Stochastic Resonance in One-Dimensional Random Media

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(Received 15 March 1973)

It is known that a nonabsorbing one-dimensional semi-infinite random medium is totally reflecting. In connection with this, when the medium is subject to monochromatic excitation with an arbitrary frequency, well-localized "stochastic resonances" may appear, at which the wave amplitude can exceed any given value, with nonzero probability in such a way as to make the mean energy density inside the medium infinite. The problem is considered both for continuous media (e.g., a random elastic bar) and discrete media (e.g., disordered linear atomic chains).

I. INTRODUCTION

During the last few years, several authors have investigated the reflection and transmission properties of a one-dimensional random medium.[1-9] The present study originates from the result obtained by P.-L. Sulem and U. Frisch[7] that, under quite general conditions, the modulus of the reflection coefficient r of a nonabsorbing one-dimensional random medium converges almost surely to one when the length L of the medium increases indefinitely. Owing to the energy flux conservation, the transmission coefficient t of the medium is related to r by

$$1-|r|^2=|t|^2 \qquad (1.1)$$

and tends to zero when $L\to\infty$. In a mechanical scope, this result means for example that if a linear elastic, nonabsorbing random bar is longitudinally excited at one end, the amplitude at the other end, assumed to be free, tends almost surely to zero when the length of the bar increases indefinitely.

This result gives no indication as to what happens inside the medium. It may be, however, that the total reflection property induces localized storages of energy inside the medium. Specifically, is there an upper bound for the amplitude of vibration inside the medium? If not, we shall say that "stochastic resonances" can take place inside the medium.

In Sec. II, the problem is formulated using the stochastic Helmholtz equation. In Sec. III, we introduce the "forward-backward method" for calculating the amplitude of vibration at any point inside the medium. In Sec. IV, the existence of stochastic resonances if proven both theoretically and numerically. Section V is devoted to a discrete version of this problem (disorder linear atomic chains). In Sec. VI, possible applications and a few open problems are mentioned.

II. STOCHASTIC HELMHOLTZ EQUATION

Let a one-dimensional, nonabsorbing medium with random refractive index $n(x)$ be subject to monochromatic excitation with angular frequency. The wave amplitude $\psi(x)$ satisfies the stochastic Helmholtz equation

$$\frac{d^2\psi}{dx^2}+\frac{\omega^2}{c^2}n^2(x)\psi(x)=0\,, \qquad (2.1)$$

where c denotes a reference speed of propagation. Assuming that the medium extends from $x=0$ to $x=L$, we impose the following boundary conditions: at $x=0$, the medium is excited with unit amplitude

$$\psi(0)=1\,, \qquad (2.2)$$

at $x=L$, we impose, the impedance Z_{end},

$$\psi'(L)=Z_{\text{end}}\psi(L)\,. \qquad (2.3)$$

Owing to the total reflection property, only stationary waves can exist in the semi-infinite medium.[7] Since we are interested in the limiting case $L\to\infty$, we impose stationarity for the finite medium by taking a real value for Z_{end}. A physical picture of this problem is provided by an elastic, nonabsorbing, cylindrical bar with random parameters, excited longitudinally at one end and free at the other end. In this case, $Z_{\text{end}}=0$.

In order to bring out the phenomenon of stochastic resonance in a simple case, we assume henceforth the refractive index $n(x)$ to be a real stepwise constant random function as pictured in Fig. 1. The values of the refractive index are chosen independently, at random, with equal probabilities between two values n' and n''. A typical picture is provided by a random stack of two kinds of homogeneous materials. To exclude pathological cases corresponding to resonances in the usual sense, we assume that neither $\omega n's/c\pi$ nor $\omega n''s/c\pi$ are rational numbers.

For each realization of the random refractive index $n(x)$, the two-point boundary-value problem (2.1–2.3) has a unique solution $\psi_L(x)$, where L denotes the length of the medium. We say that the medium can exhibit stochastic resonances if

$$\forall A>0,\ \forall x>0,\ \lim_{L\to\infty} \text{prob}\{|\psi_L(x)|>A\}>0\,; \quad (2.4)$$

in other terms, there is a nonzero probability that the amplitude at any given point exceeds any given value. There is also a weak formulation which is more convenient for numerical investigation, namely,

$$\forall A>0,\ \lim_{L\to\infty} \text{prob}\{\sup_{0\le x\le L} |\psi_L(x)|>A\}>0\,; \quad (2.5)$$

in other terms, there is a nonzero probability that, somewhere inside the random medium, the amplitude exceeds any given value.

III. FORWARD-BACKWARD METHOD

In matrix notation, Eq. (2.1) reads, with $\psi' = d\psi/dx$ and $k_0 = \omega/c$,

$$\frac{d}{dx}\begin{pmatrix}\psi(x)\\ \psi'(x)\end{pmatrix} = M(x)\begin{pmatrix}\psi(x)\\ \psi'(x)\end{pmatrix}, \quad (3.1a)$$

with

$$M(x) = \begin{pmatrix}0 & 1\\ -k_0^2 n^2(x) & 0\end{pmatrix}. \quad (3.1b)$$

In order to solve Eq. (3.1) with the boundary conditions

$$\psi(0)=1,\quad \psi'(L)/\psi(L)=Z_{\text{end}}, \quad (3.2)$$

it is useful to introduce the Green's function $G(x,y)$, i.e., the linear operator carrying $\text{col}[\psi(x),\psi'(x)]$ onto $\text{col}[\psi(y),\psi'(y)]$, which satisfies the matrix differential equation

$$\frac{d}{dx}G(x,y)=M(x)G(x,y); \qquad G(y,y)=I=\text{identity}. \quad (3.3)$$

Note that the Green's function follows a *one-point* boundary condition for $x=y$. Note also that, since $\text{Tr}M(x)=0$, the Green's function is unimodular, i.e.,

$$\det G(x,y)=1\,. \quad (3.4)$$

The solutions of the boundary-value problem (3.2) are expressible in terms of

$$G(x,0)=\begin{pmatrix}\alpha(x) & \beta(x)\\ \gamma(x) & \delta(x)\end{pmatrix}. \quad (3.5)$$

Indeed,

$$\psi(x)=\alpha(x)\psi(0)+\beta(x)\psi'(0)\,, \qquad \psi'(x)=\gamma(x)\psi(0)+\delta(x)\psi'(0)\,. \quad (3.6)$$

To obtain the value of $\psi'(0)$, we put $x=L$ in (3.6) and get, using (3.2),

$$\psi'(0)=[Z_{\text{end}}\alpha(L)-\gamma(L)]/[\delta(L)-Z_{\text{end}}\beta(L)]\,. \quad (3.7)$$

Inserting this value into (3.6), we finally obtain

$$\psi(x)=\alpha(x)+\beta(x)\frac{Z_{\text{end}}\,\alpha(L)-\gamma(L)}{\delta(L)-Z_{\text{end}}\beta(L)}\,. \quad (3.8)$$

For the study of statistical properties of $\psi(x)$, a more convenient form is obtained by introducing the impedance

$$Z(x)=\psi'(x)/\psi(x), \quad (3.9)$$

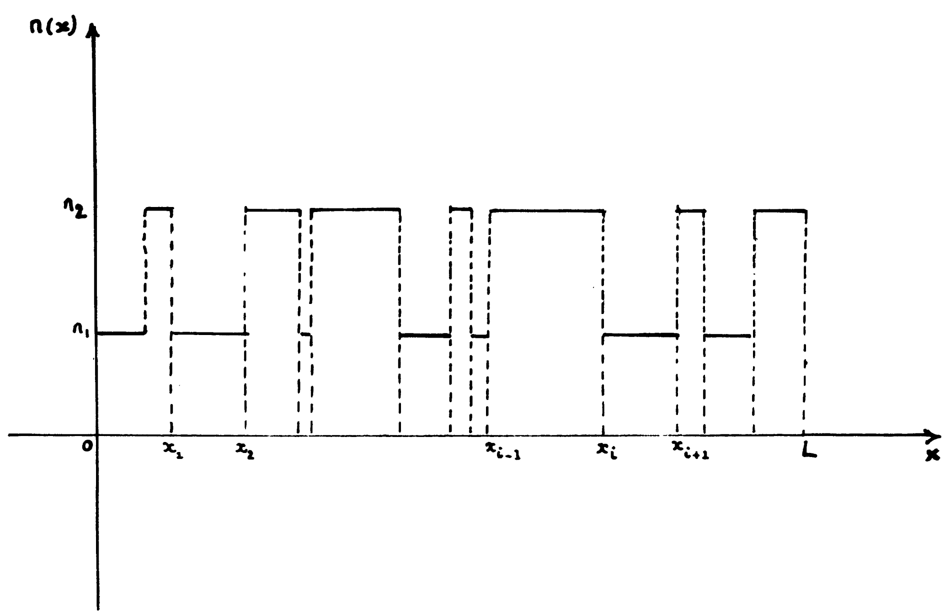

FIG. 1. The refractive index, a real stepwise constant random function of position.

which, as a consequence of (3.1), is the solution of the nonlinear Ricatti equation[4]

$$\frac{dZ}{dx}+Z^2+k_0^2n^2(x)=0;\ \ Z(L)=Z_{\text{end}}\,. \tag{3.10}$$

Note that Z obeys a boundary condition at $x=L$ only. Using (3.6) and (3.9), we obtain

$$Z(x)=[\gamma(x)+\delta(x)Z(0)]/[\alpha(x)+\beta(x)Z(0)]\,. \tag{3.11}$$

A straightforward calculation using $\det G=1$ and Eqs. (3.6) and (3.8) leads to

$$\psi(x)=1/[\delta(x)-\beta(x)Z(x)]\,. \tag{3.12}$$

As it stands, $\psi(x)$ is a functional of $G(x,0)$ which satisfies the boundary condition $G(0,0)=I$ at $x=0$, and of $Z(x)$ which satisfies the boundary condition $Z(L)=Z_{\text{end}}$ at $x=L$. Therefore, in principle, to obtain $\psi(x)$, we must integrate Eq. (3.3) forwards from 0 to x and Eq. (3.10) backwards from L to x; hence, the name "forward-backward method."

In the special case where $n(x)$ is the step process defined in Sec. II, we can calculate $G(x,0)$, using the semi-group-property

$$G(x,0)=G(x,x_p)G(x_p,x_{p-1})\ldots G(x_1,x_0)\,, \tag{3.13}$$

with $x_j=js$, where s denotes the length of the individual steps (cf. Fig. 1). For $x_j\leqslant x\leqslant x_{j+1}$, the refractive index $n(x)$ has a constant value n_{j+1}; hence, we can explicitly integrate the Helmholtz equation to obtain the expression of transfer matrices[7]:

$$G_j=G(x_j,x_{j-1})=\begin{pmatrix}\cos(k_0n_js) & \dfrac{1}{k_0n_j}\sin(k_0n_js)\\ -k_0n_j\sin(k_0n_js) & \cos(k_0n_js)\end{pmatrix}. \tag{3.14}$$

It follows that

$$Z_j=\frac{\psi'(x_j)}{\psi(x_j)}=\frac{\cos(k_0n_js)Z_{j+1}+k_0n_j\sin(k_0n_js)}{-\dfrac{1}{k_0n_j}\sin(k_0n_js)Z_{j+1}+\cos(k_0n_js)}\,. \tag{3.15}$$

Equations (3.14) and (3.15) may be used to calculate $\psi_j=\psi(x_j)$ which, from Eq. (3.12), reads

$$\psi=1/(\delta_j-\beta_jZ_j)\,. \tag{3.16}$$

It is important to notice that Z_j depends only on the n_k's for $k>j$ whereas β_j and δ_j depend only on the n_k's for $k\leqslant j$; since the different n_k's are statistically independent, Z_j is independent of both β_j and δ_j.

IV. STOCHASTIC RESONANCE

From the remark at the end of the preceding section, the statistical properties of ψ_j are easily related to the separate statistical properties of the Green's function and the impedance. We shall deal only with those statistical properties which are most relevant to the stochastic resonance.

A. Asymptotic Behavior of the Green's Function

The asymptotic behavior of random products of statistically independent identically distributed, unimodular matrices has already been considered several times in the mathematical literature[10-12]. A particular interest in the present context is taken in the following *Theorem* (Adapted from Furstenberg[10]): Assume that the group of transfer matrices G_j is noncompact and has no reducible subgroup of finite index (condition F); let $U_0=\text{col}(\psi(0),\psi'(0))$ be a nonzero initial condition, then $(1/j)\ln\|\prod_{i=1}^{j}G_iU_0\|$ tends to a finite positive limit l as $j\to\infty$, where $\|\cdots\|$ denotes the Euclidian norm.

In Ref. 13, Sec. 3, it is proved that condition F holds for a disordered linear chain; this proof is easily carried over the present case. The Furstenberg theorem means that $\alpha_j,\beta_j,\gamma_j$ and δ_j grow exponentially with j.

B. Asymptotic Properties of the Impedance

It is known that the iteration of random homographic transformations of the form (3.15) leads to a stationary and ergodic distribution for the impedance.[14-17] Since Z satisfies a deterministic boundary condition at $x=L$, the impedance will have a limiting distribution $P(Z)$ as $L\to\infty$ which is independent of x.[15,16] This distribution may be obtained either by solving the Schmidt functional equation,[17] or by making a direct numerical calculation using a Monte Carlo procedure based upon the ergodicity of Z. A histogram of the impedance distribution is plotted on Fig. 2. It must also be mentioned that when s is not a constant but a random variable distributed according to an exponential law, the distribution of Z can be calculated analytically by solving an exact Fokker-Planck equation.[8] It is important to notice that in any case, when $L\to\infty$, the values of Z become densely distributed between $-\infty$ and $+\infty$.

C. Stochastic Resonance

Using the forward-backward method of Sec. III, we have calculated numerically the amplitude ψ_j inside the medium for several realizations corresponding to the values $n_1=2$, $n_2=5$, $s=I$, $N=L/s=1000$. Two typical results are plotted in Fig. 3. Note that, in both cases, there is negligible energy left far from the excited end; this agrees well with the property of total reflection. There is, however, a striking difference between Fig. 3(a) and 3(b). In the former case, the maximum amplitude is obtained at the excited end $j=0$; in the latter case, "stochastic resonances" appear inside the medium i.e. the amplitude can exceed the exciting amplitude by a large factor. Notice that the stochastic resonances are rather well localized in position. The point where the absolute maximum is obtained has a random distribution; this point is however more likely to be located near the excited end, a result which is clearly displayed on Fig. 4 where we have plotted the position and peak amplitudes of the stochastic resonances for 70 realizations of the medium.

The existence and properties of stochastic resonances are now established on a theoretical basis. Let us consider formula (3.16) for the amplitude ψ_j inside the medium. We have already noticed that, in the limit $L\to\infty$, Z_j becomes densely distributed between $-\infty$ and $+\infty$. Since Z_j is independent of (β_j, δ_j), the denominator of (3.16) can become arbitrary close to cancellation, i.e., $\forall\epsilon>0$,

$$\text{prob}\{|\delta_j-\beta_j Z_j|<\epsilon\}>0\,,$$

and therefore $|\psi_j|$ can take arbitrary large values with nonzero probability.

For a given realization, if the resonance condition $|\delta_j-\beta_j Z_j|<\epsilon$ is satisfied for some j_0, it will usually hold only in some neighborhood of j_0; hence, the localized character of the resonance. Furthermore, if we write the resonance condition in the form

$$\left|\frac{\delta_j}{\beta_j}-Z_j\right|<\frac{\epsilon}{|\beta_j|}$$

and recall that $|\beta_j|$ grows exponentially as $j\to\infty$ (Furstenberg theorem), we see that, as j increases, the resonance condition becomes more and more stringent; hence, the probability of appearance of stochastic resonances decreases away from the excited end.

Finally, it is possible to write the expression for the probability density $\Pi_j(\psi)$ of ψ_j as $L\to\infty$. Let us denote $d\mathcal{P}_j(\beta,\delta)$ the joint probability distribution of (β_j,δ_j) and $P(Z)$ the limiting probability density of Z_j as $L\to\infty$ which is independent of j. In view of the independence

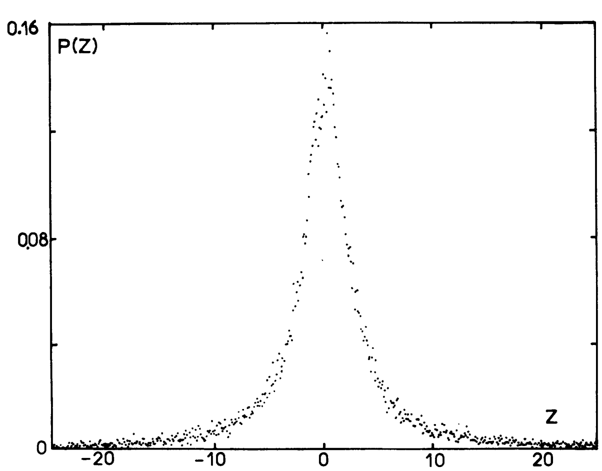

FIG. 2. Asymptotic distribution of the impedance by a Monte Carlo method.

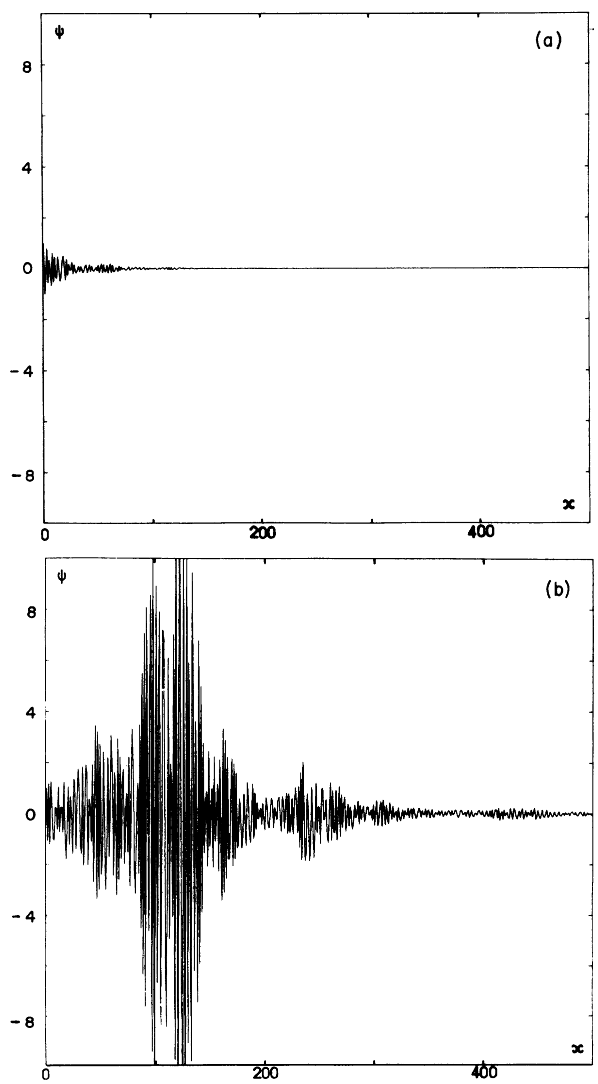

FIG. 3. Typical variations of wave amplitude Ψ_J within the random medium; (a) a nonresonant case, (b) a resonant case.

of Z_j and (β_j, δ_j) we obtain from (3.16) in the limit $L\to\infty$,

$$\Pi_j(\psi)=\frac{1}{\psi^2}\int\frac{1}{|\beta|}P\left[\left(\delta-\frac{1}{\psi}\right)\frac{1}{\beta}\right]d\mathcal{P}_j(\beta,\delta). \quad (3.17)$$

It can be checked that the integral in (3.17) has a finite limit as $\psi\to\infty$, thus

$$\Pi_j(\psi)\propto\frac{1}{\psi^2},\quad \psi\to\infty. \quad (3.18)$$

From (3.18) it follows that (in the limit $L\to\infty$) the mean and mean-square amplitudes $\langle\psi_j\rangle$ and $\langle|\psi_j|^2\rangle$ are both infinite. It must be recalled, however, that we are considering a stationary régime where energy is being continuously injected into the medium. Presumably, the total reflection property is accompanied by energy accumulations inside the medium.

V. STOCHASTIC RESONANCE IN SEMI-INFINITE DISORDERED LINEAR CHAINS

The formalism we have developed in the previous sections can also be used to study the longitudinal harmonic vibrations, with angular frequency ω, of a linear chain of N random masses, each coupled to its nearest neighbors by elastic springs with constant strength k. Let U_n be the amplitude of vibration of the mass m_n. The equation of motion of the chain

$$-\omega^2 m_n U_n = k(U_{n+1}+U_{n-1}-2U_n) \quad (5.1)$$

can be rewritten in matrix form

$$\begin{pmatrix}U_{n+1}\\ U_n\end{pmatrix}=G_n\begin{pmatrix}U_n\\ U_{n-1}\end{pmatrix}, \quad (5.2)$$

wherein

$$G_n=\begin{pmatrix}2-\frac{\omega^2}{k}m_n & -1\\ 1 & 0\end{pmatrix}. \quad (5.3)$$

Such chains have been used by several authors to figure the interaction of the atoms of disordered linear crystals. In fact, most authors[14,17] have studied the normal modes of a chain infinite in both directions, and especially its integrated distribution

$$\mathfrak{M}(\omega)=\lim\frac{1}{N}\text{(number of normal modes with frequency less than }\omega).$$

In this section, we study a quite different problem. We assume that a finite chain with N masses is excited at one end with unit amplitude

$$U_0=1 \quad (5.4a)$$

and that at the other end the impedance is given:

$$\frac{U_n}{U_{n-1}}=Z_{\text{end}}. \quad (5.4b)$$

We then study the limit as $N\to\infty$ of the statistical properties of $U_j^{(N)}$ for a finite j, the index (N) denoting the total number of masses in the chain. Using the same approach as in the preceding section, we can derive the following results:

(i) the chain is totally reflecting, a result which had already been found by Rubin[5,6];

(ii) there is a nonzero probability for U_j to be larger than any given value (stochastic resonance).

REMARK

In our approach, we have formulated a boundary value problem for a finite chain [cf. Eqs. (5.4)] and afterwards, we have taken the limit of the probability densities as $N\to\infty$. It is impossible to take the limit $N\to\infty$ on $U_j^{(N)}$ itself since this limit does not exist. Nevertheless, using a result of Matsuda and Ishii,[13] it is possible to formulate the stochastic resonance for semi-infinite chain. Indeed, in Ref. 13, it is proved that for almost every semi-infinite chain, the two-dimensional space of solutions of Eq. (5.1) has a one-dimensional subspace such as $U_n\to\infty$ for $n\to\infty$. [These solutions are used by Matsuda and Ishii[13] to construct "localized" normal modes for chains infinite in both directions, i.e., solutions of Eq. (5.1) which are bounded by a decreasing exponential for $n\to\pm\infty$. "Localization" as used by us, has a different meaning since it refers to spatial extension of the stochastic resonance of a semi-infinite chain excited at one end.] Hence, the boundary value problem

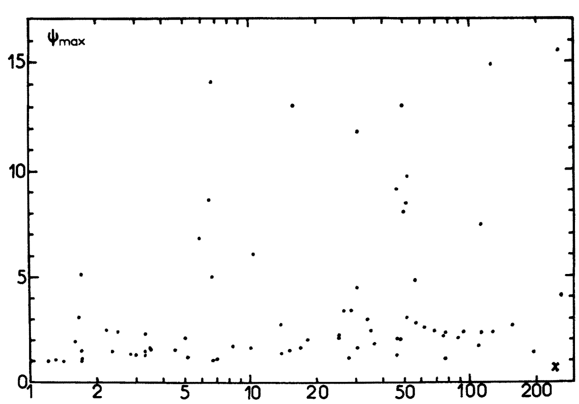

FIG. 4. Peak amplitude $\Psi_{\max}$ and its position x for 70 realizations of the medium. Notice that due to the log scale in x, the density is higher near $x=0$.

$$U_0 = 1, \tag{5.5a}$$

$$\lim_{n\to\infty} U_n = 0 \tag{5.5b}$$

has a unique solution. It may be shown that, in the limit $N \to \infty$, the statistical properties of the solution of problem (5.5) are the same as those of problem (5.5).

VI. CONCLUDING REMARKS

A new kind of resonance has been exhibited which is somewhat atypical: usual resonances are not localized in space and occur at well defined frequencies, whereas the "stochastic resonance" can occur for any exciting frequency and is sharply localized in space. Since our results are based on the one-dimensional stochastic Helmholtz equation,[18] we expect that the theory of stochastic resonance will find applications in many fields such as elastic, seismic, optical wave propagation in heterogeneous media, disordered crystals, polymeres.

Now, we mention a few open problems. As already noticed, the stochastic resonance can occur for any monochromatic excitation. What happens if the excitation is not monochromatic? A preliminary investigation indicates that, if the medium is made of a large number of independent pieces (or extends over a large number of correlation lengths), the position of the resonances are extremely sensitive to the exciting frequency, probably because small variation in the exciting frequency induces large variation of the impedance far from the end $x = L$. As a consequence, when the excitation has a finite bandwidth, the resonances may be smoothed out. This point requires further investigation.

For practical purposes (e.g. in engineering applications), it would be useful to have empirical formulas for the probability that the amplitude exceeds the exciting one by a given factor, as a function of the parameters of the medium, frequency and bandwidth of the excitation.

Finally, it would be interesting to know if stochastic resonances do also occur in two or three dimensional random media.

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