PHYSICAL REVIEW A VOLUME 8, NUMBER 4 OCTOBER 1973

Hydrodynamics of Superfluid Helium below 0.6 °K. I. Viscosity of the Normal Fluid

Humphrey J. Maris*

School of Mathematics and Physics, University of East Anglia, Norwich, England and
Department of Physics, Brown University, Providence, Rhode Island†

(Received 26 February 1973)

The normal-fluid viscosity of superfluid He^4 has been calculated for temperatures below about 0.6 °K. The calculation assumes that the dominant scattering mechanism is the three-phonon process in the first order of perturbation theory. This contrasts with the previous calculation of Landau and Khalatnikov, who assumed a second-order process involving four phonons. The theory is compared with the experiments of Whitworth on heat flow through tubes. Good agreement is obtained, as regards both the magnitude of the viscosity and its temperature dependence.

I. INTRODUCTION

In this paper I present a calculation of the normal-fluid viscosity η for He^4 in the temperature range below 0.6 °K. Measurements of the viscosity in this temperature range have been made by Whitworth.[1] A viscosity mean free path $\Lambda_{\rm expt}$ may be defined by

$$\eta = \tfrac{1}{3}\rho_n c_0 \Lambda_{\rm expt} \ , \tag{1}$$

where ρ_n is the normal-fluid density, and c_0 is the the phonon velocity for small momentum. Whitworth found that

$$\Lambda_{\rm expt} = 3.8\times 10^{-3}\, T^{-4.2}\ \text{cm} \ . \tag{2}$$

At temperatures below 0.6 °K the thermal excitations in helium are almost entirely phonons. The mean free path of the phonons is limited by three-phonon collisions, these occurring at a rate which is proportional to T^5. At first sight, this seems to be in reasonable agreement with Whitworth's result. More careful consideration reveals a difficulty, however. The three-phonon collisions are small-angle collisions,[2] whereas the mean free path relevant to viscosity must be concerned with large-angle processes. To correct for this, it is necessary to multiply the scattering rate by a factor proportional to α^4, where α is the angle of a typical collision (see the Appendix). The collision angle is determined by the phonon dispersion relation. Suppose the relation between energy ϵ and momentum p is exactly linear, i.e.,

$$\epsilon = c_0 p \ . \tag{3}$$

Then it is easy to show that the collision angle in a three-phonon process is exactly zero. If, however,

$$\epsilon = c_0 p(1+g) \ , \tag{4}$$

where g is some function of p, which is positive and tends to zero as $p\to 0$, the collision angle is finite, and is of the order of

$$\alpha \sim g^{1/2} \ . \tag{5}$$

We therefore have the interesting result that the magnitude of the viscosity η can be used to estimate α and hence the quantity g in the dispersion relation. Moreover, the *temperature* dependence of α provides information about the *momentum* dependence of g. This is because the momentum of a typical thermal phonon is proportional to temperature.

In this paper we investigate theoretically the temperature dependence of the viscosity, assuming various forms for the phonon dispersion relation. Our starting point is a formal expression[3] for the viscosity in terms of the eigenfunctions and eigenvalues of the phonon collision operator. We describe the calculation of these eigenfunctions in Sec. II. In Sec. III we compare our results with Whitworth's experiments, and in Sec. IV discuss what inferences can be made about the phonon dispersion relation.

II. CALCULATION OF THE VISCOSITY

Let the collision term in the linearized Boltzmann equation[4] for phonons be written in the form:

$$\left(\frac{\partial n_p}{\partial t}\right)_{\rm coll} = \int C(\vec{p}, \vec{p}\,')\, n_{p'}\, d\tau_{p'} \ . \tag{6}$$

Introduce the symmetrized collision operator $\tilde{C}(\vec{p}, \vec{p}\,')$, defined as

$$\tilde{C}(\vec{p},\vec{p}\,') \equiv C(\vec{p},\vec{p}\,')[n^0_{p'}(n^0_{p'}+1)/n^0_p(n^0_p+1)]^{1/2} \tag{7}$$

where n^0_p is the equilibrium distribution function,

$$n^0_p = (e^{\epsilon/k_B T} - 1)^{-1} \,. \tag{8}$$

$\tilde{C}(\vec{p},\vec{p}\,')$ is symmetric[5] with respect to interchange of $\vec{p}$ and $\vec{p}\,'$. We define eigenfunctions $\psi_i(\vec{p})$ and eigenvalues λ_i by

$$\int \tilde{C}(\vec{p},\vec{p}\,')\,\psi_i(\vec{p}\,')\,d\tau_{p'} = -\lambda_i\,\psi_i(\vec{p}) \,. \tag{9}$$

Since $\tilde{C}$ only depends on the magnitudes of $\vec{p}$ and $\vec{p}\,'$ and the angle between them, the angular part of $\psi_i(\vec{p})$ must be a spherical harmonic $Y_{lm}(\theta,\phi)$. We therefore label the states, as in spectroscopy, by S, P, D, etc., depending on the value of l. It can be shown that, because energy is conserved, there is one S state with eigenvalue zero. The eigenfunction for this state is[3]

$$\psi_{1S}(\vec{p}) = (\beta/C\rho T)^{1/2}\,\epsilon(n^0_p)^{1/2}(n^0_p+1)^{1/2} \,, \tag{10}$$

where $\beta = 1/k_B T$, C is the specific heat per unit mass, and ρ is the density. We define the radial part $S_1(p)$ of the eigenfunction $\psi_{1S}(\vec{p})$ by setting

$$\psi_{1S}(\vec{p}) = Y_{00}(\theta,\phi)S_1(p) \,, \tag{11}$$

and therefore,

$$S_1(p) = (4\pi\beta/C\rho T)^{1/2}\,\epsilon(n^0_p)^{1/2}(n^0_p+1)^{1/2} \,. \tag{12}$$

The D states are of particular interest in regard to the viscosity. Let the nth D state have eigenvalue λ_{nD} and eigenfunction

$$\psi_{nD}(\vec{p}) = Y_{2m}(\theta,\phi)D_n(p) \,. \tag{13}$$

Then it can be shown[3] that the viscosity η is given by

$$\eta = \frac{C\rho T}{15}\sum \frac{1}{\lambda_{nD}}\left[S_1\left|\frac{\rho}{\epsilon}\frac{\partial\epsilon}{\partial p}\right|D_n\right]^2 . \tag{14}$$

Here, the sum is over all D states. The fivefold degeneracy of the D states has been taken into account in deriving this result. Thus, only one of the five D states with a given radial quantum number n is to be included in the sum. The matrix element is defined by

$$\left[S_1\left|\frac{p}{\epsilon}\frac{\partial\epsilon}{\partial p}\right|D_n\right] = h^{-3}\int S_1(p)\frac{p}{\epsilon}\frac{\partial\epsilon}{\partial p}D_n(p)p^2\,dp. \tag{15}$$

The calculation of η reduces, therefore, to the problem of finding the eigenfunctions and eigenvalues of the D states. Consider first the form of the collision operator. The matrix element for phonon-phonon scattering has been derived by Landau and Khalatnikov.[6] Using their result, we find, after a straightforward perturbation calculation, that

$$\begin{aligned}\int \tilde{C}(\vec{p}_1,\vec{p}_2)f(p_2)\,\phi(\vec{p}_2)\,d\tau_{p_2} = -[2\pi^2 c_0(u_0+1)^2/\rho h^4]\iint p_1p_2p_3 f(p_2)f(p_3)\,d\vec{p}_2\,d\vec{p}_3 \\ \times\Big\{\tfrac{1}{2}[\phi(\vec{p}_1)-\phi(\vec{p}_2)-\phi(\vec{p}_3)]\,\delta(\vec{p}_1-\vec{p}_2-\vec{p}_3)\,\delta(\epsilon_1-\epsilon_2-\epsilon_3) \\ +[\phi(\vec{p}_1)+\phi(\vec{p}_2)-\phi(\vec{p}_3)]\,\delta(\vec{p}_1+\vec{p}_2-\vec{p}_3)\delta(\epsilon_1+\epsilon_2-\epsilon_3)\Big\},\end{aligned} \tag{16}$$

where

$$f(p) \equiv (n^0_p)^{1/2}(n^0_p+1)^{1/2} \tag{17}$$

$$u_0 \equiv \frac{\rho}{c_0}\frac{\partial c_0}{\partial\rho} \,, \tag{18}$$

and $\phi(\vec{p})$ is an arbitrary function of $\vec{p}$. In deriving this result, we have simplified the form of the matrix element by assuming that the angle between the momenta of the colliding phonons is small. We have also assumed that the phonons have long wavelengths. This allows us to approximate the Grüneisen constant by u_0, its limiting value for small p. In the same spirit, we may assume, for the purposes of estimating the *magnitude* of the matrix element, that

$$\epsilon = c_0 p \,. \tag{19}$$

We discuss the uncertainties introduced by these approximations in Sec. IV. We must, of course, keep the complete form of the dispersion relation [Eq. (4)] for calculating the angle of scattering.

Consider now the eigenvalue equation (9). Let $\vec{p}$ be in the z direction and let $\psi_i(\vec{p})$ be a D state with magnetic quantum number zero. Then if

$$D_n(p) \equiv f(p)d_n(p) \,, \tag{20}$$

we find that the eigenvalue equation (9) becomes

$$[2\pi^2 c_0(u_0+1)^2/\rho h^4] \int\int p p_2 p_3 f(p_2) f(p_3)\, d\vec{p}_2\, d\vec{p}_3$$

$$\times \left\{ \tfrac{1}{2}[d_n(p) - d_n(p_2)P_2(\theta_2) - d_n(p_3)P_2(\theta_3)]\delta(\vec{p}-\vec{p}_2-\vec{p}_3)\delta(\epsilon-\epsilon_2-\epsilon_3)\right.$$

$$\left. +[d_n(p) + d_n(p_2)P_2(\theta_2) - d_n(p_3)P_2(\theta_3)]\,\delta(\vec{p}+\vec{p}_2-\vec{p}_3)\delta(\epsilon+\epsilon_2-\epsilon_3)\right\} = \lambda_{nD} f(p) d_n(p)\ . \tag{21}$$

θ_2 and θ_3 are the angles between the z axis and $\vec{p}_2$ and $\vec{p}_3$, respectively. The geometry of the collision is shown in Figs. 1(a) and 1(b). $P_2(\theta)$ is the Legendre polynomial for $l=2$, normalized so that $P_2(0)=1$. After performing the integrals over the directions of $\vec{p}_2$ and $\vec{p}_3$, we obtain

$$\{c_0(u_0+1)^2/4\pi\rho\hbar^4\}\int\int p_2^2\, dp_2 p_3^2 dp_3 f(p_2)f(p_3)\left\{\tfrac{1}{2}[d_n(p) - d_n(p_2)P_2(\theta_2) - d_n(p_3)P_2(\theta_3)]\,\delta(\epsilon-\epsilon_2-\epsilon_3)\right.$$

$$\left. +[d_n(p) + d_n(p_2)P_2(\theta_2) - d_n(p_3)P_2(\theta_3)]\,\delta(\epsilon+\epsilon_2-\epsilon_3)\right\} = \lambda_{nD} f(p)\, d_n(p)\ . \tag{22}$$

It is more convenient now to consider f and d_n as functions of energy rather than of momentum. Next, we use Eq. (19) to convert the momenta *outside* the square brackets into energies. We may then perform the integral over ϵ_3. Finally, we express the result as

$$\int A(\epsilon,\epsilon_2)\, D_n(\epsilon_2)\, \epsilon_2\, d\epsilon_2 = \lambda_{nD} D_n(\epsilon)\epsilon\ , \tag{23}$$

where

$$A(\epsilon,\epsilon_2) = \Gamma(\epsilon)\delta(\epsilon-\epsilon_2) - B\epsilon\epsilon_2(\epsilon-\epsilon_2)^2 f(|\epsilon-\epsilon_2|)P_2(\theta')$$

$$+B\epsilon\epsilon_2(\epsilon+\epsilon_2)^2 f(\epsilon+\epsilon_2)P_2(\theta'')\ , \tag{24}$$

$$\Gamma(\epsilon) = [B/\epsilon f(\epsilon)]\int(\epsilon-\epsilon_3)^2\epsilon_3^3 f(|\epsilon-\epsilon_3|)f(\epsilon_3)\,d\epsilon_3$$

$$-[B/\epsilon f(\epsilon)]\int(\epsilon+\epsilon_3)^2\epsilon_3^3 f(\epsilon+\epsilon_3)f(\epsilon_3)\,d\epsilon_3\ , \tag{25}$$

$$B = (u_0+1)^2/4\pi\rho\hbar^4 c_0^5\ , \tag{26}$$

$$\cos\theta' = \frac{p^2(\epsilon)+p^2(\epsilon_2)-p^2(|\epsilon-\epsilon_2|)}{2p(\epsilon)p(\epsilon_2)}\ , \tag{27}$$

$$\cos\theta'' = \frac{p^2(\epsilon+\epsilon_2)-p^2(\epsilon)-p^2(\epsilon_2)}{2p(\epsilon)p(\epsilon_2)}\ . \tag{28}$$

The lower limit of the range of integration is zero in each case. The upper limit is restricted only by the requirement that θ' and θ'' be real.

The physical significance of $\Gamma(\epsilon)$ is the reciprocal lifetime of a phonon of energy ϵ. We can make the result for Γ look more conventional by writing it as

$$\Gamma(\epsilon) = \tfrac{1}{2}B\int_0^{\epsilon}\epsilon_3^2(\epsilon-\epsilon_3)^2[n^0(\epsilon_3)+n^0(\epsilon-\epsilon_3)+1]\,d\epsilon_3$$

$$+B\int_0^{\infty}\epsilon_3^2(\epsilon+\epsilon_3)^2[n^0(\epsilon_3)-n^0(\epsilon+\epsilon_3)]\,d\epsilon_3\ , \tag{29}$$

where

$$n^0(\epsilon) = (e^{\epsilon/k_BT}-1)^{-1}\ . \tag{30}$$

The first term comes from processes in which a phonon of energy ϵ decays into two phonons of energy ϵ_3 and $\epsilon-\epsilon_3$. The second term arises from collisions of a phonon of energy ϵ with a phonon of energy ϵ_3. Although in this case we have explicitly indicated the limits of integration, it is to be understood that the range is still limited by the requirement that θ' and θ'' be real.

We may now find the D-state eigenvalues and eigenfunctions from Eq. (23). Consider first what happens if the dispersion relation is linear, i.e., if Eq. (19) holds exactly. Then from Eqs. (27) and (28) we find that

$$P_2(\theta') = P_2(\theta'') = 1. \tag{31}$$

Suppose we try to find an eigenfunction $\psi_{lD}(\vec{p})$ with radial part:

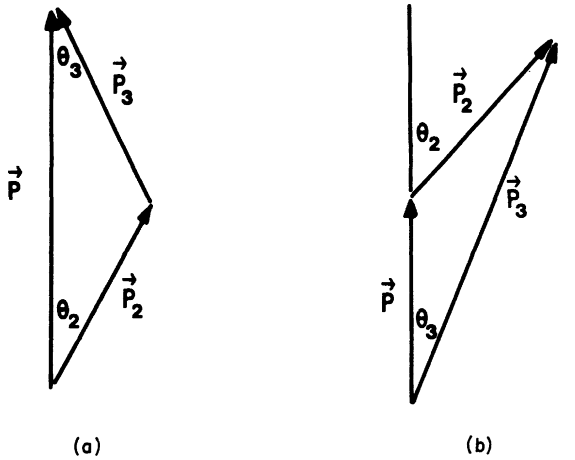

FIG. 1. Geometry of the collisions.

$$D_1(\epsilon)=\epsilon f(\epsilon)\ . \tag{32}$$

This is equivalent to setting $d_1(\epsilon)=\epsilon$. This choice, together with the result (31), leads to the vanishing of the expressions inside both square brackets on the left-hand side of Eq. (22). Hence, (32) is an eigenfunction, and its eigenvalue λ_{1D} is zero. Any other radial function must be orthogonal to $D_1(\epsilon)$. Thus, all of the other radial functions must have at least one node. It follows that for the other eigenfunctions, the expressions inside the square brackets within the curly brackets in Eq. (22) are of the order of $d_n(\epsilon)$. The order of magnitude of the eigenvalues of these eigenfunctions must be

$$B(k_BT)^5\ .$$

If the dispersion is now allowed to be finite but small, the eigenfunction $D_1(\epsilon)$ will have a small but finite eigenvalue. The eigenvalues of the other eigenfunctions will be changed only slightly. We therefore expect that the $1D$ state will dominate the sum (14) involved in calculating the viscosity. This dominance is made more certain because the radial functions $S_1(\epsilon)$ and $D_1(\epsilon)$ are very nearly equal. Thus

$$\left[S_1\left|\frac{p}{\epsilon}\frac{\partial\epsilon}{\partial p}\right|D_n\right]\approx\left[D_1\left|\frac{p}{\epsilon}\frac{\partial\epsilon}{\partial p}\right|D_n\right]. \tag{33}$$

But, if the dispersion is small,

$$\frac{p}{\epsilon}\frac{\partial\epsilon}{\partial p}\approx 1.$$

Hence we have

$$\left[S_1\left|\frac{p}{\epsilon}\frac{\partial\epsilon}{\partial p}\right|D_n\right]\approx[D_1|D_n]\ . \tag{34}$$

But this last matrix element must vanish by orthogonality for $n\neq 1$.

The sum in Eq. (14) may now be simplified to

$$\eta=C\rho T/15\lambda_{1D}\ . \tag{35}$$

When the only excitations in helium are long-wavelength phonons, it is easy to show that

$$C\rho T=3\rho_n c_0^2\ . \tag{36}$$

Hence, if we define a "theoretical" mean free path by

$$\eta=\tfrac{1}{3}\rho_n c_0\Lambda_{\text{theor}}\ , \tag{37}$$

then

$$\Lambda_{\text{theor}}=3c_0/5\lambda_{1D}\ . \tag{38}$$

Numerical calculations of λ_{1D} were made by replacing the ϵ_2 integral in Eq. (23) by a sum over the set of points $j\Delta\epsilon$, where $j=1,\ldots,j_{\max}$. By considering only discrete values of ϵ, Eq. (23) is converted to a matrix eigenvalue problem. Various values of $j_{\max}$ and $\Delta\epsilon$ were used. With $j_{\max}\geqslant 15$ and a suitable choice of $\Delta\epsilon$ (generally between $0.6k_BT$ and $1.25k_BT$), the results were found to be independent of the details of the mesh to better than 1%. A further check on the accuracy of the calculation of λ_{1D} is described in the Appendix.

III. COMPARISON WITH EXPERIMENT

We now have to decide on a form for the phonon dispersion relation. The principal sources of experimental information are as follows.

a. Specific-heat measurements. Phillips *et al.*[7] have measured the specific heat down to about 0.3 °K. They were able to fit their results by assuming

$$\epsilon=c_0p(1+\gamma p^2) \tag{39}$$

with $c_0=2.397\times10^4$ cm sec^{-1} and $\gamma=4.1\times10^{37}$ cgs units. In their analysis both c_0 and γ were treated as adjustable parameters. However, ultrasonic measurements give[8]

$$c_0=2.383\pm0.001\times10^4\ \text{cm sec}^{-1}\ .$$

If the specific heat is analyzed using this value of c_0, one finds[9] $\gamma=8\times10^{37}$ cgs units. However, the uncertainty in the data is such that values of γ between about 6 and 10×10^{37} are not ruled out. It should be noted that the specific-heat data are sensitive only to the part of the dispersion relation below an energy of around 2 °K.[10]

b. Neutron scattering. Woods and Cowley[11] have measured the dispersion relation down to about 4°K energy. The phase velocity of a phonon is

$$c\equiv\epsilon/p=c_0(1+g)\ . \tag{40}$$

Woods and Cowley's results for c are shown in Fig. 2 as a function of q^2, where

$$q=p/\hbar\ .$$

Their results show that for q^2 between 0.4 and 1.0 Å^{-2}, c is approximately given by

$$c=c_0(A-Bq^2)\ , \tag{41}$$

where A is between 1.1 and 1.2 and B is approximately 0.3 Å^2. This range of q corresponds to en-

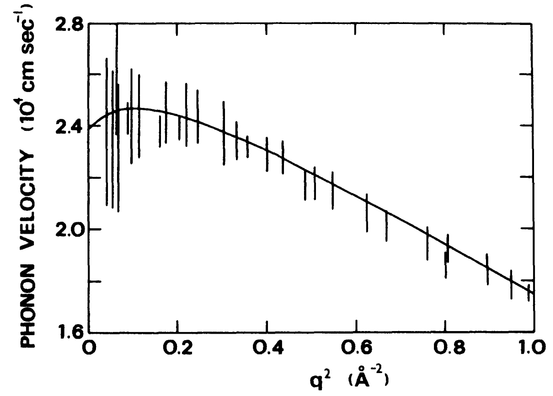

FIG. 2. Phonon velocity c as a function of q^2(q = wave number). The experimental points are the measurements of Woods and Cowley (Ref. 11). The solid line is from dispersion curve C [Eq. (42)], using the parameters listed in Table I.

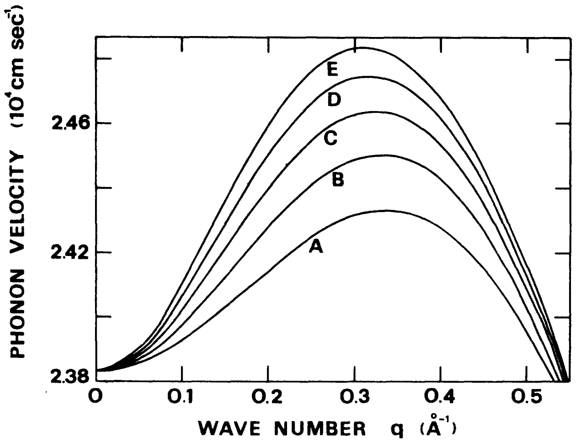

FIG. 3. Phonon velocity c as a function of wave number q. The parameters defining these curves are listed in Table I.

ergies between 11 and 18 °K. For smaller energies the neutron results become too scattered to provide useful information about g.

A simple form for the dispersion relation which is consistent with the neutron scattering results and the specific-heat measurements is

$$\epsilon = c_0 p\left(1 + \gamma p^2 \frac{1-(p/p_A)^2}{1+(p/p_B)^2}\right). \tag{42}$$

For the moment, let us take γ to be 8×10^{37} cgs units, and choose p_A and p_B to give the best fit to the neutron data. A simple way of doing this is to demand that Eq. (42) give the correct velocity c at $q^2 = 0.4$ and at 1.0 Å^{-2}. From Fig. 2,

$$c_{0.4} = 2.30\times10^4 \text{ cm sec}^{-1},$$

$$c_1 = 1.75\times10^4 \text{ cm sec}^{-1}.$$

Then, we find

$$p_A \equiv \hbar q_A, \quad p_B \equiv \hbar q_B,$$

where $q_A = 0.5384$ Å^{-1} and $q_B = 0.3727$ Å^{-1}. The dispersion relation is plotted in Fig. 2, and the small-p region is shown in more detail in Fig. 3 (curve C).

We have calculated λ_{1D} and Λ_{theor} using this dispersion relation. A value of 2.84 was used for the Grüneisen constant.[12] The results for the viscosity mean free path are shown in Fig. 4 (curve C), together with the experimental values. The theory is in excellent agreement with the temperature dependence of the experimental results. The magnitude of the predicted mean free path is about 20% greater than the experimental values.

Since there is some uncertainty in the correct value for γ, we have repeated these calculations using $\gamma = 4, 6, 8, 10$, and 12×10^{37} cgs units. The values of q_A and q_B are shown in Table I, together with $c_{\max}$, the maximum value of the phase velocity. For $q^2 \geq 0.3$ Å^{-2} the dispersion curve only changes slightly as γ is varied. Thus all of these dispersion curves are consistent with the neutron results. The small-momentum part of the dispersion relation is shown in Fig. 3, and the calculated values of the viscosity mean free path are included in Fig. 4.

IV. DISCUSSION

The comparison between theory and experiment (Fig. 4) indicates that $\gamma = 10\times10^{37}$ cgs units is the best choice for the dispersion parameter. It is not obvious, however, that the theory and the ex-

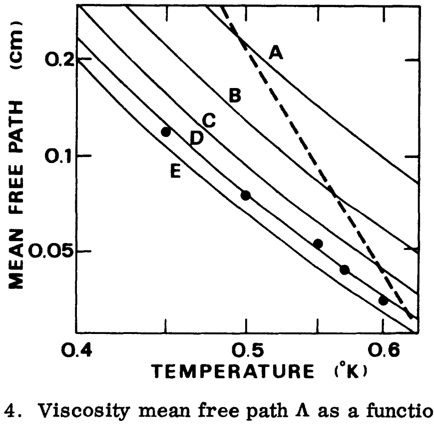

FIG. 4. Viscosity mean free path Λ as a function of temperature. Solid lines are the results of calculations using the dispersion curves A–E whose parameters are listed in Table I. The dashed line is the Landau-Khalatnikov theory. The solid circles are the experimental points of Whitworth.

TABLE I. Parameters defining the dispersion curves in Fig. 3.

Curve	γ	$c_{0.4}$	c_1	q_A	q_B	$c_{\max}$
	(10^{37} cgs units)	(10^4 cm sec^{-1})		(Å^{-1})	(Å^{-1})	(10^4 cm sec^{-1})
A	4	2.30	1.75	0.522	0.537	2.433
B	6	2.30	1.75	0.533	0.433	2.450
C	8	2.30	1.75	0.538	0.373	2.464
	8	2.31	1.75	0.551	0.388	2.469
	8	2.30	1.76	0.537	0.368	2.463
D	10	2.30	1.75	0.542	0.332	2.475
E	12	2.30	1.75	0.544	0.303	2.484

periment are sufficiently accurate to clearly favor $\gamma = 10 \times 10^{37}$ over $\gamma = 8 \times 10^{37}$, for example. Consider first the uncertainties in the experiment. The viscosity was measured by heat flow along a tube. Corrections had to be applied to allow for slip at the boundary,[13] partial specular reflection of the phonons, end effects, and the contribution of rotons to the heat flow at the higher temperatures. Despite these difficulties, a number of cross checks on the data were possible, and the results for the mean free path are probably accurate to ± 20%.

There are a number of approximations in the theory. Consider first the approximation of using the long-wavelength small-angle limit for the phonon-phonon matrix element. If we attempt to go beyond this approximation, we will find that $(u_0+1)^2$ in Eq. (16) is replaced by a function of $\vec{p}_1$, $\vec{p}_2$ and $\vec{p}_3$, which will now appear inside the integral over $\vec{p}_2$ and $\vec{p}_3$. The error in this approximation is roughly of the same order of magnitude as the difference between u_0, and u_p, the Grüneisen constant at finite p, which is defined by

$$u_p = \frac{\rho}{\epsilon}\frac{\partial \epsilon}{\partial \rho} .$$

Some information about how u_p varies with p may be obtained from the neutron scattering measurements of Svensson, Woods, and Martel.[14] They measured the dispersion curves $\epsilon_0(p)$ at zero pressure and $\epsilon_{24}(p)$ at 24 atm. A *very* rough estimate of the ratio of u_p to u_0 at $p=0$ can be obtained from

$$\frac{u_p}{u_0} \approx \frac{\epsilon_{24}(p) - \epsilon_0(p)}{\epsilon_0(p)} \; \frac{\epsilon_0(p\to 0)}{\epsilon_{24}(p\to 0) - \epsilon_0(p \to 0)} .$$

In the present context, we are most concerned with momenta in the region where the phase velocity has its maximum value. Phonons of around this energy are able to be scattered in collisions having the largest possible angle. For $q = 0.3$ Å^{-1}, we find

$$u_p/u_0 \approx 0.8 \pm 0.1 .$$

Thus, if we replaced u_0 by u_p in the $(u_0+1)^2$ factor, the scattering rate would be reduced by between 15% and 40%. This would *increase* the theoretical mean free path by the same factor. Ignoring the finite angle between the momenta of the colliding phonons introduces a much smaller error, of the order of 1%. This error, if corrected, would also increase Λ_{theor}. Another uncertainty in the matrix element arises from using ϵ and $c_0 p$ interchangeably in several places. The same approximation is implicit in Eq. (36), and also occurs in the calculations performed by Whitworth in determining Λ_{expt} from his data. At least some of these approximations cancel one another. The total error is probably 10%, or less, but could be in either direction.

Uncertainties in the neutron data of Woods and Cowley[11] will change the values assumed for the velocities $c_{0.4}$ and c_1. From the error bars in Fig. 2 we see that these velocities could be varied by 0.01 or 0.02×10^4 cm sec^{-1} without becoming inconsistent with the data. Variation of the velocity c_1 by these amounts has very little effect on the dispersion curve for small p, or on the mean free path. Changing $c_{0.4}$ to 2.31×10^4 cm sec^{-1} shifts the dispersion curve for $\gamma = 8 \times 10^{37}$ cgs units upwards for small p (see Table I). Λ_{theor} is decreased by 13% at 0.55 °K and by smaller amounts at lower temperatures.

In their original papers on the viscosity of helium, Landau and Khalatnikov[6,15] assumed that the viscosity mean free path was governed by four-phonon processes, these arising from the three-phonon interaction in the second order of perturbation theory. They had assumed that the dispersion was "normal" in the sense that the group and phase velocities at finite p were always less than c_0. The three-phonon process is then unallowed. While their assumption about the dispersion must now be regarded as wrong, it is still possible that the second-order process that they considered makes a significant contribution to the scattering rate. We think, however, that this is unlikely. According to their calculations, the viscosity mean free path should vary with temperature according to the law

$$\Lambda_{\mathrm{LK}} = AT^{-9} .$$

Although they do provide an explicit expression for the constant A in terms of known quantities, they also admit that there is considerable uncertainty in A due to the various approximations they were forced to make. Their value of A gives a mean free path 30% above the experimental value at 0.6 °K and about six times Λ_{expt} at 0.45 °K (see Fig. 4). The fact that the experimental results do not have a T^{-9} dependence[16] strongly suggests that Landau and Khalatnikov have somewhat underestimated the constant A, and that consequently their process can be neglected below 0.6 °K. If the correct value of A is three times their value, the four-phonon process would make a contribution of only about 25% of the three-phonon process at 0.6 °K, and only 5% at 0.45 °K. It would be interesting to attempt a more accurate calculation of the mean free path for four-phonon processes.

For phonons of energy above a critical energy ϵ_c, the three-phonon scattering rate vanishes. This is because the conditions of conservation of energy and momentum can no longer be satisfied. The critical energy is approximately 8 °K but varies slightly for the different dispersion curves. The mean free path of phonons with energy greater than ϵ_c will be limited by four-phonon processes. We have ignored these phonons in calculating λ_{1D}. This was achieved by choosing $j_{\max}$ and $\Delta\epsilon$ so that the mesh of points used in solving the eigenvalue problem had its upper limit below ϵ_c. The error resulting from this approximation should be very small. Even at 0.6 °K the fraction of the total energy of the phonon system which is carried by phonons of energy greater than 8 °K is less than 0.5%.

One final assumption that should be mentioned is that we have taken a particular analytic form for the dispersion curve. We have not investigated the effect of choosing other forms. It is clear, however, that a low-order polynomial such as

$$\epsilon = c_0 p(1 + ap^2 - bp^4)$$

will not give a good fit to the neutron data and the specific-heat results. One could, of course, choose any number of expressions such as

$$\epsilon = c_0 p \left(1 + \gamma p^2 \frac{1-(p/p_A)^4}{1-(p/p_B)^4}\right) .$$

However, in the absence of any strong theoretical arguments, choosing the dispersion relation (42) seems simpler and most natural.

Putting together these pieces of information, we can draw the following conclusions: (i) The theory describes the temperature dependence of the viscosity very well. (ii) Using the value $\gamma = 8\times10^{37}$ cgs units for the dispersion parameter gives agreement in the magnitude to within 20%. (iii) The combined uncertainties in the theory and the experiment are such that one cannot determine a definite value for γ. The most likely value appears to be somewhere between 8 and 12×10^{37} cgs units. The value of 12×10^{37}, however, does not appear to be consistent with the specific-heat results.

Finally, we note that measurements of the viscosity under pressure would be interesting. The specific-heat measurements of Phillips *et al.*[7] indicate that the dispersion parameter γ becomes smaller under pressure. Thus the scattering angle should decrease and the viscosity mean free path should increase. Near the freezing pressure the specific-heat measurements imply that the dispersion curve is normal. The three-phonon process should then be unallowed and the Landau-Khalatnikov theory should apply, giving a mean free path proportional to T^{-9}.

ACKNOWLEDGMENT

I should like to thank Dr. R. W. Whitworth for a very helpful discussion of his experimental results.

APPENDIX

If the dispersion has the form

$$\epsilon = c_0 p(1 + \gamma p^2) \tag{A1}$$

and $\gamma p^2 \ll 1$, it is possible to calculate the eigenvalues in the series $\lambda_{1D}, \lambda_{1F}$, etc. as a power series in the parameter

$$\gamma(k_B T/c_0)^2 .$$

The lowest nonvanishing term is[17]

$$\begin{aligned}\lambda_{1l} &= (135/1012\pi^5)(l-1)\,l\,(l+1)(l+2)\left[\gamma^2(u_0+1)^2/\rho\hbar^4\right](k_B T/c_0)^9 \int_0^1 dx\, x^4(1-x)^4 \int_0^\infty dy\, y^{14} \operatorname{csch}^2 \tfrac{1}{2}y \coth \tfrac{1}{2}xy \\ &= 17433\,(l-1)\,l\,(l+1)\,(l+2)\left[\gamma^2(u_0+1)^2/\rho\hbar^4\right](k_B T/c_0)^9 .\end{aligned} \tag{A2}$$

We have confirmed that for sufficiently small T ($T \leqslant 0.1$ °K for $\gamma = 8\times 10^{37}$ cgs units) the results obtained numerically agree with this analytic result.

It is very interesting to consider the way the analytic result depends upon γ and T. The collision rate Γ is proportional to T^5. For the dispersion law (A1) the collision angle is of the order of

$$\alpha \sim (k_B T/c_0)\gamma^{1/2}. \qquad \text{(A3)}$$

Hence, we may interpret Eq. (A2) as suggesting that

$$\lambda_{1D} \sim \Gamma\alpha^4 . \qquad \text{(A4)}$$

At first sight this is surprising, since one might expect

$$\lambda_{1D} \sim \Gamma\alpha^2 . \qquad \text{(A5)}$$

A naive argument leading to (A5) is the following. The small-angle collisions may be considered as a random-walk process in angle space. After N collisions the direction of a phonon will have changed by an angle

$$\theta \sim \alpha\sqrt{N} .$$

One therefore needs a number of steps of the order of

$$N_\perp = \pi^2/4\alpha^2 .$$

before the direction of a phonon will have changed by an angle of $\frac{1}{2}\pi$. Hence, the effective large-angle collision rate is

$$\lambda_{1D} \sim \Gamma/N_\perp \sim \Gamma\alpha^2 .$$

This argument is fallacious. To see this, consider a gas of phonons that have a net drift velocity in the z direction. The diffusion model, in which each phonon's direction follows an independent random walk, leads to the inescapable conclusion that after sufficient time the distribution of phonon wave vectors is isotropic. But this clearly violates conservation of momentum. The key point, of course, is that when a phonon is deflected in one direction in a collision, another particle must be deflected in the opposite direction. Thus the "diffusion" is highly correlated and proceeds much more slowly than the simple picture would predict.

---

*Science Research Council Senior Visiting Fellow for the year 1972–1973.

†Permanent address.

[1] R. W. Whitworth, Proc. R. Soc. A **246**, 390 (1958).

[2] H. J. Maris and W. E. Massey, Phys. Rev. Lett. **25**, 220 (1970).

[3] H. J. Maris, Phys. Rev. A **7**, 2074 (1973).

[4] For a review of kinetic processes in liquid helium, see I. M. Khalatnikov, *Introduction to the Theory of Superfluidity* (Benjamin, New York, 1965); J. Wilks, *The Properties of Liquid and Solid Helium* (Oxford U.P., London, 1967).

[5] G. Leibfried, in *Encyclopaedia of Physics*, edited by S. Flügge (Springer-Verlag, Berlin, 1955), Vol. 7.

[6] L. D. Landau and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **19**, 637 (1949). Translations of this paper and Ref. 15 are included in *Collected Papers of Landau*, edited by D. ter Haar (Pergamon, London, 1965).

[7] N. E. Phillips, C. G. Waterfield, and J. K. Hoffer, Phys. Rev. Lett. **25**, 1260 (1970).

[8] W. M. Whitney and C. E. Chase, Phys. Rev. **158**, 200 (1967).

[9] H. J. Maris, Phys. Rev. Lett. **28**, 277 (1972).

[10] Measurements in this region have also been made by C. H. Anderson and E. S. Sabisky [Phys. Rev. Lett. **28**, 80 (1972)]. These results are inconsistent with ultrasonic measurements by P. R. Roach, B. M. Abraham, J. B. Ketterson, and M. Kuchnir [Phys. Rev. Lett. **29**, 32 (1972)], and we have not taken them into account in estimating the dispersion curve.

[11] A. D. B. Woods and R. A. Cowley, Phys. Rev. Lett. **24**, 646 (1970); Can. J. Phys. **49**, 177 (1971).

[12] B. M. Abraham, Y. Eckstein, J. B. Ketterson, M. Kuchmir, and P. R. Roach, Phys. Rev. A **1**, 250 (1970); Phys. Rev. A **2**, 550 (1970).

[13] The problem of slip at the tube boundary has recently been considered in more detail by S. Simons [Proc. R. Soc. A **315**, 239 (1970)].

[14] E. C. Svensson, A. D. B. Woods, and P. Martel, Phys. Rev. Lett. **29**, 1148 (1972).

[15] L. D. Landau and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **19**, 709 (1949).

[16] The temperature dependence of the mean-free path does appear to increase above 0.6 °K. Whitworth emphasizes, however, that in this temperature range the results from different size tubes are no longer consistent with each other. Also, the roton contributions to the entropy and the scattering rate are no longer negligible.

[17] This result may be derived by the method used by S. Ma [J. Math. Phys. **12**, 2157 (1972)] for the dilute Bose gas. Note that there is an error in Ma's final result [Eq. (5.43)].