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Multiple-Time-Scale Analysis of Spontaneous Radiation Processes. I. One- and Two-Particle Systems*

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In this first article of work on radiation processes, a multiple-time-scale expansion method specifically for these processes is developed, based on the two distinct time scales corresponding to the natural frequency and the linewidth of the radiation. The method is then demonstrated by treating systems of one and two two-level atoms. It is seen that the exponential-decay behavior of the excited-state amplitude, assumed *a priori* by Weisskopf and Wigner, follows naturally from the mathematical procedure of eliminating the secularity in each order of the expansion. The results of two-atom systems also agree with those of previous workers.

I. INTRODUCTION

It is well known that owing to the coherence effect, the radiation from a system of many particles may be quite different from that of an isolated particle. Such a coherence effect was first noticed and formulated by Dicke.[1]

The first experimental evidence of the coherence effect appeared in 1964 as Kuhn and Vaughan[2] reported a measurement on the oscillation strength of the resonance transition $2P^1$-$1S^1$ in helium. Correspondingly, the radiation width of $2P^1$ was determined as $13.1\pm1.2\times10^{-3}$ cm^{-1}, in contrast to the theoretically calculated value 9.5×10^{-3} cm^{-1}.[3] A possible explanation of this descrepancy of about 30–40% has been suggested[4] in terms of coherence enhancement. In the years following 1964, more experimental works have been carried out by Kuhn and Lewis on Ne I,[5] by Vaughan on He[6] and on krypton.[7] Similar anomalies have been found in all these results. On the other hand, experiments performed by Korolev, Odintsov, and Fursova[8] on Ne, and by Williams and Fry[9] on the $2P^1$-$1S^1$ transition of helium atoms showed only ~8% deviation from the theoretical results. This makes it interesting to study this problem more carefully.

Since 1964, the coherence effect and related topics have received much attention from theorists. Schwabl and Thirring[10] discussed a system effectively containing countless atoms, with the atoms replaced by a field coupled to the radiation field in such a way as to lead to a soluble problem. Ali and Griem[11] calculated the resonance broadening of spectral lines by atom-atom impact. Stephen[12] and Hutchinson and Hameka[13] calculated the coherence broadening for a stationary two-atom system. In 1966, Omont[14] reviewed all the prior theoretical predictions. Tavis and Cummings[15] discussed N two-level systems interacting with a single mode of quantized radiation field. Dialetis[16] discussed the spontaneous emission from a set of N closely packed two-level identical atoms. Ernst and Stehle[17(a)] used the extended Wigner-Weisskopf method to study the N-particle system with all particles excited initially and found that extreme coherence properties are associated with this system. By using the result of Ref. 17(a), Ernst [17(b)] studied the problem of the origin of laser coherence. Lee and Lin[18] studied the averaged coherence broadening and also the effect of random motion on it. More recently, they[19] also studied the renormalized frequency shift. Lehmberg[20] and Eberly and Rehler,[21] studied the ray-forming properties and superradiant systems.

Recently, Chang and Stehle[22] studied the system using the two-particle Green's function; Haake and Glauber[23] studied the statistical aspect of the cooperative emission by a many-atom system; Stroud, Eberly, Lama, and Mandel[24] discussed the superradiant effects in two-level atom systems; and Shoemaker and Bremer[25] reported on two-photon superradiance.

In this series of articles, we apply the multiple-time-scale perturbation theory (MTSPT) to the analysis of spontaneous radiation processes. The MTSPT was first introduced by Krylov and Bogoliubov[26] to solve problems in nonlinear mechanics. Frieman[27] and Sandri[28] extended this method and applied it to the kinetic theory of gases and plasmas.

In the conventional time-dependent perturbation theory, an expansion is made in powers of the coupling constant ϵ. While ϵ itself may be small, it often happens that the nth-order term diverges like $\epsilon^n t^\alpha$ at large t, where α is generally a function of n. For example, this secular behavior at large t occurs in the calculation of transition rates in quantum mechanics. When t is small, the first few terms in the expansion may suffice for the description of the behavior of the system; but at large t, an appropriate sum over an infinite number of terms must be carried out in order to obtain a finite result. Since it is usually difficult to sum an infinite number of terms, particularly when it is not even known which terms should be included or left out, it would be advantageous if some other perturbation series can be found such that the first few orders will correctly describe the behavior of the system at large t. In the kinetic theory of gases and plasmas, this was achieved first by recognizing the existence of several distinct time scales in that theory[29] and then by constructing a multiple-time-scale perturbation expansion (MTSPE) based on these distinct time scales.[27,28]

In problems concerned with radiative processes, we realize that there also exist two very different time scales. One corresponds to the inverse of the frequency of the radiative transition, ω_0^{-1}, the other is the inverse of the radiation linewidth, γ^{-1}. Physically, the detailed evolution of the system in the fine time scale of ω_0^{-1} is not of much interest. In the MTSPE constructed for radiation problems essentially according to these time scales, such fine details will be averaged over in the rougher time scale of γ^{-1}. Mathematically, the MTSPE is actually a rearrangement of the terms in the conventional perturbation series to eliminate the secular behavior in each order. This rearrangement is facilitated by exploiting the extra degree of freedom provided by the multiple time variables.

In the present paper, the method of analysis will be demonstrated by deriving known results of spontaneous radiation from one- and two-atom systems. In the other papers of this series, the same method is applied to obtain new results of radiation from many-atom systems and to a system of particles with random motion.

Section II treats the single-atom system in considerable detail. To make our works reasonably self-contained, the method of MTSPT will be explained in detail in the present context of spontaneous radiation. Section III deals with a system of two stationary atoms. Results in both Secs. II and III are in agreement with those of previous workers.

II. SPONTANEOUS EMISSION FROM AN ISOLATED ATOM

Consider an atom with two nondegenerate levels, separated by an energy $\hbar\omega_0$. The radiation from such an atom initially in the excited state $|\uparrow\rangle$ was first calculated by Weisskopf and Wigner[30] by assuming in the beginning an exponential decay of the amplitude in the excited state, and then showing that the resulting expression is consistent with the assumption at large times. As we shall see, our method of analysis will lead to this exponential decay behavior at large times naturally, without having to assume it *a priori* on intuitive ground.

The interaction Hamiltonian of a system of two-level atoms with the traverse radiation field is given, in the resonance approximation, by[19]

$$H' = \sum_i h'(i)$$
$$= \sum_i \sum_{\vec{k}} [\epsilon \mathfrak{A}_{\vec{k}}^* \exp(i\vec{k}\cdot\vec{x}_i) \times C_{\vec{k}} R_+(i) + \epsilon \mathfrak{A}_{\vec{k}} \exp(-i\vec{k}\cdot\vec{x}_i) C_{\vec{k}}^\dagger R_-(i)], \quad (1)$$

where C_k and $C_k^\dagger$ denote, respectively, the photon annihilation and creation operators; $R_+(i)$ and $R_-(i)$ represent, respectively, the raising and lowering operators[1] for the ith atom; $\mathfrak{A}_{\vec{k}}$ and $\mathfrak{A}_{\vec{k}}^*$ are essentially the atomic dipole matrix elements. Specifically, for one-electron atoms, $\mathfrak{A}_{\vec{k}}$ is given by

$$\epsilon\mathfrak{A}_{\vec{k}} = -\left(\frac{2\pi\hbar\omega_k}{V}\right)^{1/2} e\hat{\mathcal{E}}_{\vec{k}} \cdot \frac{\langle\downarrow|\vec{p}/m|\uparrow\rangle}{\omega_k}, \quad (2)$$

where $\vec{p}$ is the momentum of the electron relative to the center of mass of the atom.

The Schrödinger equation for the state vector,

$$|\Psi(t)\rangle = a(t)|\uparrow; 0_{\vec{k}}\rangle + \sum_{\vec{k}} b_{\vec{k}}(t)|\downarrow; 1_{\vec{k}}\rangle, \quad (3)$$

can be written in terms of the amplitudes $a(t)$ and $b_{\vec{k}}(t)$ as

$$i\frac{d}{dt}a(t) = \sum_{\vec{k}} \epsilon\mathfrak{A}_{\vec{k}}^* e^{i\omega_{0k}t} b_{\vec{k}}(t) \quad (4)$$

and

$$i\frac{d}{dt}b_{\vec{k}}(t) = \epsilon\mathfrak{A}_{\vec{k}}^* e^{-i\omega_{0k}t} a(t) \quad (5)$$

with the initial conditions that

$$a(0) = 1, \quad b_{\vec{k}}(0) = 0, \quad (6)$$

where $\omega_{0k} \equiv \omega_0 - \omega_k$. In the above equations, ϵ is the coupling parameter, introduced here to keep track of the order of perturbation and is to be put equal to unity at the end of the calculation.

As is well known, the ratio of the two time scales ω_0^{-1} and γ^{-1} in this problem is of the order of $\epsilon^2(e^2/\hbar c)(v^2/c^2)$, where v is roughly the electron velocity within the atom. This ratio is much smaller than unity and it is this smallness parameter which renders here the successful application of the multiple-time-scale method. Although the above smallness parameter is proportional to ϵ^2, it is more systematic to introduce the multiple time scales[27,28] according to each power of ϵ. Thus we replace the original single time variable t by τ, which represents collectively the variables $\tau_0, \tau_1, \tau_2, \ldots$, i.e., $t \to \tau = (\tau_0, \tau_1, \tau_2, \ldots,)$. The τ_n's are defined by

$$\frac{\partial}{\partial t}\tau_n = \epsilon^n \tag{7}$$

or

$$\tau_n = \epsilon^n t + C_n . \tag{7'}$$

For convenience we specify that $C_0 = 0$ so that $\tau_0 = t$. All τ_n's are independent of each other because the integration constants C_n can be arbitrary. Correspondingly, new functions $a(\tau)$, $b_{\vec{k}}(\tau)$ of the many variables τ_n are introduced. They are required to satisfy equations similar to Eqs. (4) and (5), namely,

$$i\left(\frac{\partial}{\partial t}a(\tau)\right)_{C_1, C_2 \ldots} = \sum_{\vec{k}} \epsilon \alpha_{\vec{k}}^{*} e^{i\omega_{0k}\tau_0} b_{\vec{k}}(\tau), \tag{4'}$$

$$i\left(\frac{\partial}{\partial t}b_{\vec{k}}(\tau)\right)_{C_1, C_2, \cdots} = \epsilon \alpha_{\vec{k}} e^{-i\omega_{0k}\tau_0} a(\tau). \tag{5'}$$

The initial conditions (6) are now replaced by

$$\begin{aligned} &a(\tau_0 = 0, \tau_1 = 0, \tau_2 = 0, \ldots) = 1, \\ &b_{\vec{k}}(\tau_0 = 0, \tau_1 = 0, \tau_2 = 0, \ldots) = 0 . \end{aligned} \tag{6'}$$

From Eqs. (4')–(7') it is clear that when all $C_n = 0$, the new functions $a(\tau_0 = t, \tau_1 = \epsilon t, \tau_2 = \epsilon^2 t, \ldots)$ and $b_{\vec{k}}(\tau_0 = t, \tau_1 = \epsilon t, \tau_2 = \epsilon^2 t, \ldots)$ become exactly equal to the original amplitudes $a(t)$ and $b_{\vec{k}}(t)$, respectively. However, owing to the additional variables $\tau_1, \tau_2, \ldots$ introduced into these amplitudes, Eqs. (4')–(6') alone will no longer be sufficient to determine them uniquely when the C_n's are not all zeros. Additional conditions must then be imposed. In the present case, the introduction of every new variable τ_n (or equivalently, C_n) must be accompanied by new conditions on the partial derivatives $[(\partial/\partial\tau_n)a]_{\tau_{m\neq n}}$ and $[(\partial/\partial\tau_n)b_{\vec{k}}]_{\tau_{m\neq n}}$ {or equivalently, $[(\partial/\partial C_n)a]_{t, C_{m\neq n}}$ and $[(\partial/\partial\tau_n)b_{\vec{k}}]_{t, C_{m\neq n}}$} for the unique determination of $a(\tau)$ and $b_{\vec{k}}(\tau)$. Mathematically, it is exactly these new conditions, which are completely at our disposal, that are at the heart of the method. After we obtain $a(\tau)$ and $b_{\vec{k}}(\tau)$ through the use of these conditions we set all $C_n = 0$ to get back to $a(t)$ and $b_{\vec{k}}(t)$. Physically, the partial derivatives on the left-hand side of Eqs. (4') and (5'),

$$\left(\frac{\partial}{\partial t}\right)_{C_1, C_2, \ldots} = \left(\frac{\partial}{\partial\tau_0}\right)_{\tau_{n\neq 0}} + \epsilon\left(\frac{\partial}{\partial\tau_1}\right)_{\tau_{n\neq 1}} + \epsilon^2\left(\frac{\partial}{\partial\tau_2}\right)_{\tau_{n\neq 2}} + \cdots, \tag{8}$$

will enable us to look at the time evolution of the amplitudes according to the different time scales.

We now try to solve Eqs. (4') and (6') by expanding $a(\tau)$ and $b_{\vec{k}}(\tau)$,

$$a(\tau) = \sum_{n=0}^{\infty} \epsilon^n a^{(n)}(\tau_0, \tau_1, \tau_2, \ldots), \tag{9}$$

$$b_{\vec{k}}(\tau) = \sum_{n=0}^{\infty} \epsilon^n b_{\vec{k}}^{(n)}(\tau_0, \tau_1, \tau_2, \ldots). \tag{10}$$

The derivatives $(\partial/\partial\tau_n)a^{(m)}(\tau)$ and $(\partial/\partial\tau_n)b_{\vec{k}}^{(m)}(\tau)$ for $n \geq 1$ and for any m are then completely at our disposal. Note that even though the above expansions are just series in powers of ϵ, the coefficients depend not only on the variable $\tau_0(=t)$ but also on the other variables τ_n. When we finally set $\tau_n = \epsilon^n t$ to get back to our original problem, these expansions will not be simple power series for $a(t)$, $b_{\vec{k}}(t)$ since the coefficients $a^{(n)}(\tau_0 = t, \tau_1 = \epsilon t, \tau_2 = \epsilon^2 t, \ldots)$ and $b_{\vec{k}}^{(n)}(\tau_0 = t, \tau_1 = \epsilon t, \tau_2 = \epsilon^2 t, \ldots)$ will then also contain powers of ϵ. Hence, the terms in the conventional perturbation series for these amplitudes will have been effectively rearranged in these multiple-time-scale expansions.

Substituting (8)–(10) into (4') and (5') we obtain the following equations:

$$\epsilon^0 : i\,\frac{\partial}{\partial\tau_0}a^{(0)}(\tau) = 0, \tag{11a}$$

$$\epsilon^1 : i\left(\frac{\partial}{\partial\tau_0}a^{(1)}(\tau) + \frac{\partial}{\partial\tau_1}a^{(0)}(\tau)\right) = \sum_{\vec{k}} \alpha_{\vec{k}}^{*} e^{i\omega_{0k}\tau_0} b_{\vec{k}}^{(0)}(\tau), \tag{11b}$$

$$\epsilon^2 : i\left(\frac{\partial}{\partial\tau_0}a^{(2)}(\tau) + \frac{\partial}{\partial\tau_1}a^{(1)}(\tau) + \frac{\partial}{\partial\tau_2}a^{(0)}(\tau)\right) = \sum_{\vec{k}} \alpha_{\vec{k}}^{*} e^{i\omega_{0k}\tau_0} b_{\vec{k}}^{(0)}(\tau), \tag{11c}$$

$$\epsilon^0 : i(\partial/\partial\tau_0)b_{\vec{k}}^{(0)}(\tau) = 0, \tag{12a}$$

$$\epsilon^1 : i[(\partial/\partial\tau_0)b_{\vec{k}}^{(1)}(\tau) + (\partial/\partial\tau_1)b_{\vec{k}}^{(0)}(\tau)] = \alpha_{\vec{k}} e^{-i\omega_{0k}\tau_0} a^{(0)}(\tau), \tag{12b}$$

$$\epsilon^2 : i\left[\frac{\partial}{\partial\tau_0}b_{\vec{k}}^{(2)}(\tau) + \frac{\partial}{\partial\tau_1}b_{\vec{k}}^{(1)}(\tau) + \frac{\partial}{\partial\tau_2}b_{\vec{k}}^{(0)}(\tau)\right] = \alpha_{\vec{k}} e^{-i\omega_{0k}\tau_0} a^{(1)}(\tau). \tag{12c}$$

From (11a) and (12a), it is clear that $a^{(0)}$ and $b_{\vec{k}}^{(0)}$ do not vary in the time scale τ_0. With this fact in mind, an integration of (11b) over τ_0 leads to

$$a^{(1)}(\tau_0, \tau_1, \ldots) = \tau_0 \frac{\partial}{\partial \tau_1} a^{(0)}(\tau_1, \tau_2, \ldots) + a^{(1)}(\tau_0 = 0, \tau_1, = \ldots) - i \sum_{\vec{k}} \alpha_{\vec{k}}^{*} \int_0^{\tau_0} du e^{i\omega_{0k}u} b_{\vec{k}}^{(0)}(\tau_1, \ldots). \tag{13}$$

In the limit of large τ_0, the integral $\int_0^{\tau_0} du e^{i\omega_{0k}u}$ behaves like a ζ function,[31] i.e.,

$$\lim_{\tau_0 \to \infty} \int_0^{\tau_0} du e^{i\omega_{0k}u} = i\zeta(\omega_{0k}) = i\left(\frac{\wp}{\omega_{0k}} - i\pi\delta(\omega_{0k})\right),$$

which is independent of τ_0, while the term $\tau_0(\partial/\partial\tau_1)a^{(0)}(\tau_1, \ldots)$ diverges linearly as τ_0. Here $(\wp/\omega_{0k})$ stands for the principle value of $(\omega_{0k})^{-1}$. The secular behavior of $a^{(1)}$ due to the latter term may now be eliminated by imposing the condition $(\partial/\partial\tau_1)a^{(0)}(\tau_1, \tau_2, \ldots) = 0$. This simply means that $a^{(0)}(\tau)$ is not only independent of τ_0, but also independent of τ_1.

Using a similar argument, it is clear that $b_{\vec{k}}^{(0)}(\tau)$ is also independent of τ_0 and τ_1. We may also impose a first-order condition associated with τ_1 that $(\partial/\partial\tau_1)a^{(1)}(\tau_0 = 0, \tau_1, \ldots) = 0$. Consequently, we obtain with the help of (6′) that

$$a^{(1)} = -i \sum_{\vec{k}} \alpha_{\vec{k}}^{*} \int_0^{\tau_0} du e^{i\omega_{0k}u} b_{\vec{k}}^{(0)}(\tau_2, \ldots), \tag{14}$$

$$b_{\vec{k}}^{(1)} = -i\alpha_{\vec{k}} \int_0^{\tau_0} du e^{-i\omega_{0k}u} a^{(0)}(\tau_2, \ldots). \tag{15}$$

The above equations show that $a^{(1)}(\tau)$ and $b_{\vec{k}}^{(1)}(\tau)$ are independent of τ_1. Substitution of (15) into (11c) yields

$$\frac{\partial}{\partial \tau_0} a^{(2)}(\tau) + \frac{\partial}{\partial \tau_2} a^{(0)}(\tau_2, \ldots) = - \sum_{\vec{k}} |\alpha_{\vec{k}}|^2 \int_0^{\tau_0} du e^{i\omega_{0k}(\tau_0 - u)} a^{(0)}(\tau_2, \ldots). \tag{16}$$

An integration over τ_0 then gives

$$a^{(2)}(\tau) = -\tau_0 \frac{\partial}{\partial \tau_2} a^{(0)}(\tau_2, \ldots) - \int_0^{\tau_0} d\tau_0' \times \left\{ \sum_{\vec{k}} |\alpha_{\vec{k}}|^2 \int_0^{\tau_0'} du e^{i\omega_{0k}u} \right\} a^{(0)}(\tau_2, \ldots), \tag{17}$$

where the term $a^{(2)}(\tau_0 = 0, \tau_1, \tau_2, \ldots)$ has been set to zero by imposing $(\partial/\partial\tau_1)a^{(2)}(\tau_0 = 0, \tau_1, \tau_2, \ldots) = (\partial/\partial\tau_2)a^{(2)}(\tau_0 = 0, \tau_1, \tau_2, \ldots) = 0$ and by the use of (6′).

To investigate this equation, let us first consider

$$\int_0^{\tau_0} d\tau_0' \int_0^{\tau_0'} du e^{i\omega_{0k}u} = i\left(\frac{\tau_0}{\omega_{0k}} - i\frac{1 - e^{i\omega_{0k}\tau_0}}{\omega_{0k}^2}\right)$$

which behaves like $i\tau_0\zeta(\omega_{0k})$ at large τ_0. To eliminate the secular behavior, we have to require that

$$\frac{\partial}{\partial \tau_2} a^{(0)}(\tau_2, \ldots) + i\left\{\sum_{\vec{k}} |\alpha_{\vec{k}}|^2 \zeta(\omega_{0k})\right\} \times a^{(0)}(\tau_2, \ldots) = 0 . \tag{18}$$

If we require further that $(\partial a^{(0)}/\partial\tau_n)(\tau_2 = 0, \tau_3, \ldots) = 0$ for $n > 2$, the solution to (18) which also satisfies (6′) is

$$a^{(0)}(\tau_2) = e^{-i\omega_s\tau_2 - \gamma_s\tau_2}, \tag{19}$$

where

$$\omega_s \equiv \sum_{\vec{k}} |\alpha_{\vec{k}}|^2 \wp \frac{1}{\omega_{0k}}, \tag{20}$$

$$\gamma_s \equiv \pi \sum_{\vec{k}} |\alpha_{\vec{k}}|^2 \delta(\omega_{0k}). \tag{21}$$

Note that the $a^{(0)}(\tau)$ thus obtained is finite and valid for any τ_2, showing no secular behavior. Similar considerations in higher orders actually show that

$$a^{(2n+1)} = 0 \tag{22}$$

and

$$a^{(2n)}(\tau_0 \to \text{large}, \tau_2) = 0 . \tag{23}$$

Hence we have

$$a(\tau_0 \to \text{large}, \tau_2) = a^{(0)}(\tau_3), \tag{24}$$

which leads, upon putting $\epsilon = 1$ and all $C_n = 0$, to

$$a(t \to \text{large}) = e^{-i\omega_s t - \gamma_s t} . \tag{25}$$

Substitution of (25) into (5) yields, after integration,[32]

$$b_{\vec{k}}(t) = \alpha_{\vec{k}} \frac{e^{-i\omega_{0k}t} e^{-i(\omega_s - i\gamma_s)t} - 1}{\omega_{0k} + \omega_s - i\gamma_s}. \tag{26}$$

Equations (25) and (26) reproduce the Weisskopf-Wigner solution.[30] Several remarks are in order here: (i) Although many time variables were introduced in (7), only τ_0 and τ_2 appear in Eqs. (19)–(24). After setting all C_n equal to zero, it is apparent from (25) and (26) that these two variables $\tau_0 = t$ and $\tau_2 = \epsilon^2 t$ exactly correspond to the two physical time scales discussed in Sec. I. (ii) Rather than being assumed *a priori* on intuitive ground, the exponential decay behavior of the amplitude a in (25) follows directly as a result of the mathematical procedure of eliminating the secular behavior in each order of the MTSPE. (iii) That the amplitude a is independent of the finer time scale τ_0 in Eqs. (19)–(24) results from the use of mathematical identities such as

$$\lim_{\tau_0 \to \text{large}} \int_0^{\tau_0} du\, e^{i\omega_{0k}u} = i\zeta(\omega_{0k}), \tag{27}$$

where the right-hand side no longer depends on τ_0. Here, the limit of large τ_0 actually means in our context that t becomes so large that the variable $\tau_2 = \epsilon^2 t$ may take on finite values. Physically what happens is that the rapid fluctuations as seen in the time scale of ω_0^{-1} are averaged out in the time scale of γ^{-1}.

III. SPONTANEOUS EMISSION FROM TWO STATIONARY ATOMS

Consider two stationary atoms, each with two nondegenerate levels as in Sec. II. If we assume that only one of the atoms is excited initially at $t=0$, then the state of the system at any later time may be written

$$|\Psi(t)\rangle = b_1(t)|\uparrow\downarrow; 0_{\vec{k}}\rangle + b_2(t)|\downarrow\uparrow; 0_{\vec{k}}\rangle + \sum_{\vec{k}} b_{\vec{k}}(t)|\downarrow\downarrow; 1_{\vec{k}}\rangle , \tag{28}$$

with

$$b_1(0)=1, \quad b_2(0)=b_{\vec{k}}(0)=0 . \tag{29}$$

The Schrödinger equation in the interaction representation leads to the following equations for the amplitudes:

$$i\frac{d}{dt}b_1(t) = \epsilon \sum_{\vec{k}} \alpha_{\vec{k}}^* e^{i\omega_{0k}t} e^{i\vec{k}\cdot\vec{x}_1} b_{\vec{k}}(t), \tag{30}$$

$$i\frac{d}{dt}b_2(t) = \epsilon \sum_{\vec{k}} \alpha_{\vec{k}}^* e^{i\omega_{0k}t} e^{i\vec{k}\cdot\vec{x}_2} b_{\vec{k}}(t), \tag{31}$$

$$i\frac{d}{dt}b_{\vec{k}}(t) = \epsilon \alpha_{\vec{k}} e^{-i\omega_{0k}t} \sum_{j=1}^{2} e^{-i\vec{k}\cdot\vec{x}_j} b_j(t). \tag{32}$$

As in Sec. II, we introduce the multiple-time variables τ of (7′) and the multiple-time-scale perturbation expansions

$$b_j(\tau) = \sum_{n=0}^{\infty} \epsilon^n b_j^{(n)}(\tau), \quad j=1,2 \tag{33}$$

$$b_{\vec{k}}(\tau) = \sum_{n=0}^{\infty} \epsilon^n b_{\vec{k}}^{(n)}(\tau). \tag{34}$$

Equations for $b_j(\tau)$ and $b_{\vec{k}}(\tau)$ are just the same as Eqs. (30)–(32), except that d/dt is replaced by $(\partial/\partial t)_{c_1, c_2, \ldots}$ given in (8). These equations can then be analyzed in each order. The amplitudes $b_j^{(0)}(\tau)$ will again be independent of τ_0 and τ_1. Similar to Eqs. (16)–(18), eliminating the secular behavior in the second order leads to

$$\frac{\partial}{\partial\tau_2} b_1^{(0)} = -f_{11}b_1^{(0)} - f_{12}b_2^{(0)}, \tag{35}$$

$$\frac{\partial}{\partial\tau_2} b_2^{(0)} = -f_{21}b_1^{(0)} - f_{22}b_2^{(0)}, \tag{36}$$

where

$$f_{ij} = +i\omega_j + \gamma_{ij}, \qquad i,j=1,2 \tag{37}$$

$$\omega_{ij} \equiv \sum_{\vec{k}} |\alpha_{\vec{k}}|^2 \frac{\wp}{\omega_{0k}} e^{i\vec{k}\cdot\vec{x}_{ij}}, \tag{38}$$

$$\gamma_{ij} \equiv \pi \sum_{\vec{k}} |\alpha_{\vec{k}}|^2 e^{i\vec{k}\cdot\vec{x}_{ij}} \delta(\omega_{0k}), \tag{39}$$

and $\vec{x}_{ij} \equiv \vec{x}_i - \vec{x}_j$. Note that $\omega_{ii} = \omega_s$ of (20) and $\gamma_{ii} = \gamma_s$ of (21).

It is a simple matter to solve the two simultaneous, linear, homogeneous differential equations (35) and (36). Going through steps analogous to Eqs. (22)–(26) we finally obtain, in the limit of large t,

$$b_1(t) = \tfrac{1}{2}(e^{-\alpha_1 t} + e^{-\alpha_2 t}), \tag{40}$$

$$b_2(t) = \tfrac{1}{2}(-e^{-\alpha_1 t} + e^{-\alpha_2 t}), \tag{41}$$

$$b_{\vec{k}}(t) = \tfrac{1}{2}\alpha_{\vec{k}}\left[(e^{-i\vec{k}\cdot\vec{x}_1} - e^{-i\vec{k}\cdot\vec{x}_2}) \frac{e^{-(i\omega_{0k}+\alpha_1)t} - 1}{\omega_{0k} + i\alpha_1} + (e^{-i\vec{k}\cdot\vec{x}_1} + e^{-i\vec{k}\cdot\vec{x}_2}) \frac{e^{-(i\omega_{0k}+\alpha_2)t} - 1}{\omega_{0k} + i\alpha_2}\right], \tag{42}$$

where

$$\alpha_1 = i \sum_{\vec{k}} |\alpha_{\vec{k}}|^2 \zeta(\omega_{0k})(1 - \cos\vec{k}\cdot\vec{x}_{12}), \tag{43}$$

$$\alpha_2 = i \sum_{\vec{k}} |\alpha_{\vec{k}}|^2 \zeta(\omega_{0k})(1 + \cos\vec{k}\cdot\vec{x}_{12}). \tag{44}$$

These results, (40)–(44), agree with those in Refs. 12, 13, 18, and 19, and can be compared with those obtained by Ernst[17(b)] and Chang and Stehle.[22]

Comparing the present section with Sec. II we see that although the physical results (26) and (42) have different significance in that the radiation (42) from a two-particle system shows the coherence effect especially when $k_0 x_{12} \ll 1$, the mathematics of the MTSPT leading to them is essentially the same. We are therefore prompted to look into more complicated systems. These will be discussed in the subsequent papers of this series.

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[32] This integration is justified by noticing that $d/dt = (\partial/\partial\tau_0) + \epsilon(\partial/\partial\tau_1) + \epsilon^2(\partial/\partial\tau_2) + \cdots$ and $(\partial/\partial\tau_2) b_{\vec{k}}^{(2n+1)} + (\partial/\partial\tau_0) b_{\vec{k}}^{(2n+1)} = -i \mathfrak{a}_{\vec{k}} e^{-i\omega_0 k \tau_0} a^{(2n)}$. The second condition can be easily derived from Eqs. (12a)–(12c) and (15) together with the higher-order equations. Substituting them back into the original equations, it is readily observed that this integration over τ_0: $0 \to$ large τ_0 is exactly the same as that of a direct integration over t when t is large.

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Multiple-Time-Scale Analysis of Spontaneous Radiation Processes. II. Many-Particle Systems*

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The multiple-time-scale expansion method has been used to study a system of N-two-level atoms with one of them excited initially. It is found that, in the limit of large time, the Schrödinger equation for the system reduces to a set of algebraic equations, and can thus be solved exactly. A detailed study of a four-atom system is given. The analysis is then extended to a regular lattice of N atoms. It is found that in a large lattice, the probability for the excitation energy to be trapped is very large, even when the lattice spacing is comparable to the characteristic wavelength of the radiation.

I. INTRODUCTION

Since the pioneering work of Dicke[1] on the superradiant states of a system of atoms, there have been numerous experimental and theoretical investigations on the coherence effects in spontaneous radiation. A brief survey has been given in the preceding paper.[2]

It is well known that the coherence effect due to the radiative coupling is especially important