PHYSICAL REVIEW A VOLUME 8, NUMBER 4 OCTOBER 1973

Acoustoelectric Amplification—A Phonon-Laser Approach: Theory of a Single Acoustic Mode*

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(Received 22 February 1973)

Acoustoelectric amplification in piezoelectric semiconductors can be profitably treated as the acoustic analog of a classical laser. The nonlinear acoustic gain and frequency pulling can be found by combining Maxwell's equations with the piezoelectric equations of state and solving the resulting system to third order in a perturbation expansion. Recent experimental findings (acoustic mode locking, etc.) encourage the use of such methods, which produce results analogous to those of the Lamb theory of a classical laser. We here present solutions for the case of a single acoustic mode. Diffusion effects are found to be important, especially near threshold.

I. INTRODUCTION

An ultrasonic wave traveling in a piezoelectric semiconductor can be amplified by applying a strong electric field along the direction of propagation.[1] The acoustic wave creates lattice stresses that result in local ac modulations of the electric field. The latter cause a similar modulation of the electron density ("bunching"). When the dc drift velocity of the electrons v_d exceeds the velocity of sound v_s, the electron wave will give up energy to the lattice wave. This situation has been likened to laser action[2,3] in which the acoustic wave is augmented by the stimulated emission of phonons. In this analogy the supersonic electrons represent a population inversion.

A linear theory of the acoustoelectric effect was developed by White,[4] who coupled Maxwell's equations with the piezoelectric equations of state. Nonlinear theories of increasing complexity have been offered by a variety of authors. For example, Butcher and Ogg[5] introduce the steady-state current as an *ad hoc* parameter, and assume a sinusoidal space-charge distribution. Computer calculations by Tien[6] show, however, that once the acoustic amplitude becomes large numerous higher harmonics inevitably appear in the carrier distribution; harmonics of the acoustic wave are found to be far less important. Ridley and Wilkinson[7] use Krylov-Bogoliubov techniques to derive expressions for the high-field domains observed in CdS bar experiments.[8]

In this paper we offer a calculation which carries the laser approach to acoustoelectric amplification to its logical conclusion. We recast White's equations[4] in the form of the classical theory of the laser[9] and solve these to third order in a perturbation expansion. This procedure is particularly useful in obtaining, for example, steady-state power, frequency pulling and mode-locking in a simple fashion.

The experimental literature on acoustoelectric amplification is voluminous indeed. Early experiments used piezoelectric semiconductors like CdS[8,10] because of the strong coupling between electric and elastic waves (and thus local carrier concentrations and lattice deformations) in such materials. Bismuth was also suggested[2,11] and tried.[12] Most other materials have weak acoustic-electrical coupling and require prohibitively large carrier concentrations and current densities for acoustic amplification.

Observations on CdS bars[8,10] revealed the existence of narrow traveling domains of high-field intensity, corresponding to a wide range of amplified frequencies. More recently, experiments with thin CdS platelets (0.3 mm) have limited the output to a few discrete modes,[13] separated by the fundamental frequency of the plate thickness. These modes are analogous to the cavity modes of a laser; and they participate in mode locking, as has recently been observed.[14] Such experimental results provide considerable motivation for a theoretical investigation of acoustoelectric amplification from a laser point of view.

II. BASIC EQUATIONS

We begin by finding self-consistent solutions for the acoustic and electric fields $\vec{U}(x,t)$, $\vec{E}(x,t)$ in a piezoelectric semiconductor subjected to a constant dc field E_0 along its x axis (Fig. 1). $\vec{U}(x,t)$ represents a plane-wave displacement of lattice ions which propagates down the x axis. $\vec{U}(x)$ is perpendicular to x for a shear wave, and along x for a longitudinal wave, whereas $\vec{E}(x)$ points along the x axis. With these spatial conventions in mind, $\vec{U}(x)$ and $\vec{E}(x)$ are treated as scalar quantities hereafter, and the vector signs omitted. Various boundary conditions can be applied at the ends of the bar ($x=0, L$). We can use transducers and circuitry to transfer the signal intact from $x=L$ to $x=0$. This corresponds to a ring laser. Another approach is to allow the signal to bounce back and forth between the sample walls. This corresponds to a laser oscillator. There is an important difference between the two cases. In the former the acoustic wave always propagates in the same direction ($+x$) and is continually amplified. In the latter the sound wave travels alternately along and opposite to the electron flow. As will be shown, the acoustic wave suffers losses on the backward half of its journey, i.e., the medium is nonreciprocal. This is quite different from normal laser operation; however, if the backward loss is small enough, one can consider the sample to contain a standing wave to a good approximation. This is often true in experiments involving thin plates of CdS; and in what follows we shall emphasize the semi-standing-wave formulation.

A. Phenomenological Equations

The acoustoelectrical coupling in a piezoelectric medium is represented empirically by equations of state[4] which relate the stress T, acoustic field U, electric displacement D, and electric field E:

$$T = cU' - eE, \tag{1}$$

$$D = eU' + \epsilon E, \tag{2}$$

where $'$ represents $\partial/\partial x$, c is the elastic constant, e is the piezoelectric coupling constant, and ϵ is the dielectric constant. The electric field must also satisfy the wave equation, the continuity equation, and Gauss's law:

$$T' = \rho \ddot{U}, \tag{3}$$

$$J' = q\dot{N}, \tag{4}$$

$$D' = -q(n_c - n_0)/f = -qN, \tag{5}$$

where $\cdot$ represents $\partial/\partial t$, $\rho(x)$ is the crystal mass density, $J(x)$ is the electric current density, $N(x)$ is the electron excess density, and q is the electronic charge (absolute magnitude). Of course, not all the excess space charge $N(x)$ in a given region represents electrons in the conduction band; a fraction $(1-f)$ of these electrons are trapped in the energy gap of the semiconductor.[4] Thus in Eq. (5) the expression $fN = (n_c - n_0)$ is used to represent the deviation of the *conduction* electron density n_c from its equilibrium value n_0.

We now assume that Ohm's law holds locally, i.e., the electron mean-free path $l_e \ll$ the acoustic wavelength λ. Then the net current density J resulting from the local field E and spatial diffusion is given by

$$J = q(n_0 + fN)\mu E + q\mathcal{D}_n N', \tag{6}$$

where μ is the mobility and $\mathcal{D}_n$ is the electron-diffusion constant. Taking the time derivative of Eq. (5) and using Eqs. (4) and (6), we find

$$\dot{D}' = -J' = -\mu q[(n_0 + fN)E]' - q\mathcal{D}_n N''. \tag{7}$$

N is eliminated by using Gauss's law (5), and Eq. (7) becomes

$$-\dot{D}' = \frac{\partial}{\partial x}\,\mu(qn_0E - fD'E) - f\mathcal{D}_n D'''. \tag{8}$$

This equation is important and will hereafter be referred to as the "matter equation."[15] Using Eq. (2) to eliminate D gives E in terms of U.

The other important equation is the driven wave equation which follows from (1) and (3):

$$T' = cU'' - eE' = \rho\ddot{U}. \tag{9}$$

Expressing this as

$$\ddot{U} - (c/\rho)\,U'' = -(e/\rho)\,E', \tag{10}$$

makes clear the role of E' as a source term. It occupies the same position as the polarization P in the wave equation for the classical laser.[9]

B. Phase - Amplitude Equations

We now convert the wave equation into the phase-amplitude form used by Lamb[9] to analyze the gain and frequency pulling of a laser. The acoustic field is decomposed into traveling-wave modes:

$$U(x,t) = \sum_{n=-\infty}^{+\infty} U_n(x,t) = \sum_{n=-\infty}^{+\infty} U_n(t)\sin(k_n x - \omega_n t), \tag{11}$$

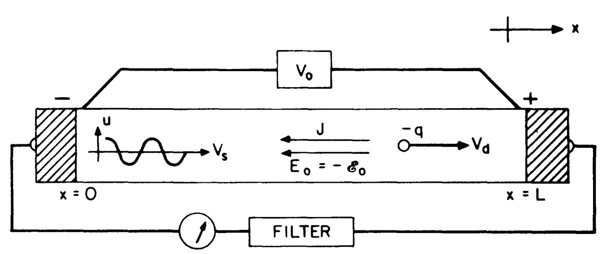

FIG. 1. Experimental setup to study acoustoelectric amplification in a long CdS bar. A constant voltage source V_0 creates a dc field $E_0 = -\mathcal{E}_0$ in the sample. This causes the electrons (charge $-q$) to travel with a drift velocity v_d in the x direction, creating a current J. Transducers transfer the acoustic signal U from $x=L$ to $x=0$ to give ring-laser-like boundary conditions. The filter restricts amplification to a single acoustic mode. Alternately, we would allow the wave to reflect back and forth. Waves traveling in the $+x$ direction will be amplified for $v_d > v_s$; those traveling in the $-x$ direction will be attenuated. If the loss for the backward trip is small, a standing wave can be approximated.

where the phase has been chosen to permit the eventual formation of standing waves upon applying oscillator boundary conditions. All variables are real, $k_n = k_{-n}$, $\omega_n = -\omega_{-n}$, and $U_0(t) = 0$; the effects of amplification are included in the time-dependent amplitudes $U_n(t)$. The wave vector k_n is fixed, while ω_n must be determined by the equations that follow; this choice of procedure differs somewhat[16] from that of White.[4] Similarly,

$$E(x,t) = \sum_{n=-\infty}^{+\infty} E_n(x,t) = \sum_{n=-\infty}^{+\infty} [S_n \sin(k_n x - \omega_n t) + C_n \cos(k_n x - \omega_n t)], \quad (12)$$

where the phase relationship to $U_n(x,t)$ can not be pre-assumed. $N(x,t)$ can be similarly decomposed.

When Eq. (11) is substituted in (10) a linear second-order differential equation results that can be separated into a set of equations for each Fourier component $U_n(t)$:

$$\ddot{U}_n(x,t) + \frac{\omega_n}{Q_n}\dot{U}_n(x,t) - \frac{c}{\rho}U_n''(x,t) = -\frac{e}{\rho}E_n'(x,t), \quad (13)$$

where a phenomenological damping term has been added to represent the cavity losses Q_n for each mode. We then use the explicit forms of Eq. (11) and make the slowly varying phase and amplitude approximation, i.e., $U_n(t)$ and ω_n do not change appreciably in a single cycle. Thus $\dot{U}_n(t)$ is small compared to $\omega_n U_n(t)$, and $\ddot{U}_n(t)$ is negligible compared to $\omega_n^2 U_n(t)$. We also assume that ω_n/Q_n is small, and thus neglect $(\omega_n/Q_n)\dot{U}_n(t)$. Upon taking $\int_0^{2\pi} \cos(k_n x - \omega_n t)$ and $\int_0^{2\pi} \sin(k_n x - \omega_n t)$ of both sides of (13), we obtain the two phase-amplitude equations

$$\dot{U}_n(t) + (\omega_n/Q_n)U_n(t) = +(ek_n/2\Omega_n\rho)S_n(t), \quad (14)$$

$$(\omega_n - \Omega_n)U_n(t) = -(ek_n/2\Omega_n\rho)C_n(t), \quad (15)$$

where Ω_n is the unpulled acoustic frequency $(c/\rho)^{1/2}k_n$. The frequency shift during operation is assumed small so that $(\omega_n + \Omega_n) \sim 2\Omega_n$ in (15) and Ω_n can be substituted for ω_n on the right-hand side of (14). These equations are equivalent to the phase-amplitude equations of a classical laser,[9] with U_n replacing E_n, and E_n replacing P_n. As in that case, the problem is reduced to finding the source terms, i.e., to solving the matter equation (8) for S_n and C_n in terms of U_n.

In Secs. III–V, we restrict our attention to a single forward-traveling acoustic mode and drop the mode subscript n; thus

$$U(x,t) = U(t)\sin(kx - \omega t). \quad (16)$$

This could be achieved in practice through the use of appropriate external filters (traveling wave-boundary conditions) or thin semiconductor samples (oscillator boundary conditions). No restrictions are placed on the possibility of higher harmonics in $E(x,t)$ and $N(x,t)$, in accordance with the results of computer calculations by Tien.[6] The following calculation will be expanded to the case of several *acoustic* modes in a future publication.[17]

III. LINEAR THEORY

A. Linear Equations

Following the procedure used by Lamb[9] for the classical laser, we expand $E(x,t)$ and $U(x,t)$ in a perturbation series:

$$E(x,t) = \hat{E}_0 + \hat{E}_1(x,t) + \hat{E}_2(x,t) + \hat{E}_3(x,t) + \cdots, \quad (17)$$

$$U(x,t) = \hat{U}_0 + \hat{U}_1(x,t) + \hat{U}_2(x,t) + \hat{U}_3(x,t) + \cdots, \quad (18)$$

where the subscripts now refer to the *order* of the term in the expansion, and $\hat{E}_3 \ll \hat{E}_2 \ll \hat{E}_1$. Boundary conditions require $\hat{E}_0 = -\mathcal{E}_0$ and $\hat{U}_0 = 0$ (Fig. 1). A look at the matter equation (8) shows that nonlinearities arise only from the term $fD'E$.

Keeping only the first-order terms gives the matter equation for *linear* theory:

$$\frac{d}{dt}\hat{D}_1' = -\mu\frac{\partial}{\partial x}[qn_0\hat{E}_1 - f\hat{E}_0\hat{D}_1'] + f\mathfrak{D}_n\hat{D}_1''', \quad (19)$$

where Eq. (2) expresses $\hat{D}_1$ in terms of $\hat{E}_1$ and $\hat{U}_1$. From (16) and (19) we see that the appropriate expressions for $\hat{U}_1$ and $\hat{E}_1$ are

$$\hat{U}_1(x,t) = U_1(t)\sin(kx - \omega t), \quad (20)$$

$$\hat{E}_1(x,t) = S_1(t)\sin(kx - \omega t) + C(t)\cos(kx - \omega t). \quad (21)$$

Substituting these into (19) and dividing the results into sinusoidal and cosinusoidal parts by integration as before gives

$$\Delta S_1 + (\omega_c + h)C_1 + hbU_1 = 0, \quad (22)$$

$$\Delta C_1 - (\omega_c + h)S_1 + \Delta bU_1 = 0, \quad (23)$$

where several new parameters have been introduced.

The dielectric relaxation frequency $\omega_c \equiv qn_0\mu/\epsilon$ is a measure of the time required to restore equilibrium in a perturbed electron distribution. A frequency $\omega > \omega_c$ is required if acoustic deformations are to create measurable piezoelectric effects, which explains the use of high-resistivity (low μn_0) samples in experimental work.[8,10,13,14] The diffusion analog of ω_c is $h \equiv f k^2 \mathfrak{D}_n$, which represents the smoothing out of the electron spatial distribution by diffusion of electrons from regions of high concentration[17]; this decreases the effectiveness of the electron-bunching process and the acoustoelectric interaction. The diffusion frequency $\omega_D \equiv \omega^2/h$.

The quantity $\Delta = (f k v_d - \omega) \equiv k(f v_d - v_s)$ resembles both the inversion and detuning parameters of a laser oscillator. It is more natural to the spirit of our calculation than the parameter $\gamma = 1 - f(v_d/v_s) = -\Delta/\omega$ used by White.[4] Here $v_s \equiv \omega/k$ is the self-consistently determined velocity of the sound wave, which may differ somewhat from $(c/\rho)^{1/2}$. The electron drift velocity $v_d \equiv \mu\,\mathcal{E}_0 = -\mu E_0$ defines the $+x$ direction. Finally, the constant $b = ek/\epsilon$ is useful, since bU has the same units as E.

B. Linear Results

Solving (22) and (23) and substituting in the phase-amplitude equations (14) and (15) gives the familiar results[4,16] for the linear gain α, and the operating frequency ω (Fig. 2):

$$\alpha = \tfrac{1}{2}\kappa^2\Omega\,\frac{\Delta\omega_c}{\Delta^2 + (\omega_c + h)^2}\,, \tag{24}$$

$$(\omega - \Omega) = \tfrac{1}{2}\kappa^2\Omega\,\frac{\Delta^2 + h(\omega_c + h)}{\Delta^2 + (\omega_c + h)^2}\,, \tag{25}$$

where we have used $\Omega = (c/\rho)^{1/2}k \simeq \omega$ and $\kappa^2 = e^2/\epsilon c$ on the right-hand sides of (24) and (25). For CdS, the electromechanical coupling constant $\kappa^2 \simeq 0.018$. The linear gain reaches its maximum value for operation at the frequency $\omega_{max} \equiv (\omega_c\omega_D)^{1/2}$ in the absence of special boundary conditions or externally imposed losses.[4]

These expressions are formally similar to those obtained for a classical laser,[9] especially for $h \to 0$. Gain exists for $\Delta > 0$, or for Δ greater than some threshold Δ_T if the cavity loss $-\omega/2Q \neq 0$. The frequency ω is pushed toward the value $\Omega(1 + \frac{1}{2}\kappa^2)$. The Lorentzian factors are reminiscent of $L(\omega_0 - \omega) = 1/[(\omega_0 - \omega)^2 + \gamma_{ab}^2]$, which appears in the laser formulation, where $\omega_0 = (\epsilon_a - \epsilon_b)/\hbar$ is the frequency difference between the upper (ϵ_a) and lower (ϵ_b) energy states of the two-level atoms, and γ_{ab} is a phenomenological decay constant characterizing the transition. This suggests the correspondence

$$(\omega_0 - \omega) \leftrightarrow \Delta = (f v_d - v_s)k, \quad \omega_c \leftrightarrow \gamma_{ab}; \tag{26}$$

however, Δ is more than the detuning parameter: it represents the source of the "population inversion" as well. (This is described from a different point of view in Ref. 2).

IV. NONLINEAR THEORY

A. The Matter Equation

The nonlinear theory proceeds by solving the matter equation (8) for increasing orders of the perturbation expansion [Eqs. (17) and (18)]: first order:

$$\frac{d}{dt}\hat{D}_1' = -\mu\frac{\partial}{\partial x}\,[qn_0\hat{E}_1 - f\hat{E}_0\hat{D}_1'] + f\mathfrak{D}_n\hat{D}_1'''; \tag{27}$$

second order:

$$\frac{d}{dt}\hat{D}_2' = -\mu\frac{\partial}{\partial x}\,[qn_0\hat{E}_2 - f\hat{E}_0\hat{D}_2' - f\hat{E}_0\hat{D}_1'] + f\mathfrak{D}_n\hat{D}_2''';$$

third order: (28)

$$\frac{d}{dt}\hat{D}_3' = -\mu\frac{\partial}{\partial x}\,[qn_0\hat{E}_3 - f\hat{E}_0\hat{D}_3' - f\hat{E}_1\hat{D}_2' - f\hat{E}_2\hat{D}_1'] + f\mathfrak{D}_n\hat{D}_3'''. \tag{29}$$

Actually these equations are valid for any number of acoustic modes, and the nonlinear bracketed terms of Eqs. (28) and (29) mix the various modes of the acoustic field in the general case. We first

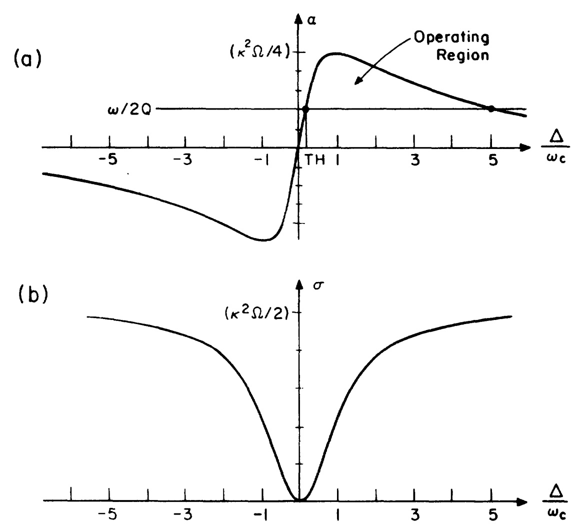

FIG. 2. (a) Linear gain α for negligible diffusion ($h \simeq 0$). For electron drift velocities v_d exceeding the velocity of sound v_s the inversion parameter $\Delta = (f v_d - v_s)k > 0$, and the acoustic wave can be amplified. However, finite cavity losses $(\omega/2Q) \neq 0$ may increase the threshold above $\Delta = 0$. For CdS, $\frac{1}{2}\kappa^2 = 0.018$; the dielectric relaxation frequency $\omega_c = qn_0\mu/\epsilon$ depends on the conductivity of the sample. (b) First-order shift $\sigma = \omega - \Omega$ in the operating frequency ω ($h = 0$). For a fixed wave vector k, this corresponds to the velocity of sound increasing away from $\Delta = 0$; the maximum operating frequency is $\Omega(1 + \frac{1}{2}\kappa^2)$, where $\Omega = (c/\rho)^{1/2}k$.

consider the second-order matter equation (28) for a single acoustic mode.

B. Second - Order Calculation

The second-order terms of (28) correspond to products of the first-order forms; by simple trigonometry these can be expressed in terms of the equivalent functions:

$$\sin(2kx-2\omega t),\quad \cos(2kx-2\omega t),\quad 1. \tag{30}$$

Since translations of the whole rod are excluded, and since acoustic waves of all frequencies except ω are suppressed in this single-mode calculation, $\hat{U}_2\equiv 0$. However, no such restriction exists on

$$\hat{E}_2(x,t)\equiv S_2(t)\sin(2kx-2\omega t)+C_2(t)\cos(2kx-2\omega t). \tag{31}$$

The dc electric field is E_0 by definition.

Using these forms, the second-order matter equation

$$\epsilon\frac{d}{dt}\hat{E}_2'=-\omega_c\epsilon\hat{E}_2'-fv_d\epsilon\hat{E}_2''+\mu f\hat{E}_1'(e\hat{U}_1''+\epsilon\hat{E}_1')+\mu f\hat{E}_1(e\hat{U}_1'''+\epsilon\hat{E}_1'')+f\mathfrak{D}_n\epsilon\hat{E}_2''' \tag{32}$$

has the solutions

$$S_2=\frac{\mu fk}{2}\,\frac{2\Delta(bUS_1+2C_1S_1)-(\omega_c+4h)(bU_1C_1-S_1^2+C_1^2)}{4\Delta^2+(\omega_c+h)^2}, \tag{33}$$

$$C_2=\frac{\mu fk}{2}\,\frac{2\Delta(bU_1C_1-S_1^2+C_1^2)+(\omega_c+4h)(bU_1S_1+2C_1S_1)}{4\Delta^2+(\omega_c+h)^2}. \tag{34}$$

C. Third - Order Calculations

Proceeding to third order (29) yields the matter equation

$$e\frac{d}{dt}\hat{U}_3''+\epsilon\frac{d}{dt}\hat{E}_3'=-\epsilon\omega_c\hat{E}_3'-fv_d(e\hat{U}_3'''+\epsilon\hat{E}_3'')+\mu f\epsilon(\hat{E}_1'\hat{E}_2'+\hat{E}_1\hat{E}_2'')+\mu f\hat{E}_2'(e\hat{U}_1''+\epsilon\hat{E}_1')+\mu f\hat{E}_2(e\hat{U}_1'''+\epsilon\hat{E}_1'')+f\mathfrak{D}_n(e\hat{U}_3''''+\epsilon\hat{E}_3''') \tag{35}$$

since $\hat{U}_2=0$ for this single-acoustic-mode calculation. The nonlinear terms consist of products with amplitudes $\hat{E}_2'\hat{U}_1$ and $\hat{E}_2'\hat{E}_1$, etc., and thus may be represented in terms of a set of functions:

$$\sin(3kx-3\omega t),\quad \cos(3kx-3\omega t),\quad \sin(kx-\omega t),\quad \cos(kx-\omega t). \tag{36}$$

Since

$$\int_0^{2\pi}\sin(3\theta)\cdot\sin(\theta)\,d\theta=0,$$

$$\int_0^{2\pi}\sin(\theta)\cdot\sin(\theta)\,d\theta=\pi,$$

etc., only those third-order terms which oscillate like the last two functions of (36) will eventually contribute to the phase-amplitude equations (14) and (15). Thus we need only consider the third-order forms

$$\hat{U}_3(x,t)=U_3(t)\sin(kx-\omega t), \tag{37}$$

$$\hat{E}_3(x,t)=S_3(t)\sin(kx-\omega t)+C_3(t)\cos(kx-\omega t). \tag{38}$$

Substituting in (35) eventually yields

$$S=\frac{(\Delta\omega_c)bU_3+(\tfrac{1}{2}\mu fk)[M\Delta-N(\omega_c+h)]}{\Delta^2+(\omega_c+h)^2}, \tag{39}$$

$$C=\frac{-[\Delta^2+h(\omega_c+h)]bU_3+(\tfrac{1}{2}\mu fk)[M(\omega_c+h)+N\Delta]}{\Delta^2+(\omega_c+h)^2}, \tag{40}$$

where

$$M=S_2(C_1-bU_1)-C_2S_1, \tag{41}$$

$$N=C_2(C_1-bU_1)+S_2S_1. \tag{42}$$

V. SELECTED CASES

One can proceed by substituting (33), (34), etc., in (41) and (42), then into (39) and (40), and finally into the phase-amplitude equations (14) and (15); however, the algebra is too protracted and complex to justify an explicit presentation of the general solution. It is more instructive to discuss a few special cases.

A. Calculation for $h=0$ (No Diffusion)

Ignoring the effects of spatial diffusion ($\mathfrak{D}_n=0$, $h=0$) dramatically simplifies the algebra, but produces results of doubtful relevance to real situations.[18] The solutions of the first-, second-, and third-order matter equations become

$$S_1=\Delta\omega_c(\ell_1)bU_1, \tag{43}$$

$$C_1=-\Delta^2(\ell_1)bU_1, \tag{44}$$

$$S_2=(\tfrac{1}{2}\mu fk)\,\Delta^2\omega_c(4\omega_c^2-2\Delta^2)(\ell_1)^2(\ell_2)b^2U_1^2, \tag{45}$$

$$C_2=(\tfrac{1}{2}\mu fk)\,\Delta\omega_c^2(\omega_c^2-5\Delta^2)(\ell_1)^2(\ell_2)b^2U_1^2, \tag{46}$$

$$S_3 = \Delta\omega_c(\ell_1) bU_3 + (\tfrac{1}{2}\mu f k)^2 \Delta\omega_c \times (2\Delta^6 - 5\Delta^4\omega_c^2 - 12\Delta^2\omega_c^4 + \omega_c^6)(\ell_1)^4(\ell_2) b^3 U_1^3, \tag{47}$$

$$C_3 = -\Delta^2(\ell_1) bU_3 + (\tfrac{1}{2}\mu f k)^2 \Delta^2\omega_c^2 \times (10\Delta^4 + 10\Delta^2\omega_c^2 - 6\omega_c^4)(\ell_1)^4(\ell_2) b^3 U_1^3, \tag{48}$$

where the Lorentzian factors (ℓ_1) and (ℓ_2) are defined by

$$\ell_1 = (\Delta^2 + \omega_c^2)^{-1}, \quad \ell_2 = (4\Delta^2 + \omega_c^2)^{-1}. \tag{49}$$

In the absence of cavity losses ($Q=\infty$), the phase-amplitude equations (14) and (15) become

$$\dot{U} = \alpha U - \beta U^3 = (\tfrac{1}{2}\kappa^2\Omega)\, \omega_c\Delta(\ell_1) U - (\tfrac{1}{2}\kappa^2\Omega) \times (\tfrac{1}{2}\mu f k b)^2 \omega_c \Delta(-2\Delta^6 + 5\Delta^4\omega_c^2 + 12\Delta^2\omega_c^4 - \omega_c^6)(\ell_1)^4(\ell_2) U^3, \tag{50}$$

$$\omega = \Omega + \sigma + \zeta U^2 = \Omega + (\tfrac{1}{2}\kappa^2\Omega)\,\Delta^2(\ell_1) - (\tfrac{1}{2}\kappa^2\Omega) \times (\tfrac{1}{2}\mu f k b)^2 \omega_c \Delta^2 (10\Delta^4 + 10\Delta^2\omega_c^2 - 6\omega_c^4) \times (\ell_1)^4(\ell_2) U^2, \tag{51}$$

where the intermediate forms follow the notation of Eqs. (81) and (89) in Ref. 9.

The net gain $\alpha - \beta U^2(t)$ is plotted in Fig. 3 as a function of the inversion Δ for several levels of acoustic intensity $I(t) = U^2(t)$ [measured in units of $\frac{1}{2}\kappa^2\Omega$, ω_c, and $a = (2\omega_c/\mu f k b)^2$, respectively]. The results are seen to be unrealistic, especially near threshold, where β is negative, and the third-order terms add to, rather than limit, the linear gain. No real steady-state acoustic amplitude $U_{ss} = (I_{ss})^{1/2} = (\alpha/\beta)^{1/2}$ exists for these values of Δ [Fig. 4(a)]. The frequency shift is similarly pathological [Fig. 4(b)]. These instabilities demonstrate that diffusion effects must not be neglected, especially near threshold. Realistic calculations must allow electrons to diffuse from regions of high negative charge once the acoustic amplitude becomes large enough to "bunch" them considerably. This becomes particularly important for electrons traveling along with the acoustic wave at velocities $v_d \sim v_s$.

B. Calculation for $h = \omega_c$

We now include diffusion effects by setting the diffusion constant $\mathfrak{D}_n = \omega_c/fk^2$, which is equivalent to considering the special case $\Omega = (\omega_c\omega_D)^{1/2} = \omega_{max}$, in which the linear gain α is a maximum.[4] The solutions of the first-, second-, and third-order matter equations become

$$S_1 = \Delta\omega_c(L_1)\, bU_1, \tag{52}$$

$$C_1 = -(\Delta^2 + 2\omega_c^2)(L_1)\, bU_1, \tag{53}$$

$$S_2 = (\tfrac{1}{2}\mu f k)\, \omega_c(20\omega_c^4 + 15\Delta^2\omega_c^2 - 2\Delta^4) \times (L_1)^2(L_2)\, b^2 U_1^2, \tag{54}$$

$$C_2 = -(\tfrac{1}{2}\mu f k)\, \Delta\omega_c^2(8\omega_c^2 + 11\Delta^2) \times (L_1)^2(L_2)\, b^2 U_1^2, \tag{55}$$

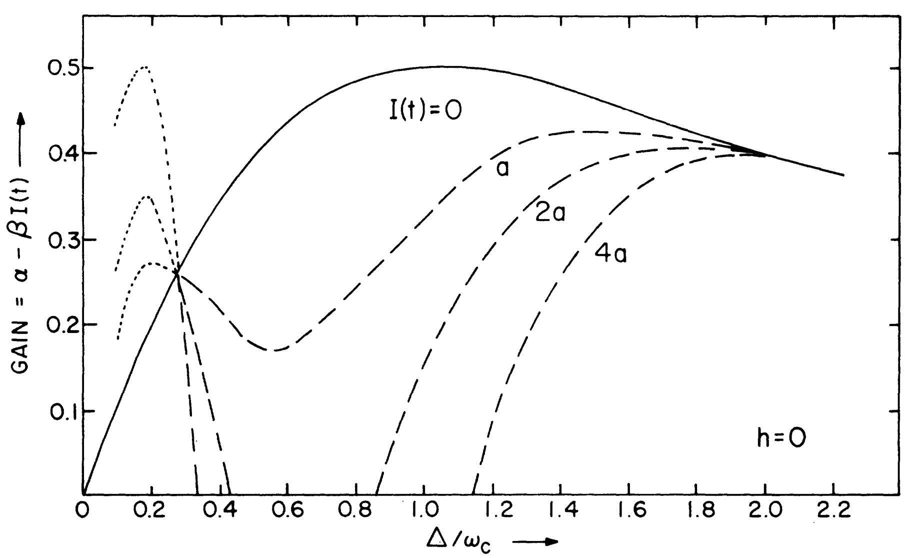

FIG. 3. Net gain as a function of Δ for the diffusionless case ($\omega_D = \infty$, $h=0$). The first-order gain curve becomes distorted (dashed lines) as the acoustic intensity $I(t) = U^2(t)$ increases; once the net gain becomes zero, steady-state operation ensues. The solution is nonphysical near $\Delta=0$ (dotted lines), demonstrating that diffusion effects cannot be ignored (see text). The gain is plotted in units of $\frac{1}{2}\kappa^2\Omega$; $I(t)$ is presented in units of $a = (2\omega_c/\mu f k b)^2$.

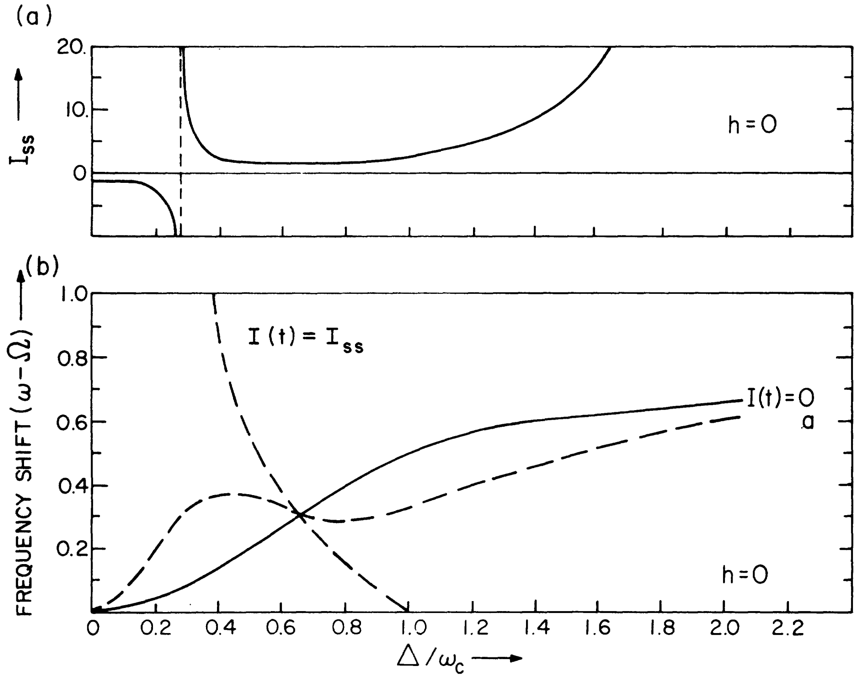

FIG. 4. (a) Steady-state acoustic intensity I_{ss} in units of $a = (2\omega_c/\mu f k b)^2$ as a function of the inversion Δ for the diffusionless case ($\omega_D = \infty$, $h = 0$). The expression diverges near $\Delta = 0.28\omega_c$; thus there is no steady-state solution for small Δ (see text). (b) The frequency shift ($\omega - \Omega$) as a function of Δ for the diffusionless case. Nonlinear contributions result in frequency pushing or pulling depending on the value of the inversion Δ. The steady-state frequency shift ($I = I_{ss}$) diverges near threshold. The ordinate units are $\frac{1}{2}\kappa^2 \Omega$.

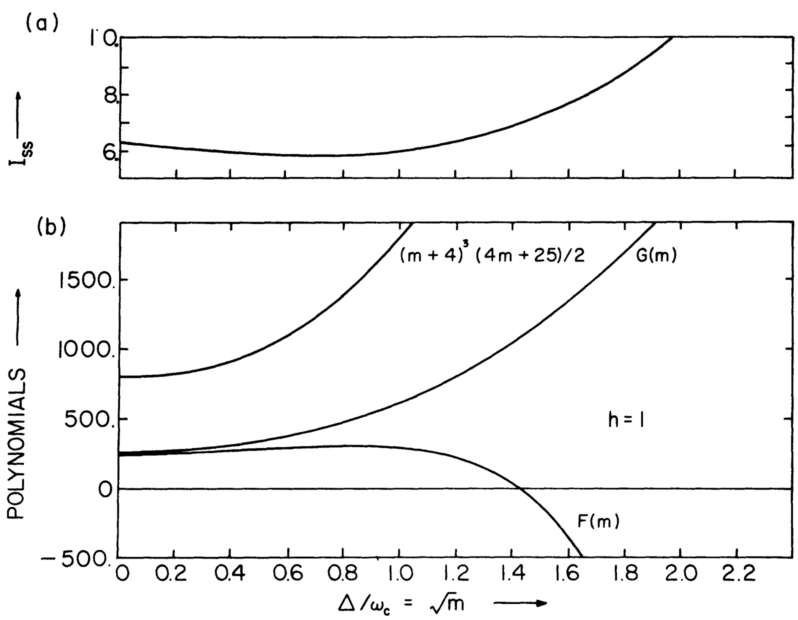

FIG. 5. Steady-state acoustic intensity I_{ss} and the dimensionless polynomials $F(m)$ and $G(m)$ as a function of the inversion Δ for the special case $h = 1$ [or $\Omega = (\omega_c \omega_D)^{1/2} = \omega_{\max}$], in which diffusion effects are included. $F(m)$ and $G(m)$ are balanced by the Lorentzian denominators $(m + 4)^3(4m + 25)$ to produce a rather constant I_{ss} for $0 < \Delta < 1.5\omega_c$. Notice that $G(m)$ is always positive, and I_{ss} does not diverge near threshold. For very large values of Δ, fifth-order terms can no longer be neglected. I_{ss} is plotted in units of $a = (2\omega_c/\mu f k b)^2$.

$$S_3 = \Delta\omega_c(L_1)\,bU_3 - (\tfrac{1}{2}\mu f k)^2 \times \Delta\omega_c^7 G(m)(L_1)^4(L_2)\,b^3U_1^3, \quad (56)$$

$$C_3 = -(\Delta^2 + 2\omega_c^2)(L_1)\,bU_3 - (\tfrac{1}{2}\mu f k)^2 \times \omega_c^8 F(m)(L_1)^4(L_2)\,b^3U_1^3, \quad (57)$$

where

$$G(m) = 256 + 316m + 47m^2 - 4m^3, \quad (58)$$

$$F(m) = 240 + 176m - 83m^2 - 28m^3 \quad (59)$$

are polynomials in $m \equiv \Delta^2/\omega_c^2$, and

$$L_1 = (\Delta^2 + 4\omega_c^2)^{-1}, \quad L_2 = (4\Delta^2 + 25\omega_c^2)^{-1}. \quad (60)$$

In the absence of cavity losses ($Q = \infty$) the phase-amplitude equations (14) and (15) become

$$\dot{U} = \alpha U - \beta U^3 = \frac{\kappa^2\Omega}{2}\,\frac{\Delta/\omega_c}{m+4}\,U - \frac{\kappa^2\Omega}{2}\,\frac{(\Delta/\omega_c)G(m)}{a(m+4)^4(4m+25)}\,U^3, \quad (61)$$

$$\omega = \Omega + \sigma + \zeta U^2 = \Omega + \frac{\kappa^2\Omega}{2}\,\frac{m+2}{m+4} + \frac{\kappa^2\Omega}{2}\,\frac{F(m)}{a(m+4)^4(4m+25)}\,U^2, \quad (62)$$

where $a = (2\omega_c/\mu f k b)^2$ and $m = \Delta^2/\omega_c^2$ as before. The polynomials $F(m)$ and $G(m)$ and the Lorentzian denominators $(m+4)^3(4m+25)$ are plotted vs Δ/ω_c in Fig. 5(b). $G(m)$ is always positive, unlike the corresponding gain polynomial in the diffusionless case (50), which is negative for small Δ. Notice that $F(m)$ still changes sign, i.e., frequency pushing becomes frequency pulling for applied voltages sufficient to give $\Delta > 1.45\omega_c$. The Lorentzian denominators $(m+4)^3(4m+25)$ counterbalance the large values of $F(m)$ and $G(m)$ for moderate values of Δ; thus the steady-state acoustic intensity $I_{ss} = \alpha/\beta$ is a rather flat function of the inversion [Fig. 5(a)] for $\Delta < 1.5\omega_c$.

The net gain $\alpha - \beta I(t)$ is presented in Fig. 6 (in units of $\frac{1}{2}\kappa^2\Omega$) as a function of Δ for several acoustic intensities [measured in units of $a = (2\omega_c/\mu f k b)^2$]. As the acoustic intensity increases from 0 to $6a$, the linear-gain curve ($I = 0$) becomes increasingly distorted by third-order terms. A net gain of zero means that steady-state operation has been attained for that value of Δ [compare the gain curve for $I = 6a$ with Fig. 5(a)]. Since $G(m)$ is always positive, the net gain is always less than or equal to α, and is well behaved near threshold; although fifith- and higher-order terms may become important for $\Delta \gg \omega_c$ [Fig. 5(a)]. Notice that the maximum gain is less than in the diffusionless case ($h = 0$) and occurs at higher values of Δ. In practice, Δ is varied by changing the external voltage and thus v_d.

Figure 7 presents the total frequency shift $(\omega - \Omega)$ for several levels of acoustic intensity. The vertical scale (measured in units of $\frac{1}{2}\kappa^2\Omega$) has been greatly expanded. The steady-state frequency shift (attained once $I = I_{ss}$) also represents the maximum value of $(\omega - \Omega)$ for a given Δ, and is well behaved near threshold. Notice that the qualitative behavior for $\Delta > \omega_c$ is similar to the diffusionless case (Fig. 4), although the crossover from frequency pushing to frequency pulling occurs at higher values of Δ/ω_c. Notice too that the presence of diffusion ($h \neq 0$) ensures $(\omega - \Omega) \neq 0$ for $\Delta = 0$, even in the case of linear theory[4] ($I = 0$).

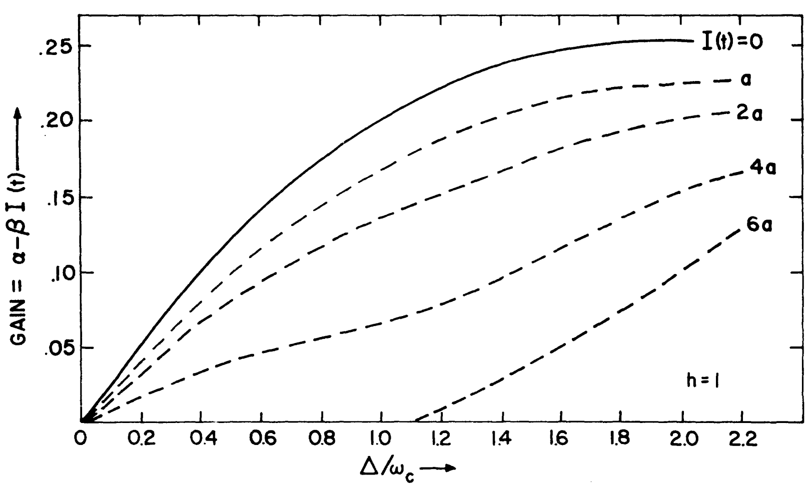

FIG. 6. Net gain as a function of Δ for the case $h = 1$ or $\Omega = (\omega_c\omega_D)^{1/2}$. As the acoustic intensity $I(t)$ increases, the net gain is reduced until steady-state operation is approached. Once $I(t) = 6a$, for example, the net gain has become zero (steady-state operation), for all acoustic amplifiers with inversions $\Delta \lesssim 1.1\omega_c$. Note that there is no instability near threshold as in the diffusionless case (Figs. 3 and 4). The ordinate units are $\frac{1}{2}\kappa^2\Omega$, and $a = (2\omega_c/\mu f b k)^2$.

VI. THE BACKWARD-TRAVELING WAVE

Throughout this presentation we have considered only a single forward-traveling wave (20). We find that it is amplified for $fv_d > v_s$ and attenuated for $fv_d < v_s$ (Fig. 1); and that the acoustic intensity approaches a steady value $I_{ss}(\Delta)$ once third-order nonlinearities are taken into account (61). We now consider the first-order backward-traveling waves for the same applied voltage:

$$U_{b1}(x,t) = U_{b1}(t)\sin(kx+\omega t), \tag{63}$$

$$E_{b1}(x,t) = S_{b1}(t)\sin(kx+\omega t) + C_{b1}\cos(kx+\omega t), \tag{64}$$

where k and ω, are positive and have the same value as in (20) and (21) but where $U_{b1} \neq U_1$, etc., since the material is nonreciprocal.

The matter (8) and wave equations (13) now yield the linear gain

$$\alpha_b = (\tfrac{1}{2}\kappa^2\Omega)\,\xi\omega_c/[\xi^2 + (\omega_c + h)^2], \tag{65}$$

which is equivalent to (24) with the substitution of $\xi = -k(fv_d + v_s)$ for Δ, where v_d still represents the unchanged positive $(+x)$ electron drift velocity. Notice that, unlike Δ, ξ is never positive; thus the "gain" α_b always represents an attenuation of the backward-traveling acoustic wave U_b. This is reasonable, since the acoustic wave is "bucking" the electron flow (negative-momentum transfer). Actually α_b is small in many cases of practical interest, e.g., for $\omega_c^2 \ll \omega^2$ (high-resistivity samples). Then $\xi = -(\Delta + 2\omega)$ dominates ω_c in the denominator of (65) and if $h \sim \omega_c$

$$\alpha_b \simeq -(\tfrac{1}{2}\kappa^2\Omega)\,\omega_c/(\Delta + 2\omega), \tag{66}$$

yielding a backward loss $\alpha_b \simeq -(\frac{1}{4}\kappa^2)\,\omega_c \simeq -0.005\omega_c$ which is considerably smaller than the forward gain (24), since $\omega_c \ll \Omega$. In such cases, applying oscillator boundary conditions (reflections at $x = 0, L$) gives $U_b \sim U$, with the slight backward losses compensated by forward gain; and a good approximation to a standing wave:

$$U(t)\sin(kx - \omega t) + \overline{U}_b(t)\sin(kx + \omega t) \simeq 2U(t)\sin(kx)\cos(\omega t) \tag{67}$$

can be maintained. This case is particularly important in light of recent experiments.[13,14]

The backward-traveling component contributes negligibly to frequency pulling in the case $h, \Delta \sim \omega_c \ll \omega$ and usually $\sigma_b \simeq \frac{1}{2}\kappa^2\Omega$. Corresponding higher-order expressions follow from the substitution $\Delta \to \xi$ in (33), (34), (39), and (40), etc.

VII. CONCLUSION

We have formulated the problem of the acoustoelectric amplifier in analogy with laser theory. Solving for a single acoustic mode to third order gives nonlinear solutions for the gain and frequency pulling. We find that diffusion effects must be included, especially near threshold.

Current experimental work supports the philosophy of this calculation.[13,14] Hopefully, future experiments will help determine to what extent a single-mode description is valid,[19] and will search

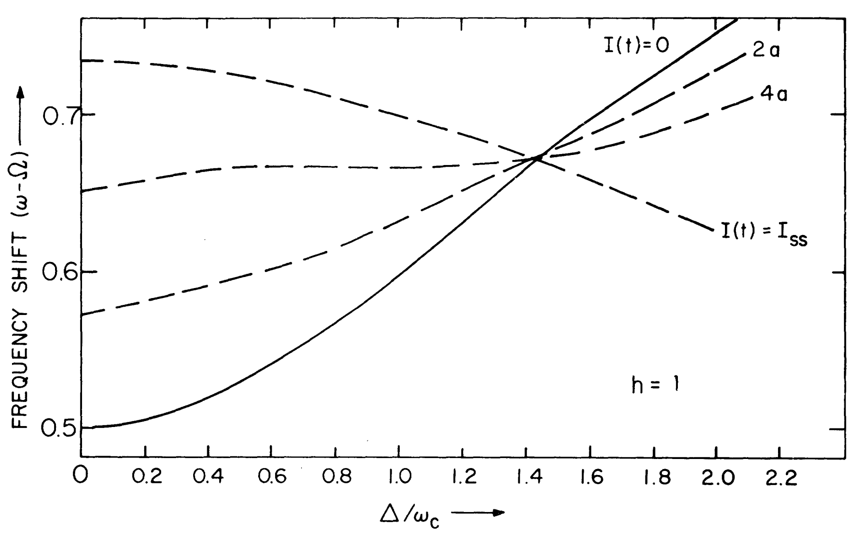

FIG. 7. Total frequency shift as a function of Δ for the case $h=1$ or $\Omega = (\omega_c\,\omega_D)^{1/2}$. Frequency pushing becomes frequency pulling near $\Delta = 1.4\omega_c$; the frequency shift is well behaved near threshold. Note that $\omega - \Omega \neq 0$ at $\Delta = 0$, even in linear theory $(I=0)$, if diffusion effects are included [$h \neq 0$ in Eq. (25)]. The ordinate units are $\frac{1}{2}\kappa^2\Omega$.

for effects like the shift from frequency pushing to frequency pulling at high Δ. We are currently applying similar methods to the multimode case[17] to provide a firm theoretical basis for recent observations of acoustic mode locking.

ACKNOWLEDGMENTS

We wish to acknowledge the interest and help provided by F. A. Hopf and the stimulating suggestions of R. Hearing.

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*Work based in part on a doctoral dissertation submitted by I. M. Asher to the Department of Physics, Massachusetts Institute of Technology (1971). Work supported in part by United States Air Force Office of Scientific Research.

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[15]The extra spatial derivative (′) is retained to avoid introducing magnetic effects; although $\nabla\times H=\dot{D}+J\neq 0$, its spatial derivative is zero as is obvious from the first part of Eq. (7).

[16]White (Ref. 4) uses the spatial formulas $-\alpha_w x$ to define the linear-spatial-gain coefficient α_w. We use the temporal form $U\sim e^{+\alpha t}$, whose linear-gain coefficient α differs from α_w by a factor of $-v_s$, i.e. $\alpha_w=\alpha/v_s$.

[17]This restriction will be relaxed in a future publication: I. M. Asher, F. A. Hopf, M. O. Scully (unpublished).

[18]Typically $\omega_D\sim 500$ MHz. Since $\omega\sim 50$ MHz and $\omega_c\sim 1$ MHz in many experiments, diffusion effects cannot be neglected in practice ($h=\omega^2/\omega_D\sim\omega_c$).

[19]In practice, one might insert electronic filters in the external circuit of a traveling-wave acoustic amplifier (Fig. 1), or make use of the wide-mode spacing in thin standing-wave CdS acoustic oscillators, to realize single-mode operation.