Phase Transitions in Some Generalized Dicke Models of Superradiance*

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The Dicke model of superradiance studied in a previous paper by Wang and Hioe is generalized (i) to include an interaction energy of the form $a^\dagger\sigma^+ + a\,\sigma^-$, where $a, a^\dagger$ and σ^-, σ^+ are the usual photon and spin operators, (ii) to include the spatial variation of the electromagnetic field, and (iii) to include the possibility of small classical motions of atoms from their equilibrium positions. The free energies of these generalized models are calculated exactly in the thermodynamic limit, and their phase-transition properties are presented.

I. INTRODUCTION

In a previous paper, Wang and Hioe[1] (hereafter called Paper I) studied and solved the Dicke model of superradiance with a new method which is quite different from the approach of Hepp and Lieb.[2] The Dicke model[3] is a simplified model of a system of N two-level atoms interacting with a quantized field. The atoms are considered to be at fixed positions within a linear cavity of volume V and the separations between the atoms are assumed to be large enough so that the direct interaction among them can be ignored. The reader is referred to Paper I for a more detailed description of the Dicke model. In this paper, we shall apply the method used in Paper I to solve a more general type of Dicke model. Specifically, we shall consider the following simple generalizations of the model: (i) The Hamiltonian includes an interaction between the photon field and the spins of the form $a^\dagger\sigma^+ + a\sigma^-$, where $a, a^\dagger$ and σ^-, σ^+ are the usual photon and spin operators; (ii) the spatial variation of the electromagnetic field, which is assumed to be of a simple sinusoidal form, is taken into consideration; and (iii) the atoms are assumed to make small classical motions about their equilibrium positions. These models, which we shall call models A, B, and C, respectively, for easy reference, will be considered separately in order to illustrate clearly the effects of the variations of the Hamiltonian on the thermodynamic properties of the system. The exact solutions of these models in the thermodynamic limit that $N, V \to \infty$, N/V is finite, and their phase-transition properties are presented in Secs. II–IV. A brief conclusion is given in Sec. V.

It should be pointed out that our calculations are based on the same two assumptions used in Paper I, namely, (i) the limits as $N \to \infty$ of the field operators $a/\sqrt{N}$ and $a^\dagger/\sqrt{N}$ are assumed to exist, and (ii) the order of the double limit in the exponential series

$$\lim_{N\to\infty}\lim_{R\to\infty}\sum_{r=0}^{R}\frac{(-\beta H_N)^r}{r!},$$

where β denotes, as usual, $1/kT$ and H_N denotes the Hamiltonian, is assumed to be interchangeable. We hope to provide a rigorous justification of these assumptions in a future publication.

II. MODEL A

We recall that the Hamiltonian of the Dicke model is given by [$\hbar = 1$, see Eq. (6) of Paper I]

$$H = a^\dagger a + \sum_{j=1}^{N}\left(\tfrac{1}{2}\epsilon\sigma_j^z + \frac{\lambda}{2\sqrt{N}}\,(a\sigma_j^+ + a^\dagger\sigma_j^-)\right), \tag{1}$$

where $a^\dagger$ and a are the photon creation and annihilation operators; σ_j^+ and σ_j^- are the spin operators for the jth atom defined by $\sigma_j^+ = \sigma_j^x + i\sigma_j^y$, $\sigma_j^- = \sigma_j^x - i\sigma_j^y$, σ^x and σ^y being the x and y components of the Pauli-spin matrices; λ measures the coupling of the interaction terms $a\sigma_j^+ + a^\dagger\sigma_j^-$ and $\epsilon = \omega/\nu$, $\hbar\omega$ being the energy difference between the two levels of the atom and ν being the frequency of the electromagnetic wave; and N is the total number of atoms.

We now consider the following Hamiltonian, which has the additional interaction terms $a^\dagger\sigma_j^+ + a\sigma_j^-$ (the so-called antiresonant terms), namely,

$$H = a^\dagger a + \sum_{j=1}^{N}\left(\tfrac{1}{2}\epsilon\sigma_j^z + \frac{\lambda}{2\sqrt{N}}\,(a\sigma_j^+ + a^\dagger\sigma_j^-) + \frac{\lambda'}{2\sqrt{N}}\,(a^\dagger\sigma_j^+ + a\sigma_j^-)\right), \tag{2}$$

where the coupling constants λ and λ' are assumed to be real and positive. We shall call the model with the above Hamiltonian model A.

Using the method described in Paper I, it follows that the expectation value of $e^{-\beta H}$ with respect to the set of Glauber's coherent states[4] α's is given, as $N\to\infty$, by

$$\langle\alpha|e^{-\beta H}|\alpha\rangle=\exp\left\{-\beta\left[|\alpha|^2+\sum_{j=1}^{N}\left(\tfrac{1}{2}\epsilon\sigma_j^z+\frac{1}{2\sqrt{N}}\left[(\lambda\alpha^*+\lambda'\alpha)\sigma_j^-+(\lambda\alpha+\lambda'\alpha^*)\sigma_j^+\right]\right)\right]\right\} \tag{3}$$

and that the partition function $Z(N,T)$ of the system, in the limit $N\to\infty$, is given by

$$Z(N,T)=\int\frac{d^2\alpha}{\pi}e^{-\beta|\alpha|^2}\left\{2\cosh\left[\frac{\beta\epsilon}{2}\left(1+\frac{4}{\epsilon^2N}\left[(\lambda^2+\lambda'^2)|\alpha|^2+\lambda\lambda'(\alpha^{*2}+\alpha^2)\right]\right)^{1/2}\right]\right\}^N, \tag{4}$$

where $\int d^2\alpha$ means $\iint d(\mathrm{Re}\alpha)\,d(\mathrm{Im}\alpha)$. By letting $\alpha=x+iy$ and $\alpha^*=x-iy$,

$$Z(N,T)=\int\int\frac{dx\,dy}{\pi}e^{-\beta(x^2+y^2)}\left\{2\cosh\left[\frac{\beta\epsilon}{2}\left(1+\frac{4}{\epsilon^2N}\left[(\lambda+\lambda')^2x^2+(\lambda-\lambda')^2y^2\right]\right)^{1/2}\right]\right\}^N. \tag{5}$$

Let us first assume $\lambda>\lambda'$. Let $\lambda+\lambda'=a$, $\lambda-\lambda'=b$, $ax=r\cos\theta$, (6)

and

$$by=r\sin\theta; \tag{7}$$

then

$$a^2x^2+b^2y^2=r^2 \tag{8}$$

and

$$x^2+y^2=r^2(\cos^2\theta/a^2+\sin^2\theta/b^2), \tag{9}$$

$$dx\,dy=\frac{\partial(x,y)}{\partial(r,\theta)}dr\,d\theta=\begin{vmatrix}\dfrac{\partial x}{\partial r}&\dfrac{\partial x}{\partial\theta}\\ \dfrac{\partial y}{\partial r}&\dfrac{\partial y}{\partial\theta}\end{vmatrix}dr\,d\theta \tag{10}$$

$$=\begin{vmatrix}\dfrac{\cos\theta}{a}&-\dfrac{r\sin\theta}{a}\\ \dfrac{\sin\theta}{b}&\dfrac{r\cos\theta}{b}\end{vmatrix}dr\,d\theta=\frac{r}{ab}\,dr\,d\theta.$$

Therefore, substituting the transformed variables r and θ into Eq. (4), we obtain

$$Z(N,T)=\frac{1}{\pi ab}\int_0^{2\pi}\exp\left\{-\beta r^2\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)\right\}d\theta\times\int_0^\infty r\,dr\left\{2\cosh\left[\frac{\beta\epsilon}{2}\left(1+\frac{4}{\epsilon^2N}r^2\right)^{1/2}\right]\right\}^N. \tag{11}$$

Let $r^2/N=y$, then

$$Z(N,T)=\frac{N}{2ab\pi}\int_0^{2\pi}\exp\left\{-N\beta y\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)\right\}d\theta\times\int_0^\infty dy\exp\left\{N\ln 2\cosh\left[\frac{\beta\epsilon}{2}\left(1+\frac{4}{\epsilon^2}y\right)^{1/2}\right]\right\}. \tag{12}$$

Consider the integral

$$\int_0^{2\pi}\exp\left[-N\beta y\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)\right]d\theta=4\int_0^{\pi/2}\exp\left\{-N\beta y\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)\right\}d\theta. \tag{13}$$

This integral can be readily evaluated using the Laplace method.[5] Let

$$\psi(\theta)=\beta y(\cos^2\theta/a^2+\sin^2\theta/b^2), \tag{14}$$

$$\frac{d\psi(\theta)}{d\theta}=\beta y(+2\sin\theta\cos\theta)\left(-\frac{1}{a^2}+\frac{1}{b^2}\right), \tag{15}$$

$$\frac{d^2\psi(\theta)}{d\theta^2}=2\beta y\left(\frac{1}{b^2}-\frac{1}{a^2}\right)(\cos^2\theta-\sin^2\theta). \tag{16}$$

Thus, $\sin\theta=0$ corresponds to the maximum of $\psi(\theta)$, and

$$\psi(\theta)_{\max}=\beta y/a^2. \tag{17}$$

Therefore, as $N\to\infty$,

$$Z(N,T)=\frac{2CN}{\pi ab\sqrt{N}}\int_0^\infty\exp N\left\{-\frac{\beta y}{a^2}+\ln 2\cosh\left[\frac{\beta\epsilon}{2}\left(1+\frac{4}{\epsilon^2}y\right)^{1/2}\right]\right\}dy$$
$$=\frac{2aCN}{\pi b\sqrt{N}}\int_0^\infty\exp N\left\{-\beta y+\ln 2\cosh\left[\frac{\beta\epsilon}{2}\left(1+\frac{4a^2}{\epsilon^2}y\right)^{1/2}\right]\right\}dy, \tag{18}$$

where C is some constant. Applying the Laplace method for the evaluation of the integral (18) and proceeding as for the case $\lambda'=0$ discussed in Paper I, the equation which determines the critical temperature of the system becomes

$$(\epsilon/a^2)\eta=\tanh(\tfrac{1}{2}\beta\epsilon\eta), \tag{19}$$

where

$$\eta=[1+(4a^2/\epsilon^2)y]^{1/2}. \tag{20}$$

Thus, the phase-transition property of the systems is precisely the same as for the case $\lambda'=0$ considered in Paper I with, however, λ replaced by $a=\lambda+\lambda'$ [see Eq. (29) of Paper I].

In the case $\lambda'=\lambda$, the partition function $Z(N,T)$ becomes

$$Z(N,T)=\frac{1}{\pi}\int_{-\infty}^{\infty}dy\,e^{-\beta y^2}\int_{-\infty}^{\infty}e^{-\beta x^2}\left\{2\cosh\left[\frac{\beta\epsilon}{2}\left(1+\frac{4\Lambda^2}{\epsilon^2 N}x^2\right)^{1/2}\right]\right\}^N dx\,, \tag{21}$$

where $\Lambda=2\lambda$.

In this case, instead of letting $x^2/N=u$, let us put $x^2/N=u^2$. Then

$$Z(N,T)=C\sqrt{N}\int_{-\infty}^{\infty}du\,\exp\left(N\left\{-\beta u^2+\ln 2\cosh\left[\frac{\beta\epsilon}{2}\left(1+\frac{4\Lambda^2}{\epsilon^2}u^2\right)^{1/2}\right]\right\}\right). \tag{22}$$

The integral can be evaluated by the Laplace method as before. Thus, letting

$$\phi(u)=-\beta u^2+\ln 2\cosh\left[\frac{\beta\epsilon}{2}\left(1+\frac{4\Lambda^2}{\epsilon^2}u^2\right)^{1/2}\right], \tag{23}$$

then

$$\frac{d\phi(u)}{du}=-2\beta u+\frac{2\beta\Lambda^2 u}{\epsilon}\left(1+\frac{4\Lambda^2}{\epsilon^2}u^2\right)^{-1/2}\times\tanh\left[\frac{\beta\epsilon}{2}\left(1+\frac{4\Lambda^2}{\epsilon^2}u^2\right)^{1/2}\right]. \tag{24}$$

$\phi(u)_{\max}$ is determined by putting $\phi'(u)=0$ as before. It is easy to show that the free-energy and the phase-transition property of the system are the same as in the previous case with $\Lambda=2\lambda$ in the place of $a=\lambda+\lambda'$.

Summarizing the thermodynamic properties of model A: (i) if $a^2<\epsilon$, no phase transition occurs in the system at any temperature, and the free energy per atom $f(T)$ is given by $-\beta f(T)=\ln[2\cosh(\frac{1}{2}\beta\epsilon)]$; (ii) If $a^2>\epsilon$, there is a critical temperature T_c given by

$$\epsilon/a^2=\tanh(\tfrac{1}{2}\beta_c\epsilon),\quad \beta_c=1/kT_c\,, \tag{25}$$

at which the system undergoes a phase transition. At $\beta<\beta_c$, the free energy per atom $f(T)$ of the system is given by

$$-\beta f(T)=\ln[2\cosh(\tfrac{1}{2}\beta\epsilon)]\,. \tag{26}$$

At $\beta>\beta_c$, however, $f(T)$ is given by

$$-\beta f(T)=\ln[2\cosh(\beta a^2\sigma)]-\beta a^2\sigma^2+\beta\epsilon^2/4a^2\,, \tag{27}$$

where

$$2\sigma=\tanh(\beta a^2\sigma)\neq 0\,. \tag{28}$$

It will be noted from Eq. (5) that the partition function $Z(N,T)$ is symmetric with respect to $\lambda+\lambda'$ and $\lambda-\lambda'$. Thus, the thermodynamics of the system with the Hamiltonian given by

$$H=a^\dagger a+\sum_{j=1}^{N}\left(\frac{\epsilon}{2}\sigma_j^z+\frac{\lambda}{2\sqrt{N}}(a\sigma_j^+ +a^\dagger\sigma_j^-)-\frac{\lambda'}{2\sqrt{N}}(a^\dagger\sigma_j^+ +a\sigma_j^-)\right) \tag{29}$$

is precisely the same as that for the system with the Hamiltonian given by Eq. (2).

It is clear that in general, systems with the following Hamiltonian can be solved in a similar way by the same method:

$$H=a^\dagger a+\sum_{j=1}^{\infty}\left[\frac{\epsilon}{2}\sigma_j^z+\sigma_j^+ f\left(\{\lambda_s\},\frac{a}{\sqrt{N}},\frac{a^\dagger}{\sqrt{N}}\right)+\sigma_j^- g\left(\{\lambda_s\},\frac{a}{\sqrt{N}},\frac{a^\dagger}{\sqrt{N}}\right)\right], \tag{30}$$

where $f(\{\lambda_s\},a/\sqrt{N},a^\dagger/\sqrt{N})$ and $g(\{\lambda_s\},a/\sqrt{N},a^\dagger/\sqrt{N})$ are some arbitrary functions of $\{\lambda_s\}, a/\sqrt{N}$ and $a^\dagger/\sqrt{N}$ ($\{\lambda_s\}$ represents a set of coupling parameters $\lambda_1,\lambda_2,\ldots$).

III. MODEL B

As pointed out in Paper I, the assumption that all atoms see exactly the same field even in the thermodynamic limit is somewhat unrealistic. To relax this restriction, we shall take into consideration the spatial variation of the electromagnetic wave. We shall consider only the simplest case in which the spatial variation is of a simple sinusoidal form.

Let us consider the Hamiltonian of the system to be given by

$$H=a^\dagger a+\sum_{j=1}^{N}\left\{\tfrac{1}{2}\epsilon\sigma_j^z+(\lambda/2\sqrt{N})\left[\left(\sin\frac{kL}{N}j\right)(a\sigma_j^+ +a^\dagger\sigma_j^-)\right]\right\}, \tag{31}$$

where L is the length of the cavity containing the atoms and

$$k=n\pi/L,\quad n=0,1,2,\ldots\,. \tag{32}$$

For the case $n=0$, the problem becomes trivial. In the following discussion, it is assumed that n is not to be equal to zero. We shall call the model with the above Hamiltonian model B.

Following the method used in Paper I, it is easy to show that the partition function $Z(N, T)$ of the system, as $N\to\infty$, is given by

$$Z(N,T)=\int\frac{d^2\alpha}{\pi}e^{-\beta|\alpha|^2}\prod_{j=1}^{N}2\cosh\left[\frac{\beta\epsilon}{2}\left(1+\frac{4\lambda^2}{\epsilon^2N}|\alpha|^2\sin^2\frac{\pi n}{N}j\right)^{1/2}\right]. \tag{33}$$

Replacing $\int d^2\alpha/\pi$ by $2\int_0^\infty r\,dr$ and letting $y=r^2/N$ as before, we obtain

$$Z(N,T)=N\int_0^\infty dy\exp N\left\{-\beta y+\frac{1}{N}\sum_{j=1}^{N}\ln 2\cosh\left[\frac{\beta\epsilon}{2}\left(1+\frac{4\lambda^2}{\epsilon^2}y\sin^2\frac{n\pi}{N}j\right)^{1/2}\right]\right\}. \tag{34}$$

Let

$$\phi(y)=-\beta y+\frac{1}{N}\sum_{j=1}^{N}\ln 2\cosh\left[\frac{\beta\epsilon}{2}\left(1+\frac{4\lambda^2}{\epsilon^2}y\sin^2\frac{n\pi}{N}j\right)^{1/2}\right]. \tag{35}$$

As $N\to\infty$,

$$\begin{aligned}\phi(y)&=-\beta y+\frac{1}{\pi}\int_0^\pi\ln\left\{2\cosh\left[\frac{\beta\epsilon}{2}\left(1+\frac{4\lambda^2}{\epsilon^2}y\sin^2 n\theta\right)^{1/2}\right]\right\}d\theta\\&=-\beta y+\frac{2n}{\pi}\int_0^{\pi/2n}\ln\left\{2\cosh\left[\frac{\beta\epsilon}{2}\left(1+\frac{4\lambda^2}{\epsilon^2}y\sin^2 n\theta\right)^{1/2}\right]\right\}d\theta\\&=-\beta y+\frac{2}{\pi}\int_0^{\pi/2}\ln\left\{2\cosh\left[\frac{\beta\epsilon}{2}\left(1+\frac{4\lambda^2}{\epsilon^2}y\sin^2\omega\right)^{1/2}\right]\right\}d\omega=-\beta y+\frac{2}{\pi}\int_0^{\pi/2}\ln\left(2\cosh\frac{\beta\epsilon}{2}\eta(y,\omega)\right)d\omega,\end{aligned} \tag{36}$$

where

$$\eta(y,\omega)=[1+(4\lambda^2/\epsilon^2)y\sin^2\omega]^{1/2}, \tag{37}$$

thus,

$$\frac{d\phi(y)}{dy}=-\beta+\frac{2\beta\lambda^2}{\pi\epsilon}\int_0^{\pi/2}\frac{\sin^2\omega}{\eta(y,\omega)}\tanh\left(\frac{\beta\epsilon}{2}\eta(y,\omega)\right)d\omega. \tag{38}$$

Putting $d\phi/dy=0$, the equation which determines the phase transition property of the system is given by

$$\frac{\epsilon\pi}{2\lambda^2}=\int_0^{\pi/2}\frac{\sin^2\omega}{\eta(y,\omega)}\tanh\left(\frac{\beta\epsilon}{2}\eta(y,\omega)\right)d\omega,\quad 1\le\eta<\infty. \tag{39}$$

Denoting the integral on the right-hand side of Eq. (39) by $I(\eta)$, it is seen that $I(\eta)<\int_0^{\pi/2}\sin^2\omega\,d\omega=\frac{1}{4}\pi$ for $1\le\eta<\infty$, since the function $\tanh x<1$ for $x<\infty$. It is also easy to see that $I(\eta)$ is a decreasing function of η as η increases from 1 to ∞ $(I(\infty)=0)$. Thus, for (i), $\lambda^2<2\epsilon$, the integral equation (39) has no solution and the free energy $f(T)$ per atom of the system is given by

$$-\beta f(T)=\ln(2\cosh\tfrac{1}{2}\beta\epsilon) \tag{40}$$

for it is easy to show that the function

$$\exp\left\{-\beta y+\frac{2}{\pi}\int_0^{\pi/2}\ln\left[2\cosh\frac{\beta\epsilon}{2}\times\left(1+\frac{4\lambda^2}{\epsilon^2}y\sin^2\omega d\omega\right)^{1/2}\right]\right\}$$

is a monotonically decreasing function of $y(0\le y<\infty)$ in this case. For (ii), $\lambda^2>2\epsilon$ and $\beta<\beta_c$, where β_c is given by

$$2\epsilon/\lambda^2=\tanh\tfrac{1}{2}\beta_c\epsilon, \tag{41}$$

the integral equation (39) again has no solution and the free energy per atom of the system is given as in the previous case by

$$-\beta f(T)=\ln(2\cosh\tfrac{1}{2}\beta\epsilon). \tag{42}$$

For (iii) $\lambda^2>2\epsilon$ and $\beta>\beta_c$, the integral equation (39) has one (and only one) solution $y_0(>0)$ which satisfies

$$\frac{\pi\epsilon}{2\lambda^2}=\int_0^{\pi/2}\frac{\sin^2\omega}{[1+(4\lambda^2/\epsilon^2)y_0\sin^2\omega]^{1/2}}\times\tanh\left[\frac{\beta\epsilon}{2}\left(1+\frac{4\lambda^2}{\epsilon^2}y_0\sin^2\omega\right)^{1/2}\right]d\omega \tag{43}$$

and the free energy per atom $f(T)$ of the system is given by

$$-\beta f(T) = -\beta y_0 + \frac{2}{\pi}\int_0^{\pi/2} \ln\left[2\cosh\frac{\beta\epsilon}{2} \times\left(1+\frac{4\lambda^2}{\epsilon^2} y_0 \sin^2\omega\right)^{1/2}\right] d\omega\,. \tag{44}$$

Summarizing the thermodynamic properties of model B: (i) If $\lambda^2 < 2\epsilon$, no phase transition occurs in the system at any temperature, the free energy per atom is given Eq. (40). (ii) If $\lambda^2 > 2\epsilon$, there is a critical temperature T_c given by Eq. (41) ($\beta_c = 1/kT_c$). At $\beta < \beta_c$, the free energy per atom of the system is given by Eq. (42). At $\beta > \beta_c$, however, the free energy per atom of the system is given by Eq. (44), where y_0 is given by Eq. (43). It will be noted that these solutions are independent of the value of $n (\neq 0)$.

The above results can be immediately extended to the case in which the electromagnetic wave assumes a more general form $g(n\pi j/N)$, say, provided that $g(\theta)$ has the main characteristic of $\sin\theta$ required in the above formulation, namely, it is a function of period 2π and is a monotonically increasing function of θ from 0 to $\frac{1}{2}\pi$, etc.

IV. MODEL C

Suppose that the atoms are not held rigidly in their positions but are held to their equilibrium positions by a force which is proportional to any displacements from these positions. Suppose also that the small motions of the atoms can be treated classically such that the momentum and the displacement variables are assumed to commute. If the direct interactions among the atoms are ignored as in the previous cases, the effect of the atoms taking up positions in the neighborhood of their equilibrium positions is still significant if the spatial variation of the electromagnetic field is taken into consideration.

Consider the Hamiltonian of the system to be given by

$$H = a^\dagger a + \sum_{j=1}^{N}\{(\tfrac{1}{2}\epsilon)\sigma_j^z + (\lambda/2\sqrt{N}) \times[\sin k(z_j + x_j)(a\sigma_j^+ + a^\dagger\sigma_j^-)] + \gamma x_j^2\}\,, \tag{45}$$

where z_j denotes the distance of the jth atom from the first atom, x_j denotes the displacement of the jth atom from its equilibrium position, and γ represents the strength of the force which keeps the atoms to their equilibrium positions. We shall call the model with the above Hamiltonian model C. Following the calculations in Sec. III for model B, it is easily seen that the partition function $Z(N,T)$ for model C, in the limit $N\to\infty$, is given by

$$Z(N,T) = \int\frac{d^2\alpha}{\pi} e^{-\beta|\alpha|^2} \prod_{j=1}^{N}\frac{1}{L}\int dx_j\, e^{-\beta\gamma x_j^2}\, 2\cosh\left\{\frac{\beta\epsilon}{2}\left[1+\frac{4\lambda^2}{\epsilon^2 N}|\alpha|^2 \sin^2\left(\frac{n\pi j}{N}+kx_j\right)\right]^{1/2}\right\} \tag{46}$$

and the equation which determines the phase transition property of the system is given by

$$\frac{\pi\epsilon}{2\lambda^2} = \int_0^{\pi/2} d\omega\, \frac{\int dx\{e^{-\beta\gamma x^2}\sin^2(\omega+kx)\sinh[\frac{1}{2}\beta\epsilon\eta(x,y,\omega)]/\eta(x,y,\omega)\}}{\int dx\, e^{-\beta\gamma x^2}\cosh[\frac{1}{2}\beta\epsilon\eta(x,y,\omega)]}\,, \tag{47}$$

where

$$\eta(x,y,\omega) = [1+(4\lambda^2/\epsilon^2)y\sin^2(\omega+kx)]^{1/2}\,. \tag{48}$$

Let us denote the integral on the right-hand side of Eq. (47) by $I(\beta, y)$. Since $\cosh x > \sinh x$ for all x, hence $\cosh x > (\sinh x)/x$ for all $x > 1$. Moreover, since $(\tanh x)/x$ is a decreasing function of x as x increases from 1 to ∞, thus the integral $I(\beta, y)$ is a monotonically decreasing function of y as y increases from 0 to ∞, and $I(\beta,\infty) = 0$. It also follows that in the region $0 \le y < \infty$, that

$$I(\beta, y) < \max_{0\le\beta\le\infty} R(\beta)\,, \tag{49}$$

where

$$R(\beta) = \left(\tanh\frac{\beta\epsilon}{2}\right) \times\int_0^{\pi/2} d\omega\, \frac{\int_{-\infty}^{\infty} dx\, e^{-\beta\gamma x^2}\sin^2(\omega+kx)}{\int_{-\infty}^{\infty} dx\, e^{-\beta\gamma x^2}} = \frac{\pi}{4}\tanh\frac{\beta\epsilon}{2} \tag{50}$$

Therefore, $R(\beta)_{\max} = \frac{1}{4}\pi$ and

$$I(\beta, y) < \tfrac{1}{4}\pi \quad \text{for } 0 \le y < \infty\,. \tag{51}$$

Thus, the thermodynamic properties of model C can be outlined as follows: (i) If $\lambda^2 < 2\epsilon$, no phase transition occurs in the system at any temperature, the free energy per atom $f(T)$ of the system is given by

$$-\beta f(T) = \ln(2\cosh\tfrac{1}{2}\beta\epsilon)\,. \tag{52}$$

(ii) If $\lambda^2 > 2\epsilon$, there is a critical temperature T_c given by

$$2\epsilon/\lambda^2 = \tanh\tfrac{1}{2}\beta_c\epsilon\,, \quad \beta_c = 1/kT_c\,. \tag{53}$$

At $\beta < \beta_c$, the free energy per atom of the system is given as in the previous case by

$$-\beta f(T) = \ln(2\cosh\tfrac{1}{2}\beta\epsilon)\,. \tag{54}$$

At $\beta > \beta_c$, however, the free energy per atom of the system is given by

$$-\beta f(T) = -\beta y_0 + \frac{2}{\pi}\int_0^{\pi/2} d\omega \ln\left\{\frac{1}{L}\int_{-L/2}^{L/2} dx\, e^{-\beta\gamma x^2} 2\cosh\left[\frac{\beta\epsilon}{2}\left(1+\frac{4\lambda^2}{\epsilon^2}\, y_0 \sin^2(\omega+kx)\right)^{1/2}\right]\right\}, \tag{55}$$

where y_0 is given by

$$\frac{\pi\epsilon}{2\lambda^2} = \int_0^{\pi/2} d\omega \left(\frac{\int_{-L/2}^{L/2} dx\{e^{-\beta\gamma x^2}\sin^2(\omega+kx)\sinh[\frac{1}{2}\beta\epsilon\eta(x, y_0, \omega)]/\eta(x, y_0, \omega)\}}{\int_{-L/2}^{L/2} dx\, e^{-\beta\gamma x^2}\cosh[\frac{1}{2}\beta\epsilon\eta(x, y_0, \omega)]}\right). \tag{56}$$

V. CONCLUSION

We have solved and discussed some simple generalizations of the Dicke model of superradiance. Beside the obvious interest of the models themselves, it is also hoped that the solutions for these models will serve as an introduction to a broader class of related problems which can be approached and solved in a similar way.

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