Amplified Spontaneous Emission and External Signal Amplification in an Inverted Medium

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An improved version of the theory of amplified spontaneous emission (ASE) is presented and shown to be satisfied by experimental data obtained using the 3.39-μm transition of He-Ne. The new approach allows the effect of an external signal injected at one end of the system to be investigated and its influence on both the positive-going and negative-going waves to be calculated for He-Ne. A special case occurs when the injected signal is replaced by a mirror which reflects the negative-going wave back into the medium and the theory of this commonly occurring laboratory system is experimentally verified at 3.39 μm. The treatment allows the importance of the interaction between spontaneous and stimulated emission to be considered. The beam divergence predicted by the theory is evaluated and compared with experiment. The spectral distribution for a pure ASE system is considered, as is the distribution of inversion which results from the positive- and negative-going waves and which is shown to have minima at the medium extremities. Finally the implications of this work for the problem of interstellar OH emission is considered.

I. INTRODUCTION

Recently, we have published a series of papers[1] which describes the development and experimental verification of the theory of amplified spontaneous emission. Amplified spontaneous emission (ASE) is the highly directional intense radiation emitted by an extended medium with a randomly populated inversion in the absence of a cavity. In innumerable papers this process is called "superradiance," which is a word coined by Dicke[2] to describe the radiation emitted from systems of atoms which are in prepared superposition states in the absence of dephasing relaxations. We have argued elsewhere[1,3] that the Dicke approach is both unnecessary and wrong to describe the behavior of what is in many respects a "high-gain mirrorless laser system." Consequently, no further arguments will be given here about that particular point; suffice to say that we believe that ASE is a much more meaningful description of the processes that actually take place.

In our earlier work we found it necessary to make an approximation in the term describing the saturation of the medium. The validity of the approximation has been questioned[4] (see also *Note added in proof*) and although it is apparently valid in relatively low-gain laboratory systems except as regards linewidth there is serious doubt about its applicability when extrapolations are made to account for amplified spontaneous emission in, for example, the interstellar medium.[5] Secondly, some of the measurements made in the laboratory were taken with a highly reflecting mirror at one end of the active medium. It is clear now that we did not take proper account of this fact in our earlier work. We demonstrate in this paper that it is possible to generate additional photon transport equations which allow the parametric relationship between intensity, inversion density, length, and diameter of medium to be analyzed without the necessity of the previously used approximation. In addition the equations may be suitably modified to take into account what happens in such a medium when an external signal is injected into it. In this more complex case, amplification of the external signal and ASE occur simultaneously. It is possible to show by numerical computation that a threshold value for the input signal exists below which ASE gives a larger contribution to the total output and above which the dominant role of the medium is as an amplifier for the input signal. For signals vastly different than this threshold value, either ASE or injected signal amplification will dominate as appropriate. Also an approximate parametric form of this threshold condition is derived which agrees well with the numerical computations. This analysis then allows the commonly occurring laboratory system of an inverted medium with one mirror to be properly understood. In a cw system the mirror reinjects into the system, as if from an external source, the ASE wave traveling in the opposite direction. The previous experimental results obtained in the presence of the mirror are explained extremely well by this approach.

This is a more accurate and comprehensive account of the process of amplified spontaneous emission. Our approach still embraces the features which we have shown to be those essential for an accurate description of ASE. These are the presence of both positive- and negative-going traveling waves, the importance of the concept of critical length,[6] and the careful account that is taken of that fraction of the spontaneous emission produced in the medium which usefully contributes to stimulated emission.

The first theoretical approach to ASE appears to have been given by Gordon, White, and Rigden[7] who considered the growth and saturation of a δ-function signal as a function of frequency across the gain profile of an inverted medium. Although this treatment may be expected to provide a good representation of the interaction process at a particular frequency on the spontaneous emission profile, it does not appear to do so. For a δ- (or very-narrow-) signal input, it can be shown[7,8] that the saturation term, as a function of intensity I, follows $(1+I/I_s)^{1/2}$ for an inhomogeneously broadened gain profile and $1+I/I_s$ for a homogeneously broadened one, where I_s is a saturation parameter. The difference arises because of the phenomenon of "hole burning" in the former case. However, for an inhomogeneously broadened gain profile subject to an inhomogeneously broadened emission profile, the saturation behaves more like that of homogeneous broadening. The reason for this, physically, is that although each frequency unit on the emission profile burns a hole in the gain profile, of the order of a natural linewidth, all these holes overlap. The net effect is that the whole gain profile is reduced by saturation as is typical of saturation effects in homogeneously broadened gain profiles.

Yariv and Leite[9] considered the situation of a broad-band emission amplified by a broad-band gain profile. The approach used was to consider that each frequency on the emission profile had a gain coefficient associated with it deduced from an approach similar to that of Gordon, White, and Rigden. In general, this is clearly not true for inhomogeneous broadening but Yariv and Leite considered only nonsaturation conditions and so their approach is correct, although limited in scope.

The first serious approach to ASE was given by Tonks.[10] Although cumbersome, it encompassed some of the features present in our theory. He did not, however, consider threshold conditions and only considered spontaneous emission into an angle defined by the ratio of medium bore to length for all positions within the medium rather than as a function of position within the medium. This implied that for all positions in the medium, except at the extremities, the source term is underestimated.

Various authors[11] have considered ASE from the viewpoint of noise in laser amplifiers, and others[12] have considered it in terms of limitations on the attainable inversions in solid state laser systems. However, these approaches tend to ignore the counter-traveling wave and an explicit form for the threshold condition.

More recently, Casperson and Yariv[13] have extended the work of Gordon, White, and Rigden to allow for a broadband optical signal and discussed amplifiers under saturated and nonsaturated conditions, and ASE under nonsaturated conditions. They do not, however, discuss ASE under saturated conditions nor consider many of the features present in our approach: for example, threshold conditions, geometric limitations on the spontaneous emission, the presence of two traveling waves, and the degradation of the spontaneous emission by stimulated emission. Similarly, Litvak[14] considered arbitrary signal and gain profiles and saturation using a density matrix approach, and then applied this to a number of special cases. Although he uses this theory to discuss ASE, it only strictly applies for the case of an amplifier because internally generated spontaneous emission is ignored, as are many of the other features.

II. THRESHOLD CONDITIONS

Although we have already derived the threshold conditions for amplified spontaneous emission,[3,6] we should like to derive it in an alternative way to show transparently its relationship with the Schawlow and Townes threshold condition for the laser.[15] In their classic paper they calculate the rate at which $\mathfrak{N}$ atoms spontaneously radiate into the modes of free space and argue that when this rate equals or exceeds the rate of decay of radiation in a mode that the mode is excited and will grow in amplitude. This is because for one photon the rate of stimulated emission into a mode equals that of spontaneous emission.

Consider $\mathfrak{N}$ atoms radiating at frequency ν into p modes during the spontaneous lifetime of the appropriate transition τ_{21}, then

$$\mathfrak{N}h\nu/p\tau_{21} \geq h\nu/t_D ,$$

where t_D is the decay time of the particular mode and where for a Doppler broadened line,

$$p=\frac{8\pi\nu^2 V}{c^3}\left(\frac{\pi^{1/2}\Delta\nu_D}{2(\ln 2)^{1/2}}\right) .$$

In fact, this is a factor of $\frac{1}{2}$ different than Schawlow and Townes's expression because they define the linewidth as the half-width at the half-maximum intensity point. For consistency with our previous papers we define the linewidth as the full width at the half-maximum intensity point, i.e.,

$$g(\nu)=\frac{2(\ln 2)^{1/2}}{\pi^{1/2}\Delta\nu_D}\exp\left[-4\left(\frac{\nu-\nu_0}{\Delta\nu_D}\right)^2\ln 2\right] .$$

So the inversion density is

$$n=\frac{\mathfrak{N}}{V}\geq\frac{8\pi\nu^2\tau_{21}}{c^3}\left(\frac{\pi^{1/2}\Delta\nu_D}{2(\ln 2)^{1/2}}\right)\frac{1}{t_D} .$$

Schawlow and Townes demonstrated that for a laser

$$t_D \approx \frac{L}{c(1-R)} \approx \frac{L}{c\delta'},$$

where R is the reflection coefficient of the mirrors and δ' the loss per pass of the cavity. Thus, the threshold inversion for cavity oscillation is

$$n_T = \frac{8\pi\tau_{21}}{\lambda^2 L}\left(\frac{\pi^{1/2}\Delta\nu_D}{2(\ln 2)^{1/2}}\delta'\right).$$

In the case of ASE the appropriate value of t_D is the time it takes a photon to traverse the medium, i.e.,

$$t_D \approx L/c.$$

Thus, the threshold inversion for ASE is

$$n_c = \frac{8\pi\tau_{21}}{\lambda^2 L}\left(\frac{\pi^{1/2}\Delta\nu_D}{2(\ln 2)^{1/2}}\right)$$

and it immediately follows that

$$n_T/n_c = \delta'.$$

The value obtained above for n_c is the same as that previously derived[6] provided we take into account a Doppler line shape rather than the rectangular profile assumed there. If this is done the previous form for n_c is merely modified by a factor $\pi^{1/2}/2(\ln 2)^{1/2} \approx 1\cdot 1$.

Of course the Schawlow and Townes argument involves the population inversion density as if it were the population of the upper level, and it is known that the threshold inversion for a laser derived in this way is a small numerical factor different from the more rigorously derived values of Lamb[16] and of Stenholm and Lamb.[17] We have shown[1] that the latter can be expressed as

$$n_T = \frac{2}{3}\,\frac{8\pi\tau_{21}}{\lambda^2 L}\left(\frac{\pi^{1/2}\Delta\nu_D}{2(\ln 2)^{1/2}}\delta'\right),$$

but again this must be modified to allow for the different definitions of $\Delta\nu_D$. Assuming $\Delta\nu_D$ is the full width at half-maximum intensity, then strictly

$$n_T = \frac{1}{3}\,\frac{8\pi\tau_{21}}{\lambda^2 L}\left(\frac{\pi^{1/2}\Delta\nu_D}{2(\ln 2)^{1/2}}\delta'\right)$$

giving a ratio of

$$n_T/n_c = \tfrac{1}{3}\delta'.$$

III. INTENSITY AND SATURATION IN ASE SYSTEM

Consider a cylinder of atoms which has a population inversion density of $n_2 - n_1$ at position x and time t between a pair of energy levels 2 and 1. Let the cross-sectional area of the cylinder be a and its length L. Both levels are steadily populated at rates R_2 and R_1, respectively, from a source level assumed to have a population n_0, and both lose population by radiative decay to other levels with effective probabilities A_2 and A_1. Level 2 also loses population to level 1, thus helping to destroy the population inversion, with probability A. Simple rate equations may be written for the time dependence of the population densities at position x as

$$\frac{\partial n_2}{\partial t} = -(n_2 - n_1)\frac{\sigma}{a}[N_a(x) + M_a(x)] - An_2 - A_2 n_2 + R_2 n_0$$

and

$$\frac{\partial n_1}{\partial t} = (n_2 - n_1)\frac{\sigma}{a}[N_a(x) + M_a(x)] + An_2 + R_1 n_0 - A_1 n_1,$$

where $N_a(x)$ is the number of photons threading the cross section a per second at position x and traveling in the $+ve$ x direction owing to the amplification of spontaneous emission generated in the medium, and $M_a(x)$ is the number at the same position traveling in the opposite direction. The effective resonance absorption cross section σ allows for the fact that a photon cannot interact with just any atom for an inhomogeneously broadened gain profile.

If steady-state conditions are assumed, then we find

$$n_2 = \frac{B + C[N_a(x) + M_a(x)]}{1 + E[N_a(x) + M_a(x)]}$$

and

$$n_2 - n_1 = F/\{1 + E[N_a(x) + M_a(x)]\},$$

where the coefficients characterize the particular system to be investigated. The rate of increase for one of the ASE traveling waves with distance, in the steady state, is given by

$$\frac{\partial N_a}{\partial x} = (n_2 - n_1)\sigma N_a + \frac{An_2 a\Delta\Omega}{4\pi}$$

or substituting for the equilibrium inversion density, we obtain

$$\frac{\partial N_a}{\partial x} = \frac{F\sigma N_a}{1 + E[N_a(x) + M_a(x)]} + \frac{An_2 a\Delta\Omega}{4\pi},$$

where

$$\Delta\Omega = 2\pi\left(1 - \frac{L - x}{[(L-x)^2 + r^2]^{1/2}}\right)$$

and is the solid angle into which spontaneously emitted photons may be emitted and so contribute to the output power for a medium of radius r. The population of the upper level n_2 has been deemed to be a slowly varying quantity and is taken to be a constant; the justification for this is discussed in Sec. IV. The equation may be expressed in terms of measurable quantities, for example the photocurrent of a detecting photomultiplier, by replacing N_a by I_a times a constant, etc. We then have

$$\frac{\partial I_a}{\partial x} = \frac{K_2 I_a}{1 + K_3[I_a(x) + J_a(x)]} + \frac{K\Delta\Omega}{\pi r^2}, \tag{1}$$

and so K, K_2, and K_3 play the same role as in the previously published work.[1] Similarly, we may write

$$\frac{\partial J_a}{\partial x} = -\frac{K_2 J_a}{1 + K_3[I_a(x) + J_a(x)]} - \frac{K\Delta\Omega'}{\pi r^2}, \tag{2}$$

where

$$\Delta\Omega' = 2\pi\left(1 - \frac{x}{(x^2 + r^2)^{1/2}}\right)$$

for the $-ve$-going wave. These nonlinear equations do not have analytic solutions but may be solved using the Runge-Kutta method of step-by-step evaluation on a computer. Provided the values of the variables are known at $x = 0$, the values at $x = L$ can be computed. In one case the value at $x = 0$ is not known and instead the value at $x = L$ is known. Specifically, $J_a(L)$ is known to be zero and it is necessary to provide the proper initial conditions for the solution of the equations. Consequently, a range of values of $J_a(0)$ had to be tried to see which reproduced $J_a(L) = 0$ when the above equations were solved.

In our earlier work, only Eq. (1) was used, and by making the assumption $I_a(x) + J_a(x) = \bar{I}_L$, where $\bar{I}_L$ is a constant for a given length L, a quasi-analytic solution resulted of the form

$$I(L) = K\int_{L_c}^{L} \frac{e^{-X(L)y}}{y^2}dy, \tag{3}$$

where $X(L) = -K_2/(1+K_3\bar{I}_L)$. However, such a solution has no special merit since a computer fit to experiment is necessary to evaluate the coefficients K, K_2, and K_3. It is now clear that using both equations along with the boundary conditions $I_a(0) = 0$ and $J_a(0)$ equal to the experimental intensity for a medium of length L, there is no need to assume that $I_a(x) + J_a(x)$ is a constant for all values of x. This is an assumption which as has been pointed out to us[4] is, in general, unwarranted.

Experimental work was carried out using the 3.39-μm transition in He-Ne. Although the 3.39-

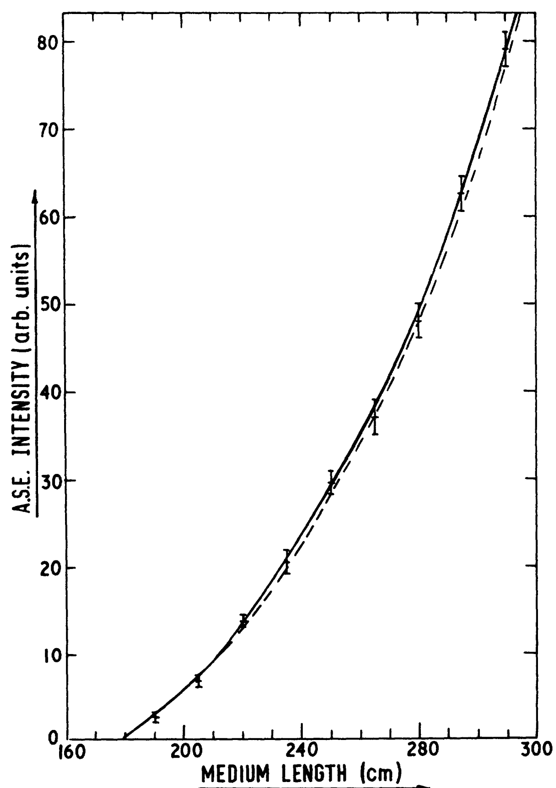

FIG. 1. ASE intensity as a function of medium length for the He-Ne system with an inversion density of 15.8 units and a medium diameter of 2.5 mm showing the experimental points and the fitted curve. The broken line shows the fitted curve for the approximate form of the saturation term.

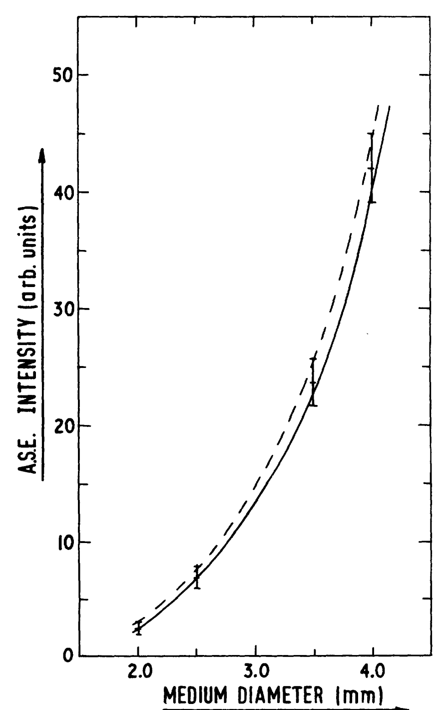

FIG. 2. ASE intensity as a function of medium diameter for the He-Ne system with an inversion density of 15.8 units and a medium length of 205 cm showing the experimental points and theoretical curve. The broken line shows the theoretical curve for the approximate form of the saturation term.

μm system is well known to have a high gain and to operate without mirrors[18] the documented information about its ASE behavior is very limited. Rigden and White[18] estimated that the system had a gain of up to about 50 dB m^{-1} for a tube 100 cm long but did not investigate gain as a function of the parameters involved. Andronova *et al.*[11] investigated the output of a 3.39-μm system as a function of gain and found that in the regions of low saturation that the theory owing to Kogelnik and Yariv,[19] which is based on analyzing the effect of noise in a laser amplifier but which neglects saturation, gave reasonably good results to within a factor of 2–2.5 for the power output. However, at high saturation, discrepancies of an order of magnitude occur between the theoretical and actual output powers.

In our work the He-Ne was rf excited and plasma tubes of 310-cm length with bores varying between 2 and 4 mm at 0.5-mm intervals, were used. The ASE intensity was measured as a function of length by disconnecting the rf clips on the far end of the tube away from the detector so that the detector geometry remained constant. The measurements were carried out for various levels of excitation and care was taken to ensure that the inversion was constant in all parts of the medium by monitoring the spontaneous emission from the upper level as seen at 0.633 μm[1]. Examination of the lifetimes, degeneracies and line intensities concerned show that the lower level will have less than 1% of the population of the upper level in the absence of heavy saturation. Consequently, the inversion density is proportional to the intensity of any transition from the upper level. Measurements of intensity as a function of medium length, diameter, and population inversion density were taken in the range L_c to 310 cm. Further measurements in the same system in the presence of a highly reflecting mirror at one end of the medium will be discussed later in this paper.

Figure 1 shows the experimental points for ASE intensity against medium length for an inversion density of 15.8 units and a diameter of 2.5 mm together with the minimum variance curve (solid line) obtained by fitting the results to Eqs. (1) and (2). It is found that $K = 2.90\times10^2$, $K_2 = 2.0\times10^{-2}$, and $K_3 = 2.2\times10^{-3}$. K and K_2 were determined using the exponential region of the results where the saturation term is negligible and then K_3 was fitted to the results in the nonexponential region. In the exponential region with $K_3 = 0$, Eqs. (1) and (2) can be decoupled and Eq. (3) is then a good solution for either I_a or J_a. Hence the new values of K and K_2 would not be expected to differ much from those previously reported[1]; nor indeed do they. In the previous work the quality of the fit possible in the high-intensity region where the role of saturation and of the mirror was improperly taken into account, necessitated a compromise to the best values of K and K_2 deduced from the exponential region. The outcome of not accounting properly for saturation in the formulation of the theory was that the numerical value of K_3 was smaller than it should have been.

The values obtained for the constants allow the intensity to be predicted as a function of medium diameter for a constant inversion density and length of medium. Figure 2 shows a typical comparison of theory with experiment for intensity against medium diameter with an inversion density of 15.8 units and a medium length of 205 cm. Similarly, the intensity may be predicted as a function of inversion density for constant length (310 cm) and medium diameter (2.5 mm) and Fig. 3 shows the comparison of theory and experiment. It may be seen that satisfactory agreement is achieved in each case. For comparison, the fitted curve pre-

TABLE I. Comparison of the intensities of a bidirectional He-Ne ASE system with the predicted intensity if one of the traveling waves were suppressed, for various medium lengths.

Medium length (cm)	ASE *o/p* single direction traveling wave only	ASE *o/p* two oppositely traveling waves present	Decrease (%)
200	0.608×10	0.601×10	1
300	0.703×10^2	0.682×10^2	3
400	0.276×10^3	0.263×10^3	5
500	0.632×10^3	0.505×10^3	25
1 000	0.350×10^4	0.218×10^4	62
2 000	0.144×10^5	0.637×10^4	79
3 000	0.199×10^5	0.107×10^5	86
4 000	0.285×10^5	0.152×10^5	88
5 000	0.373×10^5	0.196×10^5	90
10 000	0.817×10^5	0.420×10^5	94
20 000	0.172×10^6	0.871×10^5	98

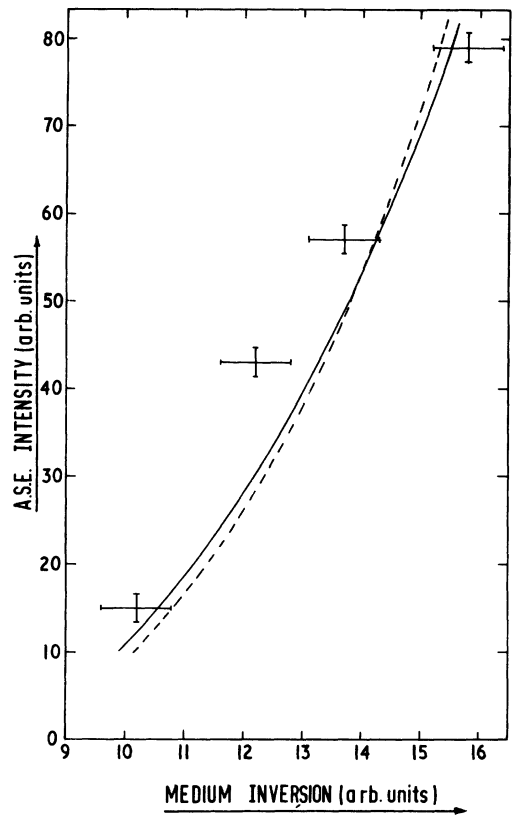

FIG. 3. ASE intensity as a function of medium inversion density for the He-Ne system using a medium length of 310 cm and a diameter of 2.5 mm showing the experimental points and the theoretical curve. The broken line shows the theoretical curve for the approximate form of the saturation term.

viously deduced using (3) is also shown as a broken line.

For medium lengths below the threshold no mode is continuously excited and there is no continuous stimulated emission output. Hence the plot of ASE intensity against medium length (Fig. 1) has a sharp cutoff at L_c. For medium lengths above L_c, when a mode is excited, the two ASE waves begin to grow within the medium from the extremities of the medium rather than at a distance L_c from them. In other words, for those lengths where a mode is continuously excited, stimulated emission must be considered over the whole medium length L rather than in just that portion L_c to L for the $+ve$ wave or in 0 to $(L-L_c)$ for the $-ve$ wave, as was assumed in our earlier treatment. Spontaneous emission is, however, still not important in the region $(L-L_c)$ to L for the $+ve$ wave because the remaining length of medium, from any point in this region to the medium end, is not sufficient to continuously maintain stimulated emission. The same is true for the $-ve$ wave for the spontaneous emission emitted in the region 0 to L_c.

The degree of interaction between the two oppositely directed waves is strongly gain dependent. Table I shows a comparison of the output intensity due to a single-noise-generated wave with that occurring in the same direction in the presence of a counter-traveling wave. For the laboratory system up to ~500 cm the intensity decreases by ~25%. It should be realized, though, that this is a relatively low-gain system and that the percentage change would be greatly increased if the gain were higher.

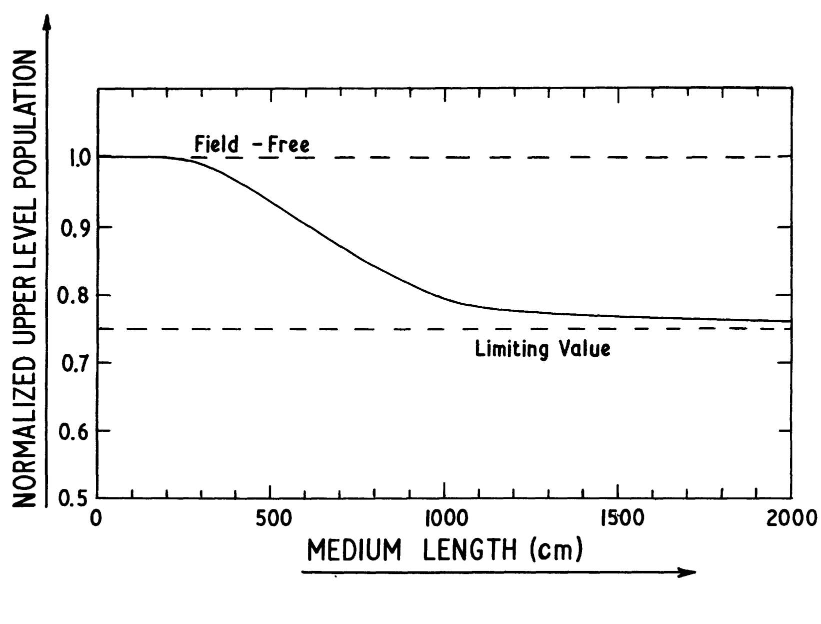

FIG. 4. Theoretical upper level population density (normalized) as a function of medium length for the He-Ne system with an inversion density of 15.8 units and a medium diameter of 2.5 mm.

IV. EFFECT OF STIMULATED EMISSION ON SPONTANEOUS EMISSION

Over the range of medium lengths used in the laboratory here (≲6 m) it was not possible to detect any change in the spontaneous emission in the presence or absence of stimulated emission to within the accuracy with which the discharge could be maintained steady (~5%). However, for extrapolations to greater lengths (e.g., in interstellar problems), this might not be the case if such observations were possible. The expression for the upper-level population contains a stimulated emission term in both the numerator and the denominator and these tend to counteract each other and consequently the effect of the stimulating field is not so pronounced as it is on the inversion. The constant in the denominator of both inversion- and upper-level population expressions is the same, but in the absence of any experimental data it was not possible to determine the constant in the corresponding term in the numerator. However, it can easily be shown that the constant C can be approximately expressed in terms of the known saturation parameter and the transition rates A, A_1, and A_2 so that

$$n_2(x)=n_2^0\left(\frac{1+K_3[I_a(x)+J_a(x)]\delta}{1+K_3[I_a(x)+J_a(x)]}\right),$$

where $\delta=(A+A_2)/(A_1+A_2)$ and n_2^0 is the zero-field upper-level population. As δ tends to unity, the effect of the stimulated emission on n_2 tends to zero. In He-Ne, the transition giving the largest contribution to A_2 is 0.633 μm, and that giving the largest contribution to A_1 is 0.359 μm. Surprisingly the tabulated data on the latter is more extensive and the detailed behavior of collisions and pressure on the broadening of the natural linewidth has been investigated by Kotlikov, Todorov, and Chaika.[20] Since such precise data are not available for the 0.633-μm transition, the value derived by Allen, Jones, and Sayers[21] of 30 MHz is used. This includes pressure-broadening contributions and, in order to minimize the discrepancy, a value for the natural linewidth at a corresponding pressure is chosen for the 0.359-μm transition. A reasonable value seems to be ~12 MHz, giving a value for $\delta\sim0.75$.

Figure 4 shows the deviation of n_2, determined at $x=0$ or L where the effect is at a maximum, from n_2^0 as a function of medium length up to 2000 cm. It may be seen, that for medium lengths up to ~3 m, the effect of stimulated emission on n_2^0 is to change it by only ~2%, and for the largest lengths used experimentally the change is still only 10%. The fact that this was not observed above the 5% fluctuations in our experiment perhaps suggests that the values of linewidth used are a little in error. It is interesting to note that when $K_3(I_a+J_a)\delta$ becomes large compared to unity, the upper-level population assumes the constant value $n_2^0(x)\delta$ and remains unchanged thereafter no matter how great the degree of saturation.

If the effect of saturation on the spontaneous emission term is taken into account it transpires that the ultimate effect on the ASE is still negligible. For the region 400–1000 cm the effect is a ~3% decrease in ASE intensity and outside this region the effect is a ~1% decrease. It must be emphasized that these deductions apply to the 3.39-μm He-Ne transition only, and for systems which have a smaller value of δ saturation of the upper-level population could lead to more pronounced effects.

V. INTENSITY AND SATURATION IN ASE SYSTEM IN THE PRESENCE OF INJECTED EXTERNAL SIGNAL

The theoretical approach used here closely follows that of Sec. IV except that Eqs. (1) and (2) must now be written

$$\frac{\partial I_a(x)}{\partial x}=\frac{K_2I_a(x)}{1+K_3[I_a(x)+J_a(x)+I_b(x)]}+\frac{K\Delta\Omega}{\pi r^2},\qquad(4)$$

$$\frac{\partial J_a(x)}{\partial x}=-\frac{K_2J_a(x)}{1+K_3[I_a(x)+J_a(x)+I_b(x)]}-\frac{K\Delta\Omega'}{\pi r^2},\qquad(5)$$

where $I_b(x)$ represents the amplifier signal which is traveling in the $+ve$ x direction due to the injection of an external signal $I_b(0)$ at $x=0$. The behavior of this third signal is described by

$$\frac{\partial I_b(x)}{\partial x}=\frac{K_2I_b(x)}{1+K_3[I_a(x)+J_a(x)+I_b(x)]}.\qquad(6)$$

There are now three equations and three unknowns, together with the conditions $I_a(0)=0$ and $J_a(L)=0$ and $I_b(0)$ is a constant for which values may be chosen. The constants K, K_2, and K_3, however, remain the same as in Sec. IV provided the same measuring devices are used. Again K is taken to be a constant, although the term involves the population of the upper level, as discussed in Sec. IV. Equations (4)–(6) represent a system capable of producing ASE and yet acting as an amplifier at the same time. This has the advantage of properly taking into account the effects of spontaneous emission which invariably in the past has been either dismissed or introduced and then neglected[13,14] in other amplifier theories. All other theories to the best of our knowledge also ignore the counter-traveling wave.

The value of input signal necessary to perturb the ASE output may be simply deduced. As an arbitrary criterion we consider the ASE to be perturbed when the ASE and amplifier emissions are of comparable magnitude. This avoids defining what one means by "perturb" as, to some extent, the two signals will always interact. The two intensities are of comparable magnitude when the source term for both types of emission are comparable. For the injected signal this is just $I_b(0)$ and for ASE this is the sum of all useful spontaneous emission $I_{spon}(L)$ throughout the medium of length L, i.e.,

$$I_{spon}(L) = \int_0^{L-L_c} \frac{K\Delta\Omega\, dx}{\pi r^2} \approx \frac{K(L-L_c)}{LL_c}$$

for the 3.39-μm transition because $L_c \gg r$ and the first term of the binomial expansion for $\Delta\Omega$ is sufficiently accurate. So the threshold value for the injected signal is

$$I_b(0)_{Thres} = K(L-L_c)/LL_c\ .$$

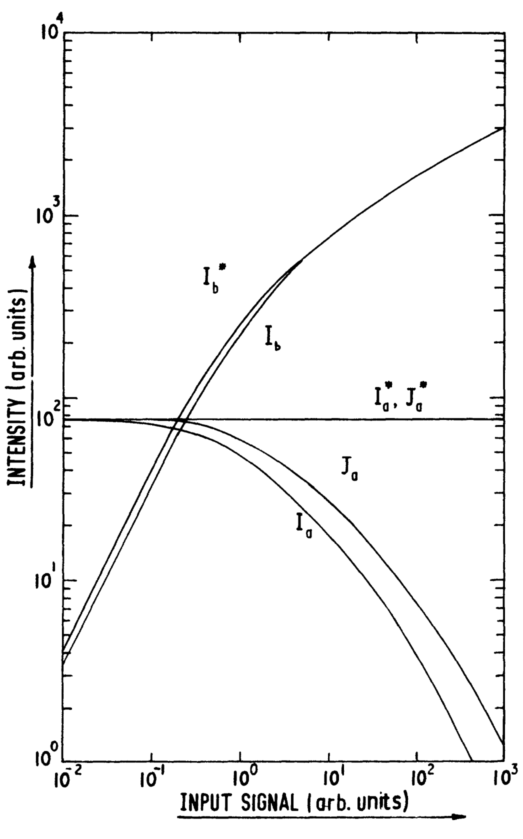

FIG. 5. Theoretical ASE and amplifier intensities as a function of input signal to the medium for the He-Ne system using a medium length of 310 cm, diameter 2.5 mm, and an inversion density of 15.8 units. I_b is the amplifier intensity, I_a the ASE intensity in the same direction, and J_a the intensity of the counter-traveling wave. Starred values represent the intensity in the absence of an input signal or of spontaneous emission.

Figure 5 demonstrates that for a medium of length 310 cm, diameter of 2.5 mm, and inversion density of 15.8 units, I_a and J_a both decrease as the input signal is increased in intensity. I_a^* and J_a^* show the level of ASE intensity in the absence of any input signal and I_b^* is the amplifier intensity if the effect of spontaneous emission is neglected. It should be noted that the ASE output in the direction opposite to that of the amplifier signal is not so heavily suppressed as that in the same direction. Physically, this is explained by the fact that the degree of saturation near $x=0$ is not as great as at $x=L$, where the amplifier signal is appreciable. Figure 6 shows, for the same system and an input of $I_b(0)=2$ units, the behavior of the different ASE components as a function of medium length. That the approximate value derived for $I_b(0)_{Thres}$ works well is shown by the fact that in He-Ne for $L_c=180$ cm, $L=310$, and $K=200$, the approximate value given by this relation is ~0.6 arbitrary units and the corresponding value from Fig. 5 is 0.3 arbitrary units.

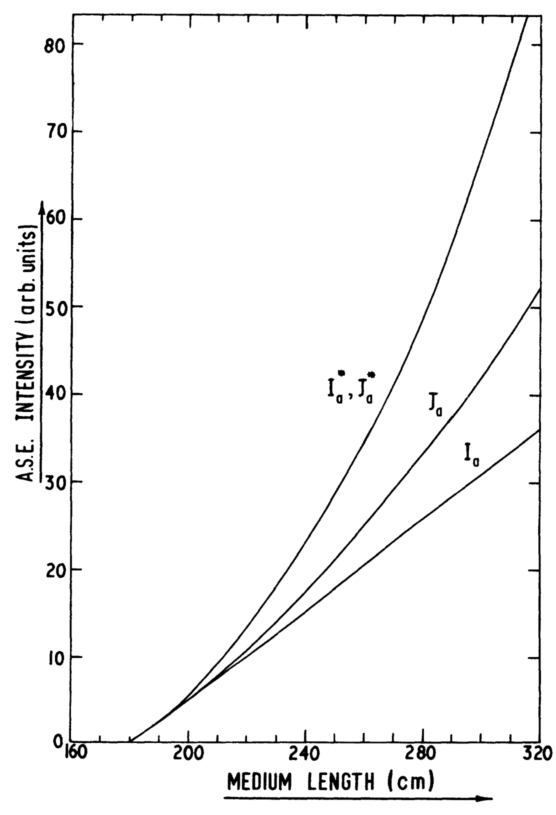

FIG. 6. Theoretical intensities as a function of medium length for the He-Ne system with an inversion density of 15.8 units, a medium diameter of 2.5 mm, and an input signal of 2 units. I_a and J_a are the intensities of the positive- and negative-going waves. Starred values represent the intensities in the absence of an input signal.

VI. AMPLIFIED SPONTANEOUS EMISSION IN THE PRESENCE OF A MIRROR

The effect of a mirror placed at one end of the inverted medium is not always merely to double the length of the medium. Its effect may be deduced from Eqs. (4)–(6), where the input signal $I_b(0)$ is the radiation that is reflected from the mirror back into the medium. It is obvious that the mirror will not usefully reflect back into the medium all of the radiation J_a. The mirror has a reflection coefficient R less than unity and the radiation described by J_a has a certain beam divergence that ensures that a major fraction of J_a is not useful as far as $I_b(0)$ is concerned. The latter loss is the most important and the precise fraction will depend upon where the spontaneous emission giving rise to the ASE originates. Simple geometric optics shows that for a position x within the medium of length L and a distance y between the mirror and the medium, the fraction $\delta(x)$ of the spontaneous emission that can act as a source

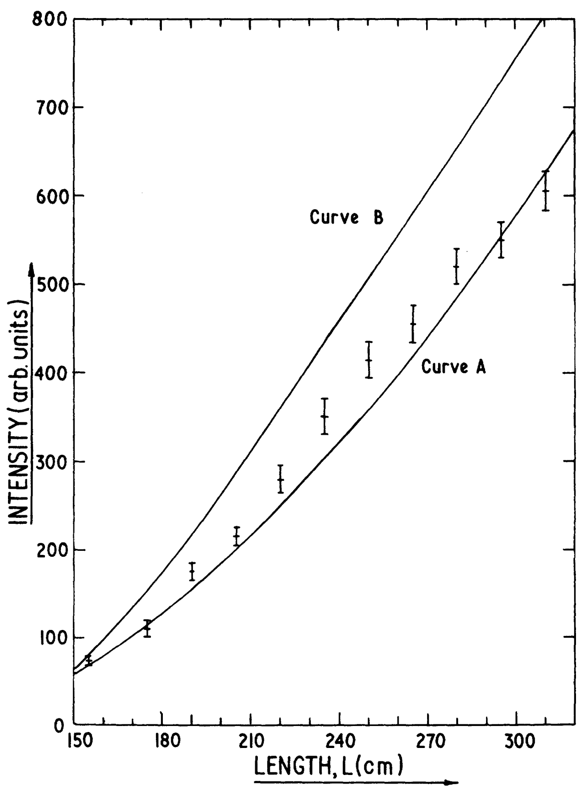

FIG. 7. Theoretical ASE intensities for a medium employing a mirror at one end (curve A) and from a medium of twice the length in the absence of a mirror (curve B) together with the corresponding experimental points, as a function of medium length for the He-Ne system with an inversion density of 15.8 units, a medium diameter of 2.5 mm, and a mirror reflection coefficient of 0.8.

for that part of the ASE beam re-entering the medium and contributing to the emission from the aperture at the opposite end is

$$\delta(x) = \frac{\pi r^2}{(L+2y+x)^2}\,\frac{1}{\Delta\Omega'}.$$

If $J_a'(x)$ is that fraction of the ASE beam which can usefully re-enter the medium then its growth is described by

$$\frac{\partial J_a'(x)}{\partial x} = -\,\frac{K_2 J_a'(x)}{1+K_3[I_a(x)+J_a(x)+I_b(x)]} - K\delta(x)\frac{\Delta\Omega'}{\pi r^2}. \tag{7}$$

The solution of all four equations (4)–(7) for self-consistency with the conditions $I_a(0)=J_a(L)=J_a'(L)=0$ and $I_b(0)=RJ_a'(0)$ allows the system of inverted medium plus mirror to be completely analyzed, where the output is the sum of $I_a(L)$ and $I_b(L)$. In this configuration the spontaneous emission in the region $x=0$ to $x=L_c$ can no longer be ignored as far as that part of the $-ve$ going wave J_a' is concerned. Although it cannot continuously maintain stimulated emission before leaving the medium at $x=0$, it can so do on reentering the medium after reflection.

In practice, the distance between the mirror and the medium y must vary when L is varied because of the necessity to keep the solid angle subtended by the detector, and hence K, a constant. That is, when the medium length is varied, rf clips are connected or disconnected at the mirror end rather than the detector end. The intensities expected when the He-Ne system is excited in the range 155–310 cm in the presence of a mirror is shown (curve A) in Fig. 7, using the values for the constants deduced previously and $R=0.80$. Also shown are the values expected from a medium of twice the length ($-L$ to $+L$), in the absence of any mirror, (curve B) and the experimental points. As can be seen the difference between the two curves becomes less as the medium length is decreased, and for values of $L\approx 150$ cm the difference in the two approaches is ~5% in agreement with the observation previously reported[1] which led us to assume that measurements with a mirror were analyzable within the limits of the then existing theory.

As discussed, only a small part of the ASE beam J_a is reflected back into the medium because of its divergence and this has the effect, as far as geometric considerations are concerned, of making the two situations discussed virtually identical. The angle subtended by the output aperture, viewed via the mirror, at a point x is the same as that which arises from the corresponding position $-x$ in a pure ASE system of length $2L$. Although the photon fluxes in the mirror system are nonsymmetric the saturation process is very similar to that of the pure ASE system. The two situations differ only in two respects: (i)The finite reflectivity of the mirror and the finite distance between mirror and medium ensure that the mirror system is more lossy and (ii) in the pure ASE system the stimulated emission in the $-ve$ direction at $x=0$ would be equal to $J_a'(0)$ but in the mirror system it would have a value $J_a(0)$ at the corresponding position, and clearly $J_a'(0) < J_a(0)$. Although the mirror acts to select only that part $J_a'(0)$ of $J_a(0)$, the saturation effect of $J_a(0)$ is still important and so the saturation effects are greater in the mirror system than in the pure ASE system. Numerical computations comparing the mirror systems with $R=0.80$ and $R=1$ for a range of lengths up to 10^4 cm shows that (i) is the dominant effect and that the small residual difference arises from saturation. Again a system possessing appreciably higher gain would be expected to show stronger saturation effects.

VII. BEAM DIVERGENCE OF ASE

The origins of beam divergence in a typical ASE system have been discussed[1] in terms of diffraction and medium geometry. Many workers, for example, Leonard, Rosenberger, and Egorov and Plekhotkin,[22] have observed the divergence of what were in effect ASE systems to be given approximately by the ratio of geometric width to length d/L of the medium. They did not, however, indicate why this should be nor did they present any detailed measurements. Andronova *et al.*[11] looked specifically at the 3.39-μm transition in He-Ne and found the value of the beam divergence to be very much greater than d/L. This was attributed to tube reflections and while this may be correct it is impossible to know for certain because the precise details of their system and how the divergence was determined are not described. We have shown for the systems investigated that diffraction cannot explain the behavior of beam divergence as a function of medium length or of inversion density, although its effect in the 3.39-μm system is of more or less the same magnitude as that arising from the medium geometry.

Consider a medium of length L and width d, then it is known that a certain critical length has to exist before continuous ASE occurs, and that only spontaneous emission in the region $x=0$ to $x=(L-L_c)$ can act as a source for continuous ASE. Spontaneous emission originating along the medium axis at $x=0$ will give rise to an ASE output with divergence given by d/L while spontaneous emission along the medium axis at $x=(L-L_c)$ will give rise to an ASE output with divergence given by d/L_c, provided $L_c \gg d$. However, as far as intensity is concerned the contributions are not equal because that spontaneous emission originating at $x=0$ has traversed a longer amplifying path than that at $x=(L-L_c)$. So it appears that to obtain the spatial intensity distribution of the ASE output, for a medium of length L, it is necessary to sum the contributions due to all elements across the medium bore and in the region $x=0$ to $x=(L-L_c)$. For each such element the angle into which the spontaneous radiation can be emitted and yet appear at the exit aperture of the medium and the ASE intensity each produces needs to be considered. This has been done and leads to what is described in the rest of this paper as an ASE-geometric theory.

The effects of single or multiple reflections, from the walls of the tube containing the medium, on the divergence of the beam ought to really be considered. These result in radiation traversing the entire length of the medium even when it is radiated into a larger solid angle than previously defined. Thus a larger value for the beam divergence would be expected than that predicted by the ASE-geometric theory, although the precise value would be difficult to predict since it depends upon the nature of the wall reflectivity. If the divergence was governed primarily by diffraction at the aperture of the medium then reflections would be of little importance in determining the value for the divergence.

Figure 8 shows a comparison of the predicted beam divergence using the ASE-geometric theory (curve A) and the diffraction approach (curve B) with that of experiment as a function of length for an inversion density of 15.8 units and medium bore of 2.5 mm. In Fig. 9 the same comparison is made for a given length of 310 cm and bore 2.5 mm as a function of inversion density. The broken line shows the theoretical prediction using Eq. (3) which again shows the insensitivity of He-Ne to the nature of the saturation term. The experimental results appear to follow the general trend of the ASE-geometric approach, although there is a discrepancy of ~1.7 in absolute magnitude, even though theoretically the diffraction approach predicts a larger value for the divergence. However, we have not taken account of reflections in the ASE-geometric approach and presumably the effect would be to increase the predicted value of the divergence above that resulting from diffraction. It has been demonstrated already[1] for pure neon that not only the trend but the absolute magnitude of the beam divergence is correctly predicted by the ASE-geometric approach. Further, we tentatively suggested that the difference between the two systems was a result of reflections from the tube walls, which exhibited different degrees of evenness and correspondingly gave rise to either diffuse (Ne) or specular (He-Ne) reflections.

VIII. SPECTRAL DISTRIBUTION OF ASE

The theory developed so far assumes a very simplified form for the spontaneous-emission spectrum and for the gain profile; whereas both should be Gaussian distributions we have so far assumed a rectangular profile.[1] The theory outlined previously can be extended to include frequency where all terms are defined exactly as before except that they now apply to a particular frequency ν on the spontaneous emission and gain profiles $g(\nu)$. The equations for the rate of increase of intensity at frequency ν and position x in the medium then have the following form:

$$\frac{\partial I_a(\nu,x)}{\partial x}=\frac{K_2\Delta\nu_D g(\nu)I_a(\nu,x)}{1+K_3\Delta\nu_D[I_a(\nu,x)+J_a(\nu,x)]}+\frac{Kg(\nu)\Delta\Omega}{4\pi}, \tag{8}$$

$$\frac{\partial J_a(\nu,x)}{\partial x}=-\frac{K_2\Delta\nu_D g(\nu)J_a(\nu,x)}{1+K_3\Delta\nu_D[I_a(\nu,x)+J_a(\nu,x)]}-\frac{Kg(\nu)\Delta\Omega'}{4\pi}, \tag{9}$$

where the equations have been written in terms of the previously evaluated constants K, K_2, and K_3.

Arguments about the inhomogeneous broadening of the spectral distribution leading to a dip in the ASE linewidth just above threshold are presented elsewhere[1] and remain true since this dip occurs in the region of negligible saturation. However, as our previous predictions concerning the behavior of the linewidth when saturation sets in are shown to be in error, because of the approximation assumed previously, it is necessary for completeness to reiterate some of those arguments.

The fact that a certain medium length has to

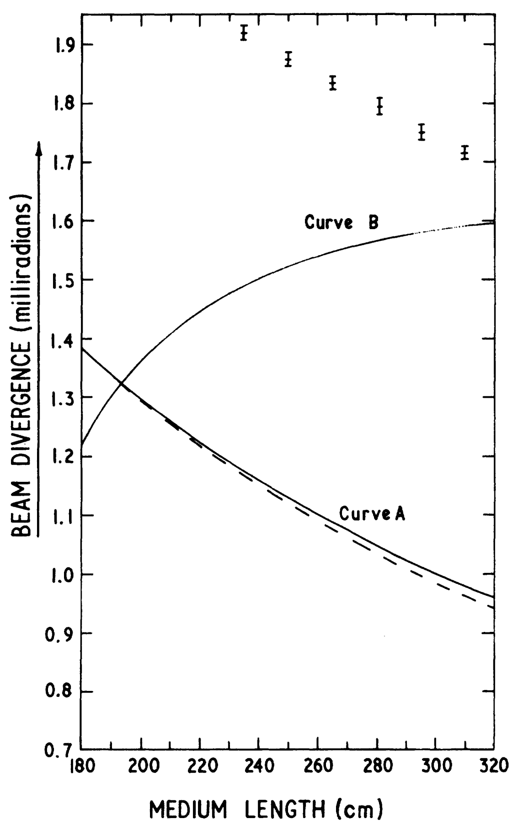

FIG. 8. Theoretical beam divergence according to the ASE-geometric (curve A) and diffraction theories (curve B) together with the experimental points for the He-Ne system as a function of medium length for an inversion density of 15.8 units and a medium diameter of 2.5 mm. The broken line shows the theoretical curve for the approximate form of the saturation term.

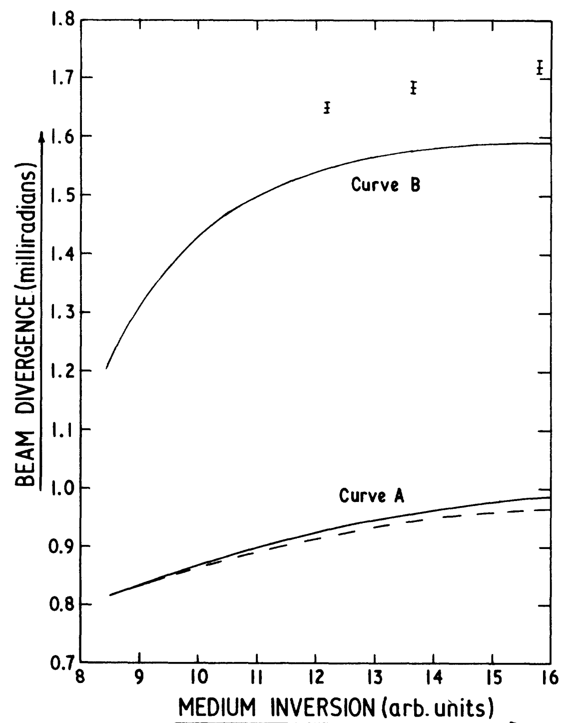

FIG. 9. Theoretical beam divergence according to the ASE-geometric (curve A) and diffraction theories (curve B) together with the experimental points for the He-Ne system as a function of inversion density for a medium length of 310 cm and a diameter of 2.5 mm. The broken line shows the theoretical curve for the approximate form of the saturation term.

exist for a given inversion density before stimulated emission occurs implies that just above the threshold length corresponding to maximum inversion $L_c(\nu_0)$, continuous ASE occurs only in a very narrow spectral region. So the spectral profile of the radiation will follow closely that of the spontaneous emission spectrum except near ν_0 where a small "bump" will occur. As the length of the medium is increased, the appropriate threshold conditions for inversions at frequencies further from the line center are achieved, and the spectral width at the base of the bump increases, as does its peak amplitude. As the peak amplitude of this bump becomes equal to the peak amplitude of the spontaneous emission spectrum, the width at half height of the total radiation is no longer determined by the spontaneous emission profile alone but increasingly by the ASE profile. Only when $L > 2L_c(\nu_0)$ will the spontaneous-emission profile become completely modified over the whole Doppler width of the gain curve. It could be argued that just above threshold the use of the term linewidth for the ASE is dubious and that the dip is a manifestation of a bad definition. However when the ASE is an order of magnitude greater than the background spontaneous emission the concept of the ASE linewidth is perfectly valid and it can be shown that this condition is achieved well within the region of the "dip."

Equations (8) and (9) were solved for the He-Ne system and Fig. 10 shows a plot of relative linewidth against medium length for an inversion of 15.8 units; as can be seen two minima occur. The first is a manifestation of critical length/inhomogeneous broadening and the second is due to competition between the narrowing process involved in convoluting two Gaussian profiles and the broadening process that sets in due to saturation. These two effects can be seen more clearly in the N_2 system previously discussed[1] as shown by Fig. 11. Here the two minima are more widely separated due to the saturation term being much smaller (i.e., the region of experimental growth above L_c is much larger). Further, the location of the second minima is determined by the saturation term rather than the gain term. This is in contrast to our earlier work[1] where we predicted that the linewidth would continue to narrow as the gain is increased, although at a very much slower rate than initially. The difference arises when the condition that $I_a(x) + J_a(x)$ is assumed constant is no longer enforced. In both Figs. 10 and 11 the broken line represents the value deduced from the theory due to Yariv and Leite[9] where saturation is not taken into account.

The over-all rebroadening that occurs is a manifestation of the inhomogeneous nature of the gain curve. An entirely homogeneous line would narrow and, as saturation increased, begin to settle down to a steady narrowed value. In a laser the atoms

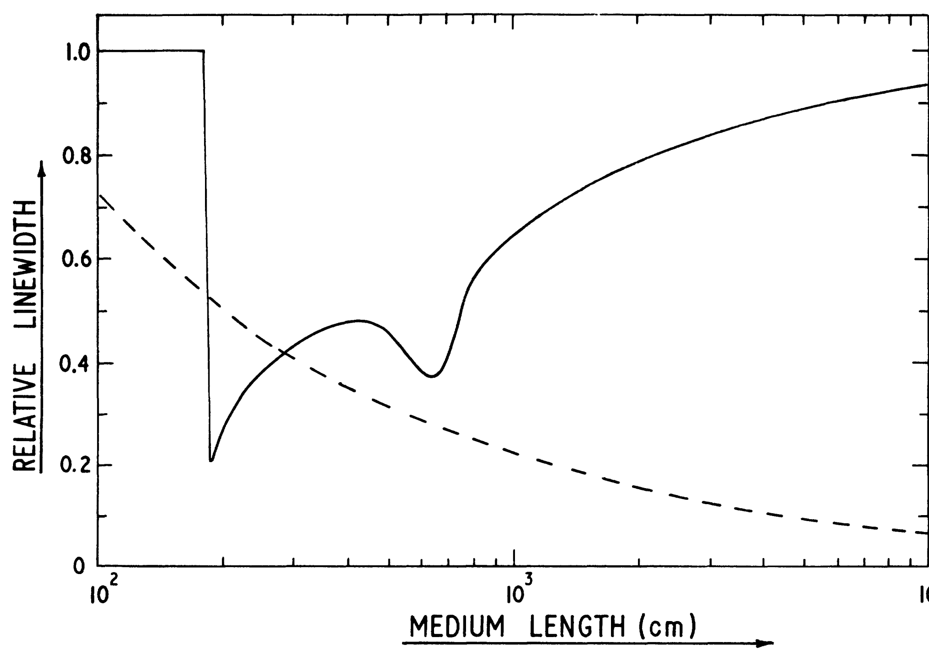

FIG. 10. Theoretical relative linewidth according to the ASE theory (full line) and the Yariv-Leite approach (broken line) for the He-Ne system as a function of medium length for an inversion density 15.8 units and a medium diameter of 2.5 mm.

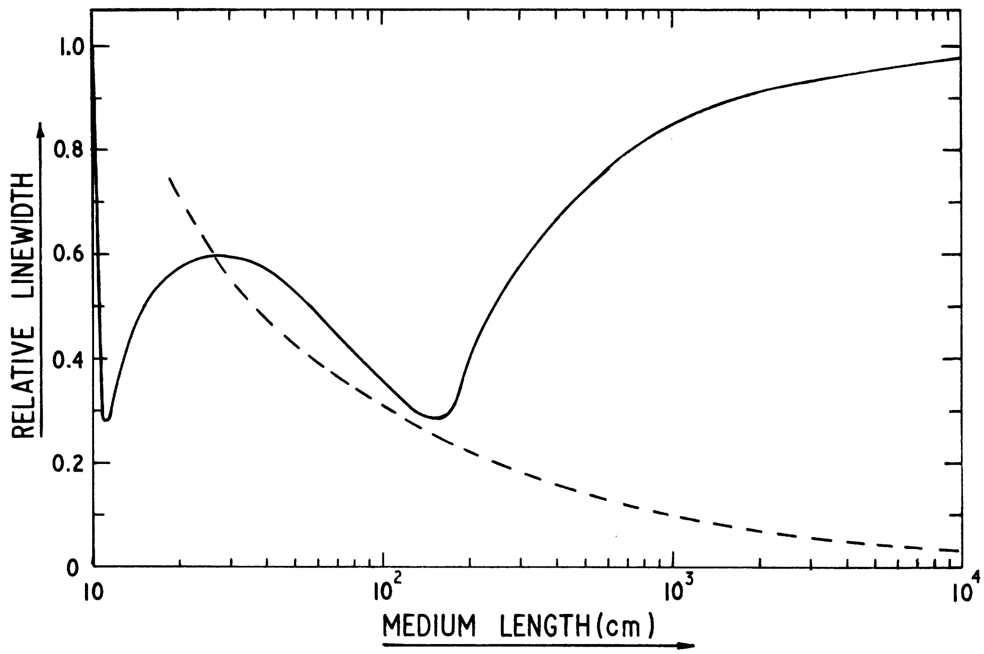

FIG. 11. Theoretical relative linewidth according to the ASE theory (full line) and the Yariv-Leite approach (broken line) for the nitrogen system as a function of medium length.

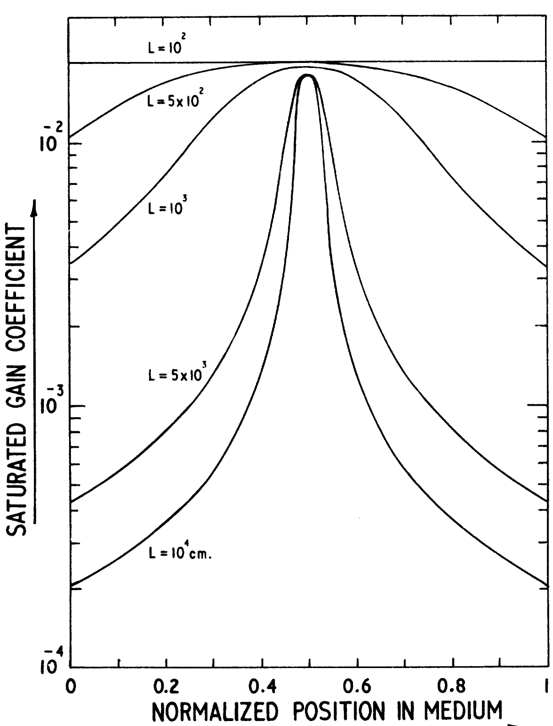

FIG. 12. Theoretical saturated gain coefficient as a function of the normalized position in the medium for the He-Ne system with an inversion density of 15.8 units and a medium diameter of 2.5 mm for various values of medium length.

in resonance with a particular cavity mode are those within a homogeneous linewidth. Thus, although a gas laser may have an inhomogeneously broadened gain curve over-all, and may appear to be in some measure a quasi-infinitely long ASE source, the linewidth of the output radiation is extremely narrow. Strangely, the only experimental observation of rebroadening in an extended medium possessing a population inversion appears to be that of Gamo and Chuang.[23] In all other cases known to us, the line continues to narrow as the gain increases, although without knowledge of the gain-saturation parameters no firm conclusions can be drawn from these observations.

IX. DISTRIBUTION OF INVERSION WITHIN THE ASE MEDIUM

Lang and Bender[4,24] have argued that the saturation within the medium is nonuniform and that the center of an ASE system will have less saturation than at the extremities. This will lead to some sort of "hot spot" at the middle of the medium. For completeness Fig. 12 shows the point-to-point value of the saturated gain coefficient $K_2/\{1+K_3[I_a(x)+J_a(x)]\}$ as a function of normalized position in the medium x/L for various medium lengths. Clearly, they are quite right and the essence of our previous approximation was to ignore this fact. The concept of a hot spot, however, is perhaps a dangerous one. It in no sense provides a region constituting a more important source of radiation than any other point in the medium; it arises as a result of the assumption that the field-free population inversion density in the medium is entirely uniform. Nonetheless it could mean that in the right environment the hot spot might well be directly observable though, clearly, not in laboratory He-Ne.

X. ASE AND INTERSTELLAR OH EMISSION

We have previously[5] attempted to understand some astronomical observations on OH emission in terms of laboratory measurements. It has been found that in no longer making the approximation that $I_a(x)+J_a(x)$ is constant for all x, little difference is to be expected in all our previous results except for the case of the spectral distribution. The effect of saturation in this new approach to ASE leads to an eventual rebroadening of the ASE profile as has been discussed previously by Litvak[14] and Casperson and Yariv[13] for an amplifier, and this leads to the need for a reappraisal of our comments about OH radiation.

It is an experimentally observed fact that the linear size[25] of the OH emission sources is $\sim 10^{15}$ cm and that line narrowing up to a factor of ~5 occurs.[26] Consequently, the OH source must have an inversion density and saturation parameter such that the resultant linewidth is somewhere around the minimum of the linewidth v length plot. If this is the case then by taking just the linear dimensions of the OH source and the observed degree of line narrowing it is possible to make a fairly accurate estimate of the inversion that must exist to give the observational facts.

Rate equations may be written for the particular electronic configuration of the OH system as has been done in this paper for He-Ne. The situation we consider is applicable to the 1720-MHz transition. Here the upper level is populated at rate R_2 and level 1 is not able to decay spontaneously, i.e., $A_1=0$, but is collisionally depopulated at rate R_1 to maintain an inversion. The relevant equations are then

$$\frac{\partial N(\nu,x)}{\partial x}=\frac{GNg(\nu)}{1+P(N+M)}+Hg(\nu)\Delta\Omega\,, \qquad (10)$$

$$\frac{\partial M(\nu,x)}{\partial x}=-\frac{GMg(\nu)}{1+P(N+M)}-Hg(\nu)\Delta\Omega'\,, \qquad (11)$$

where

$$G=\frac{n_0R_2\sigma\Delta\nu_D(1-A/R_1)}{A+A_2},$$

$$P=\frac{R_1+A_2}{A+A_2}\frac{\sigma\Delta\nu_D}{R_1a},$$

$$H=\frac{n_0R_2aA}{(A+A_2)4\pi}.$$

The spontaneous-emission rates are given by Cook,[27] the cross section σ by Allen and Peters,[1] and the Doppler width by Johnston.[28] The quantity $n_0R_2(1-A/R_1)/(A+A_2)$ in the coefficient G is equivalent to the population inversion. If the rate equations are solved for this system in the limit that $(N+M)$ is zero, which corresponds to a medium excited in the same way as the one being discussed but whose length is below the critical length, it immediately follows that $A/R_1<1$. The saturation

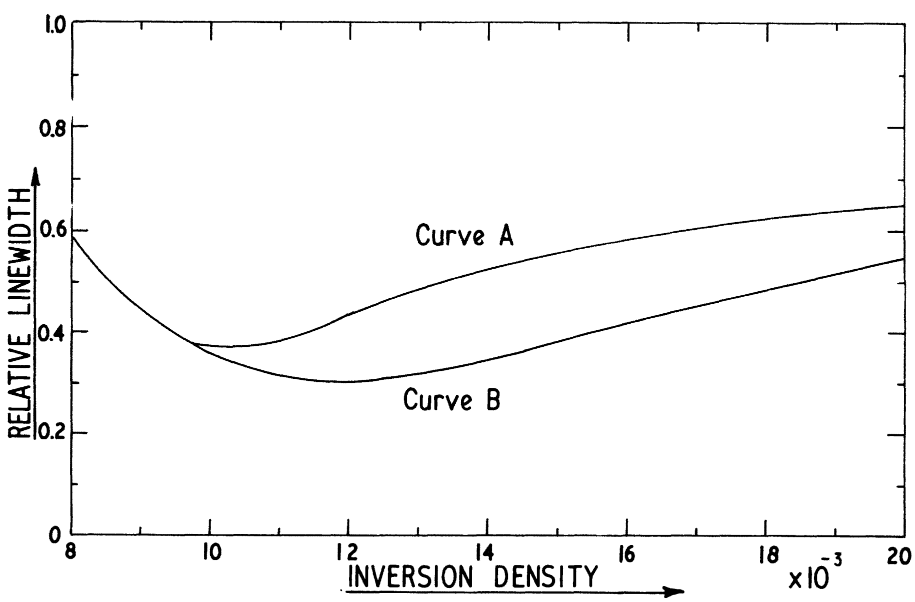

FIG. 13. Theoretical relative linewidth as a function of inversion density for the interstellar OH source assuming dimensions $\sim 10^{15}$ cm, for maximum saturation parameter (curve A) and minimum saturation parameter (curve B).

parameter P depends upon R_1 and must be carefully examined. If $R_1 \gg A_2$, then P is simply $\sigma\Delta\nu_D/(A+A_2)a$. However, if $R_1 \lesssim A_2$ then, as $R_1 > A$, the range of permissible values of R_1 is well defined. However, these inequalities imply $A_2 > A$ and in those systems where this is not the case, $R_1 > A > A_2$ is obviously the only case. In fact, for the 1720-MHz transition under consideration, $A_2 > A$ with $A \approx 10^{-11}$ sec^{-1} and $A_2 \approx 8\times 10^{-11}$ sec^{-1}, which means R_1 must lie between these values. This implies that whether R_1 is greater or less than A_2 the value of P must lie in the range 10^{-32} and 10^{-33}.

It should be noted that the numerical value of the radius in the terms $\Delta\Omega$ and $\Delta\Omega'$ is the same as that of the length of the medium. Thus, saturation due to ASE orthogonal to that observed on earth is not taken into account. but in anv case it will have a relatively little effect on the result for the probable inversion density since the resultant saturation will still lie within the range investigated.

The inversion density cannot be less than ~$5\times 10^{-3}/\mathrm{cm}^3$ or the critical length would be greater than the linear dimensions of the OH source and no continuous stimulated emission could occur. Figure 13 shows the relative linewidth as a function of inversion density for a medium length of 10^{15} cm for both the maximum saturation value (curve A) and the minimum saturation value (curve B) discussed above. Critical length/inhomogeneous broadening effects which cause the first minimum in both Figs. 10 and 11 have been ignored since this occurs in the nonsaturation region. As can be seen, curve B seems to agree better with the experimentally observed line narrowing although to obtain a really accurate description of OH emission all four transitions in the Λ doublet and their interaction would need to be considered. For instance, in the above treatment stimulated emission on other transitions involving the energy levels has been ignored. From the graph the probable inversion density existing in the interstellar medium can be estimated to be ~$1.2-1.4\times 10^{-2}/\mathrm{cm}^3$. The corresponding value of the saturation term $P(N+M)$ at the ends of the medium for the spectral line center is ~0.3, i.e., 30% of the denominator. It is not clear whether this degree of saturation is sufficient to account for the observed degree of polarization.

Although we were in error in some of our previous comments about line narrowing and OH radiation from the interstellar medium, much of what was written remains true and makes a real, if small, contribution to the understanding of the phenomenon. For example, the concept of critical length indicates that the radiation is not very highly saturated and explains also why $O^{18}H$ stimulated emission has not been observed. Secondly, the threshold value for injected photons necessary to perturb an ASE system may be determined from

$$N_b(0)_{\mathrm{Thres}} = \int_0^{L-L_c} H\Delta\Omega\,dx\,.$$

Consequently, if the observed radiation was due to a population inversion amplifying an external signal from elsewhere in space, the signal would have to be greater than 50 photons $\mathrm{cm}^{-2}\,\mathrm{sec}^{-1}$.

Evans *et al.*[29] have examined the statistics from astronomical masers and show that there is no departure from the Bose-Einstein statistics to be expected from a source of thermal radiation. They argue that for an amplifying medium without saturation the statistics should remain as those of Bose and Einstein, but that as saturation grows the statistics should become increasingly Poisson in form. If this statement is correct then their measurements suggest that the OH system has a low degree of saturation which would be in agreement with our calculations. However, in some measurements of the statistics of a laboratory system, although we believe they should be viewed with caution, Gamo and Chuang[23] have shown that there is a deviation from Bose-Einstein statistics even in the nonsaturation region. They also show that the statistics for a unidirectional ASE source tends more to Poisson than does a bidirectional source. This may seem strange at first sight because it would be anticipated that the degree of saturation in a bidirectional source is greater than in a unidirectional one, because of the two waves, and hence the statistics more Poisson. However, distinction must be made between self-saturation and mutual saturation. By removing the $-ve$-going wave, mutual saturation with the $+ve$-going wave is reduced in the region around $x \sim 0$. This, however, allows the intensity of the $+ve$ wave to grow above that expected in a bidirectional system and consequently the self-saturation increases in the region around $x \sim L$. Evidently, self-saturation is more important than mutual saturation (as has been seen earlier in this paper). The implication of this for the interstellar medium results of Evans *et al.* appears to be that OH emission arises from internally generated spontaneous emission rather than by amplifying an external source. Possibly we shall be in a better position to interpret observations of OH radiation from the interstellar medium after experiments on intensity fluctuations in ASE sources currently being carried out in this laboratory are completed. At this time no one has interpreted the coherence properties of such radiation in terms of the parameters involved, although we are optimistic of doing so.

Possibly some detailed laboratory work on the relevant transitions of the OH system itself may not be far off. Kasuya and Shimoda[30] have observed resonant absorption by the free OH radical using an H_2O laser, while Ducas *et al.*[31] have observed laser oscillation in pure rotational transitions of OH and OD free radicals although not at the wavelength of interest.

XI. CONCLUSIONS

It should be emphasized that the theory was developed for a steady state and is applicable to cw systems after transients have died out. It is applicable to pulsed systems if the amplification of the external signal and the generation of ASE occur simultaneously, assuming that the small signal inversion density is essentially constant during the time to travel along the length of the medium. However, the case of one mirror is suspect for a pulsed system in view of the fact that the "amplified input signal" is only available after ASE has occurred. The ASE output J_a that is reflected back into the medium is not in practice necessarily modified by this reflection, whereas our computation implicitly assumes the modification has been made.

As we have argued elsewhere[1] although propagation through an amplifying medium must in certain circumstances be described in terms of a coherent interaction between the field and the atoms, such a description does not appear to be necessary when describing an ASE system of the He-Ne kind. As rate equations have been used it has not been possible to take into account any coherent interaction. Clearly, a more rigorous theory than that outlined in this work could be generated using, for example, either the semiclassical laser-theory approach of Lamb[16] or perhaps by use of a master-equation technique.[32] In the case of the latter appropriate correlation functions should be calculable which would describe the state of the temporal and spatial coherence of a system as functions of the various parameters.

In pulse systems after a certain path length has been traversed the "area" of the pulse could in principle be large enough to act as a θ pulse.[2] In this case self-induced transparency[33] and even Dicke's superradiance effects[2] could begin to be important. This would be impossible in a cw system because the nature of the continuous excitation will destroy the phase relationships between dipole moments which must be achieved for collective effects to be observed. However, even in a pulse system the source of photons is the spontaneous emission which goes into the solid angle delineated by the geometry of the system. Consequently, a rigorous theory will have to take into account the finite relaxation times involved.

Although it is clear that our theory is a primitive one it appears capable of demonstrating the essential features that a good theory should have. These are the concept of critical length, the presence of $+ve$- and $-ve$-going waves and the need to take careful account of how much spontaneous emission can contribute to the amplifying field.

Although we have demonstrated in this paper that our assumption making the total photon flux at any point in the medium a constant is strictly invalid in general, it transpires that only the prediction for the spectral distribution at high saturation has been in error. However, a more accurate solution to the problem has been shown to be available and problems such as the linewidth of OH transitions in the interstellar medium have now been tackled more carefully without having to make an unsatisfactory approximation.

Note added in proof: Since submitting this paper by H. Maeda and A. Yariv [Phys. Lett. 43A, 383 (1973)] criticizing our earlier conclusions with respect to line narrowing and correctly attributing our error to the improper assumption that $I_a(x)+J_a(x)=\text{constant}$; they are, of course, quite right. As we have discussed in this paper, it is not necessary to make that assumption and our prediction with respect to linewidth in the region of saturation is now in agreement with theirs. However, we do not agree with their stricture that the concept of a critical length is not necessary. The exact meaning and the pertinence of the concept will be of some concern in a later paper.

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