i - \mathcal{C}_i)/(\mathcal{S}_i + \mathcal{C}_i)]^{-1}. \tag{8.14}$$

Equation (8.13) is a particular case of the more general (6.47), and still contains most of the observed features shown in Fig. 5. We will begin by considering the case $P(\delta Q) = 0$.

An equation (8.13) of the type

$$\dot\phi(t) = -2\kappa + a\sin\phi(t)[1 - (b/a)\cos\phi(t)],\quad \kappa \lessgtr 0,\ a \gtrless 0,\ b > 0 \tag{8.15}$$

can be handled analytically[47]; for our purposes it is sufficient to see that pulsations cease ($\dot\phi = 0$) at the locking threshold $2\kappa_L = \max|a\sin\phi - b\sin\phi \times \cos\phi|$, while for rotation rates $|\kappa| > |\kappa_L|$, the average $\langle\dot\phi(t)\rangle$ departs from the desired value (-2κ) for a rotation-rate sensor. Nevertheless, Eq. (8.15) indicates that using more detailed measurements one can still determine the rotation rate (apart from unavoidable noise fluctuations). In fact, for $|\kappa| > \kappa_L$, the oscillating $\dot\phi(t)$ will have extremes determined by the max and min values of $f(\phi) = a\sin\phi - b\sin\phi\cos\phi$. From $(\partial f/\partial\phi) = 0$, these extremes occur for

$$\cos\phi_e = (a/4b) \pm [(a/4b)^2 + \tfrac{1}{2}]^{1/2}. \tag{8.16}$$

Assuming $a > 0$, the (−) sign of (8.16) always give $|\cos\phi_e| \leq 1$, while taking the (+) sign, $|\cos\phi_e| \leq 1$ only for $b \geq 4a$. In any case, these two type of behavior can be distinguished by an examination of the experimental beat signal.

Taking $a > 0$, and the (−) sign of (8.16), we have $\cos\phi_e < 0$. Then, from $f(\phi_e) = a\sin\phi_e[1 + (b/a) \times |\cos\phi_e|]$, we see that $\sin\phi_e > 0$ (i.e., ϕ_e in the third quadrant) gives $f = f_{\max}$, while $\sin\phi_e < 0$ (i.e., ϕ_e in the fourth quadrant) gives $f = f_{\min}$. Substituting these values into (8.15), and taking into account that $f_{\min} = -f_{\max}$, one obtains the analytic result

$$\tfrac{1}{2}(\dot\phi_{\max} + \dot\phi_{\min}) = -2\kappa. \tag{8.17}$$

Thus, whenever possible, a direct or indirect measurement of $\dot\phi_{\max}$ and $\dot\phi_{\min}$ should provide, at least in this restricted case, the value of the rotation rate. In this approximation, from (8.12) one can see that $X_{\max} = -X_{\min}$.

In more complicated cases, more elaborate measurements are necessary, but the theory indicates that the possibility of determining the rotation rate is generally present. For example, the locking threshold may be shifted by introducing a calibrated nonreciprocal phase difference between the counter-rotating waves (e.g., κ_{bias}).[5] Nonreciprocal transmission ($\delta Q \neq 0$) may also be present or externally introduced.[4] In this case, and whenever the approximations (8.11)–(8.13) hold, a measurement of $\dot\phi_{\max}$ and $\dot\phi_{\min}$ will give

$$\tfrac{1}{2}(\dot\phi_{\max} + \dot\phi_{\min}) = -[2(\kappa + \kappa_{\text{bias}}) + \tfrac{1}{2}G(\mathcal{C}_r - \mathcal{S}_r)YP(\delta Q)]. \tag{8.18}$$

The term $G(\mathcal{C}_r - \mathcal{S}_r)$ depends on the losses of the resonator and relative excitation of the active medium, and varies with detuning $\xi = (\nu - \omega)/Ku$ as the dispersion curve associated with an absorption line centered on the Doppler curve [(6.32), (6.36), and Appendix A]. We see that a measurement of $X_{\max} + X_{\min} = 2YP(\delta Q)$, and Y gives the possibility of determining $P(\delta Q)$. Hence the laser gyroscope can still prove useful in the presence of reflections and nonreciprocal losses.

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APPENDIX A: $Z(\zeta), J_{\mu,\lambda}(\zeta), Y(\zeta)$

The representation

$$Z(\zeta) = -\int_{-\infty}^{+\infty} \frac{W(v)\,dv}{[\zeta \mp (v/u)]}, \quad \zeta = \xi + i\eta \tag{A1}$$

for the plasma dispersion function is valid only for $\mathrm{Im}\zeta \geq 0$; and the above integral, with ζ^* instead of ζ, represents $Z(\zeta)^*$ rather than $Z(\zeta^*)$. The complex functions $J_{\mu,\lambda}$ and Y can be expressed in terms of the plasma function $Z = [Z_r(\xi,\eta) + iZ_i(\xi,\eta)]$ by separating the complex denominators of the integral representations in partial fractions. Then,

$$J_{\pm,\pm} = -2\eta[\xi Z_i + \eta Z_r] + i[Z_i + 2\eta(1 - \eta Z_i + \xi Z_r)], \tag{A2}$$

$$J_{\pm,\mp} = -(\xi/\eta)\,\mathfrak{L}(\xi)[Z_i - (\eta/\xi)Z_r] + i\mathfrak{L}(\xi)[Z_i - (\eta/\xi)Z_r], \tag{A3}$$

$$Y = \tfrac{1}{2}(\Gamma_s/Ku)\eta \sum_\alpha \Bigg\{ \left[\frac{1}{\xi + \frac{1}{2}i(2\eta - \eta_\alpha)}\right] \left[\frac{1}{\xi + \frac{1}{2}i(2\eta - \eta_\alpha)}[Z(\zeta) - Z(\tfrac{1}{2}i\eta_\alpha)] - \frac{\partial Z}{\partial \zeta}\right] - \left[\frac{1}{\xi - \frac{1}{2}i(2\eta - \eta_\alpha)}\right] \left[\frac{1}{\xi - \frac{1}{2}i(2\eta - \eta_\alpha)}[Z(\zeta) - Z(\tfrac{1}{2}i\eta_\alpha)] - (Z_r/\xi)\right] \Bigg\}, \tag{A4}$$

where

$$\mathfrak{L}(\xi) = \frac{\gamma_{ab}^2}{(\nu - \omega)^2 + \gamma_{ab}^2} = \frac{\eta^2}{\xi^2 + \eta^2}, \tag{A5}$$

$$\frac{\partial Z(\zeta)}{\partial \zeta} = -2[1 + \zeta Z(\zeta)], \tag{A6}$$

$$\eta_\alpha = (\gamma_\alpha/Ku), \quad \Gamma_s = \frac{\gamma_a\gamma_b}{\gamma_a + \gamma_b}. \tag{A7}$$

In many gas lasers $\eta = (\gamma_{ab}/Ku) \ll 1$, so that it will be useful to expand the basic ingredient Z in power of η. Using $(\partial Z/\partial\eta) = -2i[1 + \zeta Z]$, one obtains

$$Z_r(\xi,\eta) = Z_r(\xi,0) + 2\eta\xi Z_i(\xi,0) + 2\eta^2[Z_r - 2\xi - 2\xi^2 Z_r] + \cdots,$$
$$Z_i(\xi,\eta) = Z_i(\xi,0) - 2\eta[1 + \xi Z_r(\xi,0)] + 2\eta^2 Z_i(\xi,0)[1 - 2\xi^2] + \cdots, \tag{A8}$$

where

$$Z_r(\xi,0) = -2e^{-\xi^2}\int_0^\xi e^{x^2}dx = -2\xi + \tfrac{4}{3}\xi^3 - \tfrac{8}{15}\xi^5 + \cdots,$$
$$Z_i(\xi,0) = \sqrt{\pi}\, e^{-\xi^2} \tag{A9}$$

are obtained from (A1) by using the following recipes for the δ-function and principal-value distributions:

$$\frac{1}{\pi}\lim_{\eta\to 0}\frac{\eta}{x^2+\eta^2} = \delta(x), \quad \lim_{\eta\to 0}\frac{x}{x^2+\eta^2} = \frac{\wp}{x}. \tag{A10}$$

The expansions (A8) and (A9) can be used to calculate simple expressions for Z, J, and Y valid for $\xi = [(\nu - \omega)/Ku] < 1$. In particular,

$$Y(\xi) \approx Y(0) = i4\sqrt{\pi}\,(\Gamma_s/Ku)\eta \quad \text{for } \xi < \eta, \tag{A11}$$

$Y(\xi)$ rapidly approaches zero for $\xi > \eta$.

The approximate Lorentzian ξ shape of $J_{+,-}(\xi)$, and the ξ behavior of $Y(\xi)$, may be qualitatively related with our discussion of expression (6.27) for $N(s,v,t)$.

In the case of no atomic motion ($u = 0$), the expressions for Z, J, and Y are easily found from the integral representations (6.30)–(6.32) by using

$$\lim_{u\to 0} W(v) = \lim_{u\to 0}(\pi^{1/2}u)^{-1}e^{-(v/u)^2} = \delta(v). \tag{A12}$$

APPENDIX B: THE STANDING-WAVE REPRESENTATION

1. Self-Consistency Equations in the SWR

The following forms are adopted for $E(s,t)$ and $P(s,t)$:

$$E(s,t) = \tfrac{1}{2}\sum_{\mu=c,s}\{E_\mu(t)e^{-i[\nu t + \phi_\mu(t)]}U_\mu(s)\} + \text{c.c.},$$
$$P(s,t) = \tfrac{1}{2}\sum_{\mu=c,s}\{[C_\mu + iS_\mu]e^{-i[\nu t + \phi_\mu]}U_\mu(s)\} + \text{c.c.} \tag{B1}$$

Projections onto U_c, U_s are obtained by (3.23) with $\mu = c, s$. With the origin of the s coordinate determined by (3.26), the matrix elements (3.16) and (4.5) will have the form

$$2\Delta = \frac{\nu_0}{\langle n\rangle L}\oint_0^L n^2(s)U_\mp U_\pm^* ds, \quad n^2(s) = 1 + \chi_r(s)$$

$$\Theta e^{\pm i(\delta_i - \delta_r)} = \frac{\nu_0}{\langle n\rangle L}\oint_0^L \chi_i(s)U_\mp U_\pm^* ds. \tag{B2}$$

Taking into account that $\{U_+, U_-\}$ and $\{U_c, U_s\}$ are related by a simple transformation, it is easy to find the corresponding expressions for the matrix elements in the SWR:

$$\frac{\nu_0}{\langle n\rangle L}\int_0^L n^2 U_{c,s}U_{s,c}^*\,ds=0,$$

$$\frac{\nu_0}{\langle n\rangle L}\int_0^L n^2|U_{c,s}|^2\,ds=\nu_0\pm 2\Delta,$$

$$\frac{\nu_0}{\langle n\rangle L}\int_0^L \chi_i|U_{c,s}|^2\,ds=\nu_0\frac{\langle\chi_i\rangle}{\langle n\rangle^2}\pm\Theta\cos(\delta_i-\delta_r),$$

$$\frac{\nu_0}{\langle n\rangle L}\int_0^L \chi_i U_s U_c^*\,ds=i\Theta\sin(\delta_i-\delta_r), \qquad \text{(B3)}$$

where the signs (+) and (−) are correlated with U_c and U_s, respectively.

Substituting (B1) and (B3) in Eq. (3.3) and with the same type of approximations already used in deriving (4.6), one obtains

$$i\frac{d}{dt}\begin{vmatrix}E_c e^{-i(\nu t+\phi_c)}\\ E_s e^{-i(\nu t+\phi_s)}\end{vmatrix}-\left\{\begin{vmatrix}(\nu_0-\Delta) & \kappa\\ \kappa & (\nu_0+\Delta)\end{vmatrix}-\tfrac{1}{2}i\begin{vmatrix}(\nu/Q)+\Theta\cos(\delta_i-\delta_r) & -\Lambda+i\Theta\sin(\delta_i-\delta_r)\\ -\Lambda-i\Theta\sin(\delta_i-\delta_r) & (\nu/Q)-\Theta\cos(\delta_i-\delta_r)\end{vmatrix}\right\}\begin{vmatrix}E_c e^{-i(\nu t+\phi_c)}\\ E_s e^{-i(\nu t+\phi_s)}\end{vmatrix}$$
$$=-\frac{\nu}{2\epsilon}\begin{vmatrix}(C_c+iS_c)e^{-i(\nu t+\phi_c)}\\ (C_s+iS_s)e^{-i(\nu t+\phi_s)}\end{vmatrix}. \qquad \text{(B4)}$$

2. Polarization and Population Inversion (SWR)

We begin with the pair of coupled equations (5.15) and (5.16) for $N(\hat{s},v,t;\hat{t})$ and $\rho_{ab}(s,v,t;\hat{t})$. The perturbation takes the form

$$V(\hat{s},\hat{t})=-(\wp/2\hbar)\sum_{\mu=c,s}\{E_\mu(\hat{t})e^{-i[\nu\hat{t}+\phi_\mu(\hat{t})]}U_\mu(\hat{s})+\text{c.c.}\}, \qquad \text{(B5)}$$

with

$$U_c(\hat{s})=\sqrt{2}\cos K\hat{s},\quad U_s(\hat{s})=-i\sqrt{2}\sin K\hat{s}. \qquad \text{(B6)}$$

For a small-signal theory, the solutions can be obtained by iterations beginning with zero order,

$$N^{(0)}(\hat{s},v,t;\hat{t})=N_0(\hat{s},\hat{t})W(v). \qquad \text{(B7)}$$

The calculations are straightforward and we shall only quote the results.

The velocity-dependent population $N(s,v,t)$ is given by

$$N(s,v,t)=N_0(s,t)W(v)\Big\{1-\Big[[I_c+I_s][\mathcal{L}(\nu+Kv-\omega)+\mathcal{L}(\nu-Kv-\omega)]+2(I_cI_s)^{1/2}[\mathcal{L}(\nu+Kv-\omega)-\mathcal{L}(\nu-Kv-\omega)]\cos\varphi$$
$$+2\,\text{Re}\,e^{+2iKs}\left(\sum_{\alpha=a,b}\frac{i\Gamma_s}{i\gamma_\alpha-2Kv}\right)\left(\frac{\frac{1}{2}i\gamma_{ab}}{\nu-Kv-\omega+i\gamma_{ab}}+\frac{-\frac{1}{2}i\gamma_{ab}}{\nu+Kv-\omega-i\gamma_{ab}}\right)$$
$$\times[I_c-I_s-2i(I_cI_s)^{1/2}\sin\varphi]\Big]\Big\}, \qquad \text{(B8)}$$

where $\Gamma_s=[\gamma_a\gamma_b/(\gamma_a+\gamma_b)]$, and the dimensionless intensities $I_{c,s}$ are introduced as in (6.14). The phase difference has been denoted by

$$\varphi(t)=\phi_c(t)-\phi_s(t). \qquad \text{(B9)}$$

The expression for the total polarization $P(s,t)$ is given by

$$P(s,t)=\wp[\rho_{ab}^{(1)}(s,t)+\rho_{ab}^{(3)}(s,t)]+\text{c.c.}$$
$$=-\wp N_0(s,t)\Big\{\left(\frac{\wp E_c}{2\hbar Ku}\right)e^{-i(\nu t+\phi_c)}U_c(s)(Z(\xi_c)-\{\tfrac{1}{2}(\mathcal{S}+\mathcal{C})I_c+[\mathcal{S}+\tfrac{1}{2}(\mathcal{S}-\mathcal{C})e^{2i\varphi}]I_s\})$$
$$+\left(\frac{\wp E_s}{2\hbar Ku}\right)e^{-i(\nu t+\phi_s)}U_s(s)(Z(\xi_s)-\{\tfrac{1}{2}(\mathcal{S}+\mathcal{C})I_s+[\mathcal{S}+\tfrac{1}{2}(\mathcal{S}-\mathcal{C})e^{-2i\varphi}]I_c\})\Big\}+\text{c.c.}, \qquad \text{(B10)}$$

where the functions $\mathcal{S}\equiv J_{++}$ and $\mathcal{C}\equiv(J_{+-}+Y)$ have been defined by the integral representations (6.36) and (6.30)–(6.32). In Appendix A, the functions $\mathcal{S}$ and $\mathcal{C}$ are evaluated in terms of the plasma dispersion function Z.

The amplitude- and phase-determing equations are obtained by using (B4) and (B10). It is convenient to express them in terms of the dimensionless intensities $I_{c,s}=[\wp^2E_{c,s}^2(\gamma_a+\gamma_b)/4\hbar^2\gamma_{ab}\gamma_a\gamma_b]$ by using the steps indicated in (6.39)–(6.42). For simplicity we consider the case $\Delta_+=\Delta_-=\Delta$ and $\Theta=0$:

$$\dot{I}_c = +2\kappa(I_cI_s)^{1/2}\sin\varphi + \Lambda(I_cI_s)^{1/2}\cos\varphi - (\nu/Q)I_c + GI_c(Z_i(\xi_c) - \{\tfrac{1}{2}(\mathcal{S}+\mathcal{C})_i I_c + [\tfrac{1}{2}(\mathcal{S}+\mathcal{C})_i + \tfrac{1}{2}(\mathcal{S}-\mathcal{C})_i(1+\cos2\varphi) + \tfrac{1}{2}(\mathcal{S}-\mathcal{C})_r\sin2\varphi]I_s\}), \quad \text{(B11)}$$

$$\dot{\phi}_c I_c = -\Delta I_c + \kappa(I_cI_s)^{1/2}\cos\varphi - \tfrac{1}{2}\Lambda(I_cI_s)^{1/2}\sin\varphi + \tfrac{1}{2}GI_c(Z_r(\xi_c) - \{\tfrac{1}{2}(\mathcal{S}+\mathcal{C})_r I_c + [\tfrac{1}{2}(\mathcal{S}+\mathcal{C})_r + \tfrac{1}{2}(\mathcal{S}-\mathcal{C})_r(1+\cos2\varphi) - \tfrac{1}{2}(\mathcal{S}-\mathcal{C})_i\sin2\varphi]I_s\}), \quad \text{(B12)}$$

and two more equations obtained by interchanging c and s and replacing Δ by $-\Delta$ and φ by $-\varphi$. In the case where $I_c \neq 0$ and $I_s \neq 0$, from (4.11) we can obtain a differential equation for the phase difference $\varphi = (\phi_c - \phi_s)$. This is given by

$$\dot{\varphi} = (\dot{\phi}_c - \dot{\phi}_s) = -2\Delta - \tfrac{1}{4}G(I_s - I_c)(\mathcal{S}-\mathcal{C})_r(1+\cos2\varphi) + \tfrac{1}{4}G(I_s+I_c)(\mathcal{S}-\mathcal{C})_i\sin2\varphi + \kappa(I_s - I_c)(I_cI_s)^{-1/2}\cos\varphi - \tfrac{1}{2}\Lambda(I_s+I_c)(I_cI_s)^{-1/2}\sin\varphi, \quad \text{(B13)}$$

where the approximations $Z_r(\xi_{c,s}) \approx Z_r(\xi)$ has been made.

We shall limit our discussion of the above equations to a simple case, which can be handled analytically:

$$|\xi| > \eta, \quad \kappa = 0, \quad \Lambda = 0, \quad \Delta \neq 0$$

(see Ref. 14). The results for this case are easily obtained in both the RWR and the SWR. In the weak-coupling case, when $(2\Delta/GA)$ is much less than $[(\mathcal{S}_i - \mathcal{C}_i)/(\mathcal{S}_i + \mathcal{C}_i)]$, the approximations of Sec. VIII apply, and one can write $I_\pm$ and $\dot{\phi}$ in the form

$$I_\pm \approx \tfrac{1}{2}Y[1 \mp P(\Delta)\sin\phi(t)], \quad Y \approx \frac{2A}{\mathcal{S}_i + \mathcal{C}_i},$$

$$\dot{\phi}(t) \approx -\tfrac{1}{2}G(\mathcal{C}_r - \mathcal{S}_r)YP(\Delta)\sin\phi - \tfrac{1}{2}\Delta P(\Delta)\sin2\phi. \quad \text{(B14)}$$

Making use of the fact that $\mathcal{C}_i \approx \mathcal{L}(\xi)\mathcal{S}_i$, $\mathcal{S}_i \approx Z_i$ ($\mathcal{S}_i \gg \mathcal{C}_i$ for $|\xi| > \eta$), and the series expansions of the different functions (see Appendix A), one can write

$$\dot{\phi} \approx +2\Delta(\eta/\xi)\sin\phi - \Delta(2\Delta/GA)\sin2\phi. \quad \text{(B15)}$$

If the second term is much smaller than the first, the phase angle will lock at

$$\phi = \pm\pi \text{ for } (\xi/\Delta) > 0, \qquad \phi = 0 \text{ for } (\xi/\Delta) < 0. \quad \text{(B16)}$$

Substituting (B16) into (B14), one finds

$$I_+ = I_- \approx \frac{A}{\mathcal{S}_i + \mathcal{C}_i} \approx \frac{A}{\mathcal{S}_i}. \quad \text{(B17)}$$

Furthermore, substitution of (B16) and (B17) into Eqs. (6.45) for $\dot{\phi}_\pm$ give us

$$\dot{\phi}_+ = \dot{\phi}_- = +\Delta \text{ for } (\xi/\Delta) > 0, \qquad \dot{\phi}_- = \dot{\phi}_+ = -\Delta \text{ for } (\xi/\Delta) < 0. \quad \text{(B18)}$$

Using (B16) and (B18) one can write, for $(\xi/\Delta) > 0$,

$$E(s,t) \propto (I^0)^{1/2}\{e^{-i(\nu+\Delta)t}e^{iKs} + e^{\pm i\pi}e^{-i(\nu+\Delta)}e^{-iKs}\} \propto (I^0)^{1/2}e^{-i(\nu+\Delta)t}\sin Ks, \quad \text{(B19)}$$

and, for $(\xi/\Delta) < 0$,

$$E(s,t) \propto (I^0)^{1/2}e^{-i(\nu-\Delta)t}\cos Ks. \quad \text{(B20)}$$

The same results also follow from Eqs. (B11) and (B12) of the SWR. These equations for $|\xi| > \eta$, $\Delta \neq 0$, $\kappa = 0$, and $\Lambda = 0$ can be written as (we have taken $\mathcal{C}_i = 0$, $\mathcal{S}_r = 0$, $\mathcal{C}_r/\mathcal{S}_i = -\eta/\xi$)

$$\dot{I}_c = \alpha_c I_c - \beta_c I_c^2 - \theta_{cs}I_sI_c, \qquad \dot{I}_c = \alpha_s I_s - \beta_s I_s^2 - \theta_{sc}I_cI_s, \quad \text{(B21)}$$

where α_μ, $\mu = c$ or s are given by

$$\alpha_\mu = G\{Z_i(\xi)[1 - 2\xi(\dot{\phi}_\mu/Ku)] - (\nu/Q)G^{-1}\}, \quad \text{(B22)}$$

and

$$\beta_c = \beta_s = \beta \equiv \tfrac{1}{2}\mathcal{S}_i, \quad \text{(B23)}$$

$$\theta_{cs} = \beta[1 + (1+\cos2\varphi) + (\eta/\xi)\sin2\varphi], \qquad \theta_{sc} = \beta[1 + (1+\cos2\varphi) - (\eta/\xi)\sin2\varphi]. \quad \text{(B24)}$$

The beat note $\dot{\varphi}$ is given by

$$\dot{\varphi} = -2\Delta + \tfrac{1}{4}G(I_s - I_c)\mathcal{C}_r(1+\cos2\varphi) + \tfrac{1}{4}G(I_s+I_c)\mathcal{S}_i\sin2\varphi. \quad \text{(B25)}$$

For $|\xi| > \eta$, the second term on the right-hand side is small compared with the third and will be neglected. Under these conditions, one can see that if two-mode oscillation were possible (SWR), the phase angle φ would lock at a value φ_L for which

$$\sin2\varphi_L = \frac{2\Delta}{\tfrac{1}{4}G(I_s+I_c)\mathcal{S}_i}. \quad \text{(B26)}$$

For weak coupling, (B26) is much less than one in magnitude, so that $\varphi_L \approx \pi/2$. Substituting (B26) into (B24), the coupling parameters take the form

$$\theta_{cs} = \beta[1 + (\eta/\xi)\sin2\varphi_L], \qquad \theta_{sc} = \beta[1 - (\eta/\xi)\sin2\varphi_L]. \quad \text{(B27)}$$

The expressions for the steady-state two-mode solution of (B21) are now given by

$$I_c = \frac{\alpha_c\beta - \alpha_s\theta_{cs}}{\beta^2 - \theta_{cs}\theta_{sc}} \approx \frac{\alpha(\beta - \theta_{cs}}{\beta^2 - \theta_{cs}\theta_{sc}}, \qquad I_s = \frac{\alpha_s\beta - \alpha_c\theta_{sc}}{\beta^2 - \theta_{cs}\theta_{sc}} \approx \frac{\alpha(\beta - \theta_{sc})}{\beta^2 - \theta_{cs}\theta_{sc}}. \tag{B28}$$

The range of validity of these equations should be examined. First of all, one can see that the denominator D is

$$D = \beta^2 - \theta_{cs}\theta_{sc} = \beta^2 - \beta^2[1 - (\eta/\xi)^2 \sin^2 2\varphi_L] > 0,$$

but when one of the numerators is positive, the other is negative, i.e., for $(\xi/\Delta) > 0$, $(\beta - \theta_{cs}) < 0$ and $(\beta - \theta_{sc}) > 0$, while for $(\xi/\Delta) < 0$, $(\beta - \theta_{sc}) < 0$ and $(\beta - \theta_{cs}) > 0$. Consequently, two-mode oscillation is unstable, and we shall have only single-mode operation:

$$\{I_c = 0,\ I_s \neq 0\} \text{ for } \xi > 0$$

and (B29)

$$\{I_c \neq 0,\ I_s = 0\} \text{ for } \xi < 0,$$

which coincides with results (B19) and (B20).

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