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Radiation Forces and Momenta in Dielectric Media

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There has existed a continuing dialogue concerning the proper identification of radiation forces and momenta in dielectric media. I argue herein that a sensible and consistent picture of these forces and momenta is available. That is, the density of electromagnetic momentum $\vec{G}$ is given by $\vec{G} = \vec{S}/c^2$, where $\vec{S}$ is Poynting's vector. The forces which are exerted on material objects in dielectric media are associated with changes in both the electromagnetic and mechanical momenta of the media. In fairly broad circumstances, such forces may be found from the rate of change in a pseudomomentum $\vec{K}$ given by $\vec{K} = \epsilon\vec{G}$, where ϵ is the dielectric constant of the medium.

I. INTRODUCTION

There has existed a continuing dialog concerning the proper identification of the momentum of electromagnetic waves in dielectric media. As Blount[1] has commented, "The argument has not, it is true, been carried on at high volume, but the list of disputants is very distinguished." In brief, the question is whether the momentum density for electromagnetic waves in a material medium has the form $(\vec{D}\times\vec{B})/4\pi c$ or $(\vec{E}\times\vec{H})/4\pi c$. The first form, due to Minkowski,[2] has good credentials, for all of the experimental results relating to radiation pressures seem consistent with its conservation. The second form, due to Abraham,[3] also has good credentials, for it gives the proper theoretical result with regard to the motion of the center of mass of a system containing both matter and radiation field,[4] and corresponds to a symmetric energy-momentum tensor. For a summary, see Refs. 5 and 6.

For nearly monochromatic plane waves in a nondispersive medium, Blount[1] has commented on the intimate connection between the Minkowski form and the "crystal momentum" $\hbar\vec{k}$ associated with elementary excitations of energy $\hbar\omega$. Here $\vec{k}$ is the wave vector, and ω the angular frequency of the excitation. Indeed, for the Minkowski form the ratio of momentum density to energy density is just $\vec{k}/\omega$. It is well known that the crystal momentum is definitely different from the true momentum which is associated relativistically with mass transport. Abraham's form gives a ratio of momentum density to energy density equal to $\vec{v}/c^2$, where $\vec{v}$ is the wave's group velocity. This corresponds exactly to the relativistic ratio of momentum to energy for material particles, as may be seen by simply reexpressing the above ratio as $m\vec{v}/mc^2$.

In this work we demonstrate for nondispersive dielectric media that Abraham's form, which may be expressed as

$$\vec{G} = \vec{S}/c^2, \tag{1.1}$$

where $\vec{G}$ is the momentum density and $\vec{S}$ is Poynting's vector, does indeed represent the true momentum density of electromagnetic fields. We also discuss the circumstances under which Minkowski's form, which one can label "crystal momentum," or more generally "pseudomomentum," and has the form

$$\vec{K} = \epsilon\vec{G}, \tag{1.2}$$

may be used to compute the radiation pressure on objects embedded in such dielectric media. Here ϵ is the dielectric constant of the medium. This "radiation" pressure is actually a combination of the ponderomotive force exerted directly by the field in the object and the force exerted on the object by mechanical pressures induced in the dielectric by the presence of the field. We conclude that experiment and theory are in complete accord.

II. A GASEOUS MEDIUM

The essential features of the problem are to be found through consideration of a simple model dielectric, namely, a gas of heavy atoms (heavy so they do not accelerate rapidly in response to radiation forces) which are weakly polarizable. The permeability of the medium is assumed to be dispersionless, given by

$$\epsilon - 1 = 4\pi N\alpha \ll 1, \tag{2.1}$$

where N is the density of atoms, and α is the atomic polarizability. This section of the paper will be devoted to such a medium. Later, we consider more dense media such as liquids and solids. Gaussian units are used throughout.

One apparently common misconception which clouds thinking on this problem is that if a pulse of radiation is well within a medium, one need not be concerned with the mechanical properties of the medium. Let us immediately dispose of this idea, and at the same time provide a proof of (1. 1) to first order in $\epsilon - 1$, which is sufficient to distinguish between the two forms for the momentum. Consider a short plane-wave pulse of radiation traveling through the gas, as illustrated in Fig. 1. In the presence of the radiation field, forces are exerted on the atoms. In a dilute medium, this "ponderomotive" force is simply the Lorentz force

$$\vec{f}_{\text{atom}} = (\vec{p} \cdot \vec{\nabla})\vec{E} + \frac{1}{c}\frac{d\vec{p}}{dt} \times \vec{B}, \qquad (2.\ 2)$$

where $\vec{p}$ is the dipole moment of the atom. Taking $\vec{p} = \alpha\vec{E}$, and neglecting the distance traveled by the atoms during the pulse, we can rewrite (2. 2) as

$$\vec{f}_{\text{atom}} = \alpha\left((\vec{E} \cdot \vec{\nabla})\vec{E} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \times \vec{B}\right). \qquad (2.\ 3)$$

For future reference, note that using the identity

$$(\vec{E} \cdot \vec{\nabla})\vec{E} = \nabla(\tfrac{1}{2}E^2) - \vec{E} \times \text{curl } \vec{E}$$

and Maxwell's equation

$$\text{curl } \vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0,$$

we can rewrite (2. 3) in the form

$$\vec{f}_{\text{atom}} = \alpha\left(\nabla(\tfrac{1}{2}E^2) + \frac{1}{c}\frac{\partial}{\partial t}(\vec{E} \times \vec{B})\right). \qquad (2.\ 4)$$

If z is taken as the direction of travel of the pulse, and x as the direction of polarization of its electric field, then the fields of the pulse have the form

$$nE_x = B_y = \xi(t - nz/c), \qquad (2.\ 5)$$

where n is the refractive index

$$n = \epsilon^{1/2} = (1 + 4\pi N\alpha)^{1/2} = 1 + 2\pi N\alpha, \qquad (2.\ 6)$$

and ξ is an arbitrary function of its argument. The force per unit volume exerted on the gas atoms is [from (2. 3) and (2. 5)]

$$\vec{F} = N\vec{f}_{\text{atom}} = \hat{z}\,\frac{N\alpha}{2nc}\,\frac{\partial\xi^2}{\partial t}, \qquad (2.\ 7)$$

where $\hat{z}$ is the unit vector in the z direction.

Hence the *mechanical* momentum density $\vec{M}$ of the gas is given by

$$\vec{M}(z, t) = \int_{-\infty}^{t} \vec{F}(z, t')\,dt' = \hat{z}\,(N\alpha/2nc)\,\xi^2. \qquad (2.\ 8)$$

Note that this mechanical momentum travels with the pulse. No momentum is left in the gas after the pulse has passed. Since the momentum $\vec{G}$ [see (1. 1)] is given by

$$\vec{G} = \frac{1}{4\pi c}(\vec{E} \times \vec{B}) = \hat{z}\,\frac{1}{4\pi nc}\,\xi^2, \qquad (2.\ 9)$$

we have

$$\vec{M} = 2\pi N\alpha\vec{G} = (n - 1)\vec{G}. \qquad (2.\ 10)$$

Thus the mechanical momentum of the atoms in the pulse is proportional to $n - 1$ and may not be ignored.

We are now in a position to show that $\vec{G}$ is the electromagnetic momentum in the medium as well as in the vacuum. In the vacuum, the ratio of momentum density to energy density U in a z-directed plane wave is

$$(\vec{G}/U)_{\text{vac}} = \hat{z}/c, \qquad (2.\ 11)$$

where $\hat{z}$ is the z-directed unit vector. A pulse, originally in vacuum, can enter the medium considered here with negligible reflection. That is, the power reflection coefficient is $[(n - 1)/(n + 1)]^2$ even for an abrupt boundary, and hence is of second order of smallness. Hence the total energy of the pulse is conserved. In addition, total momentum is conserved on the passage of the pulse into the medium, because the net impulse given to each atom by the pulse is zero. Hence the ratio of total momentum density to total energy density must be conserved. The additional kinetic *energy* of the atoms due to the presence of the pulse is negligible, and thus the total energy density in the medium is

$$U_{\text{med}} = \frac{1}{8\pi}(\epsilon E^2 + B^2) = \frac{1}{4\pi}\,\xi^2. \qquad (2.\ 12)$$

Using (2. 11), the total momentum density must then be

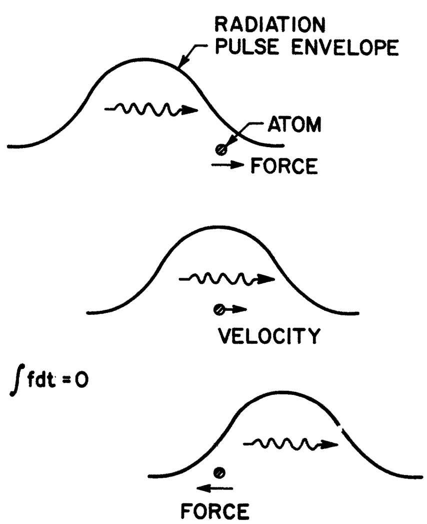

FIG. 1. A plane-wave pulse passing through a gas of atoms. The force on a representative gas atom is illustrated.

$$(\text{total momentum density}) = \hat{z}\xi^2/4\pi c = n\vec{G}, \qquad (2.13)$$

where we have used (2.9). But now from (2.10),

$$n\vec{G} = \vec{M} + \vec{G}, \qquad (2.14)$$

and since $\vec{M}$ is the mechanical momentum density of the gas, it follows that $\vec{G}$ is the electromagnetic momentum density.

This result supports the strong arguments based on the relativistic relation between momentum density and energy flow which favor (1.1) as the proper result for electromagnetic momentum density in a material medium. An argument similar to the above has been given by Haus.[7]

One point in regard to the above discussion is worthy of emphasis. If under the conditions assumed, namely, with negligible motion of the atoms during the pulse, the pulse passes through a thin transparent membrance from vacuum to the gaseous medium, *no force is exerted on the membrane*. The gas near the membrane is left essentially undisturbed as the pulse passes, since as we have pointed out above the total impulse given to any particular atom by the pulse is zero. This would appear to violate the pseudomomentum concept, for in this case it is the total true momentum $\int n\vec{G}dz$ which is conserved rather than the pseudomomentum $\int \epsilon\vec{G}dz$. We shall return to this point later. However, it illustrates that the pseudomomentum concept is *not* universal.

Let us now imagine that the pulse, in the gaseous medium, is reflected at normal incidence from a perfectly conducting plane at $z = 0$, and compute the impulse of radiation pressure on the conductor. We shall do this by finding the total momentum change of the field *and* the atoms. For finding the total force on the atoms, the form (2.4) is most useful. During the reflection process when the incident and reflected waves overlap, the detailed dependence of the forces on distance from the conductor is complicated, but we can integrate (2.4) over the whole volume of gas rather easily. We obtain for the force

$$\mathfrak{F} = \int_{\text{gas}} N\vec{f}_{\text{atom}}dz = \frac{N\alpha}{c}\frac{d}{dt}\int(\vec{E}\times\vec{B})dz. \qquad (2.15)$$

The gradient term of (2.4) does not contribute because $\vec{E}$ is zero at the surface of the conductor. Thus the total impulse $\mathfrak{g}$ given to the gas atoms during the reflection process is

$$\begin{aligned}\mathfrak{g} &= \int \mathfrak{F}\, dt \\ &= (\epsilon - 1)\{[\int \vec{G}dz]_{\text{after reflection}} \\ &\qquad - [\int \vec{G}dz]_{\text{before reflection}}\} \\ &\equiv (\epsilon - 1)\,\Delta(\int \vec{G}dz), \qquad (2.16)\end{aligned}$$

where we have used (2.9) and (2.6); the symbol Δ stands for "the change of." Only half of this impulse is necessary to reverse the mechanical momentum which accompanies the pulse. The other half remains as a net backward impulse to atoms near the conductor.

Now we can apply the conservation of momentum to find the impulse that the conductor receives. The sum of the mechanical impulse given to the gas, the electromagnetic momentum change, and the impulse to the conductor must be zero. Since $\Delta(\int G\, dz)$ is the change in the electromagnetic momentum, we see indeed that the impulse given to the conductor is

$$\mathfrak{g}_{\text{cond}} = -\Delta(\int \epsilon\vec{G}dz) = -\Delta(\int \vec{K}dz). \qquad (2.17)$$

One can find the result (2.17) by directly evaluating the Lorentz force $c^{-1}\vec{J}\times\vec{B}$ on the surface current of the conductor, but we wished to show how it follows from conservation-of-momentum considerations. In this case, we see that the force may be evaluated directly from the change in the pseudomomentum $\vec{K}$, without the necessity for evaluating the impulse given to the atoms.

Let us now return to consider the entry of the pulse into the gas through the thin transparent membrane. To find application of the pseudomomentum here, we must examine a different situation, as illustrated in Fig. 2. Rather than a plane wave, we consider a beam of radiation of finite cross section. And rather than a pulse we consider a continuous wave. These changes make the gradient term in the force equation (2.4) the only one of importance, and they require that one consider motion of the gas atoms. As before, we can neglect the small reflection of radiation at the membrane. But because the atoms experience a trans-

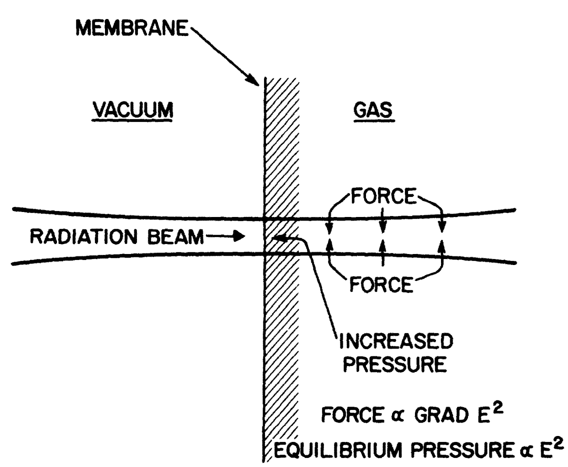

FIG. 2. A beam of radiation continuously passing a vacuum-gas interface.

verse force (in the x-y plane) inward toward the center of the beam, they are pulled in until the gas pressure is increased there. The equilibrium pressure p is simply[8]

$$p=\tfrac{1}{2}N\alpha E^2+p_0, \tag{2.18}$$

so that the average outward force due to the pressure gradient (− grad p) balances the inward force due to the field gradient. The additional pressure in the beam must be contained by the membrane, which therefore experiences an increased push outward from the medium toward the vacuum. We now need to express that force in terms of the wave momentum. If the beam is many wavelengths in width so that it is very like a plane wave, we can apply $nE_x=B_y$ and $\vec{G}=\vec{E}\times\vec{B}/4\pi c$ as in (2.5) and (2.9). The force per unit area on the membrane owing to the increased gas pressure in the beam is therefore equal to

$$\vec{f}=-\hat{z}p=-(n-1)(c/n)\vec{G}, \tag{2.19}$$

where $\vec{G}$ is the electromagnetic momentum in the medium. This force is equal to the rate of change of the pseudomomentum $\vec{K}$ as the beam passes thru the membrane, as we shall immediately show. The momentum density $\vec{G}$ is continuous across the membrane by virtue of its connection (1.1) with the energy flow and our neglect of the reflection. Thus, pseudomomentum leaves the vacuum at a rate $c\vec{G}$, and appears in the medium at a rate $(c/n)\epsilon\vec{G}=nc\vec{G}$. The increase in pseudomomentum $(n-1)c\vec{G}$ balances the force (2.19) to first order in $\epsilon-1$, the accuracy our present considerations allow. If the reflection is taken into account, the pseudomomentum balance becomes exact. However, there are other effects of order $(\epsilon-1)^2$ which we have ignored, and which we will treat in Sec. III.

This completes the discussion of the gas case. We have established that to first order in $\epsilon-1$, the quantity $\vec{G}=\vec{S}/c^2$ is the momentum density of electromagnetic waves in a material medium. The pseudomomentum density $\epsilon\vec{G}$ is useful for force calculations, but this result is not universal; the calculated forces are combinations of radiation and mechanical pressures. Section III will generalize these results.

III. GENERALIZATION

We shall now try to give these ideas a more general flavor. For simplicity, we shall restrict our attention to isotropic dielectrics, and will assume a magnetic permeability of unity, so that $\vec{H}=\vec{B}$. We will also continue to neglect dispersion, hence our considerations apply to "low" frequencies. We shall show that the concept of a "true" electromagnetic momentum density $\vec{G}$ [(1.1)] is quite consistent with force computations based on the rate change of the pseudomomentum $\vec{K}$ [(1.2)], and that there is no contradictory experimental evidence.

For the "low-frequency" case, Landau and Lifshitz[9] give the force-momentum relation in the form

$$F_i=\sum_k \frac{\partial\sigma_{ik}}{\partial x_k}-\frac{\partial G_i}{\partial t}, \tag{3.1}$$

where F_i is the ith component of the volume force on the material medium, G_i is the ith component of $\vec{G}$, and σ_{ik} is the (i,k) component of the stress tensor $\overleftrightarrow{\sigma}$, which for the case of fluid dielectrics is[10]

$$\sigma_{ik}=-p\delta_{ik}-\frac{E^2}{8\pi}\left(\epsilon-\rho_m\frac{\partial\epsilon}{\partial\rho_m}\right)\delta_{ik}+\frac{E_iD_k}{4\pi}-\frac{B^2}{8\pi}\delta_{ik}+\frac{B_iB_k}{4\pi}. \tag{3.2}$$

In (3.2), the quantity p is the pressure that would exist in the fluid in the absence of the field, but at the ambient conditions of density and temperature as they are in the presence of the field. Also, ρ_m is the mass density of the fluid, and it is assumed that ϵ is a known function of ρ_m and temperature. Substituting (3.2) into (3.1) and making use of Maxwell's equations, one arrives at the force equation for fluids, namely,[11]

$$\vec{F}=-\operatorname{grad}p-\frac{E^2}{8\pi}\operatorname{grad}\epsilon+\operatorname{grad}\left[\rho_m\left(\frac{\partial\epsilon}{\partial\rho_m}\right)\frac{E^2}{8\pi}\right]+(\epsilon-1)\frac{\partial\vec{G}}{\partial t}. \tag{3.3}$$

It is instructive to compare (3.3) with the force that the macroscopic field exerts on the atoms of the fluid. Using (2.2) and the relations

$$\vec{F}_{\text{macro}}\equiv N\vec{f}_{\text{atom}} \quad\text{and}\quad 4\pi N\vec{p}=(\epsilon-1)\vec{E},$$

one can arrive at a relation similar to (2.4), namely,

$$\vec{F}_{\text{macro}}=(\epsilon-1)\left[\operatorname{grad}\left(\frac{E^2}{8\pi}\right)+\frac{\partial\vec{G}}{\partial t}\right]. \tag{3.4}$$

Using (3.4), (3.3) may be expressed as

$$\vec{F}=\vec{F}_{\text{macro}}-\operatorname{grad}p+\operatorname{grad}\left\{\left(\frac{E^2}{8\pi}\right)\left[\rho_m\left(\frac{\partial\epsilon}{\partial\rho_m}\right)-(\epsilon-1)\right]\right\}. \tag{3.5}$$

The quantity $\vec{F}_{\text{macro}}$ is the force exerted on the dipoles by the macroscopic field. The third term

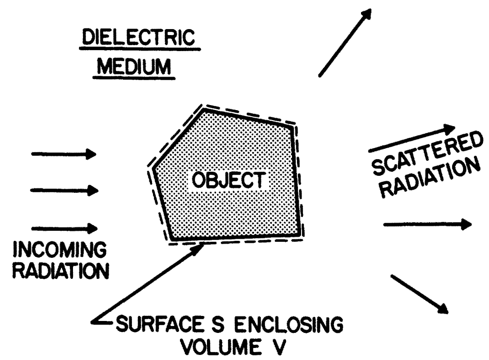

FIG. 3. Geometry of an object in a volume V surrounded by a closed surface S. If the support is present, it is assumed that the radiation does not exert any appreciable force directly on the support.

of (3.5) exists because the microscopic field in the medium is not the same as the macroscopic field, and the fluctuations of the microscopic field are correlated with the charge fluctuations. This term is of second order in $\epsilon - 1$, [12] and is a result of the dipole-dipole forces in the medium. It becomes negligible for dilute media such as discussed above in Sec. II. Being of the form of a gradient, it disappears when integrated over the whole space containing the fields. This is a symptom of forces which are balanced simultaneously by equal and opposite forces elsewhere.

If G is truly the electromagnetic momentum density, as is implicit in (3.1), then under what circumstances do we arrive at the usefulness of the pseudomomentum $\vec{K} = \epsilon \vec{G}$ for force calculations? Integrating (3.1) over a finite region of space V surrounded by a closed surface S yields

$$\int_V \vec{F} dV + \frac{d}{dt}\int_V \vec{G} dV = \int_S \overleftrightarrow{\sigma} \cdot \hat{n}\, dS , \qquad (3.6)$$

where $\hat{n}$ is the outward unit normal on the surface. Now suppose that an object is immersed in dielectric fluid, except that it may be attached by some supports, as shown in Fig. 3. The fluid need not be of uniform composition. We draw the surface S outside of the object (perhaps just outside). The condition that makes the pseudomomentum useful is that $\vec{F} = 0$ in the fluid on S. Let us see how this happens.

The condition $\vec{F} = 0$ on S may be examined through (3.3). It implies that $\partial \vec{G}/\partial t = 0$ on S, which can happen either for a steady-state situation, or if $\vec{G} = 0$ on S by virtue of the conditions of the problem. This latter situation occurs, for example, for reflection at normal incidence from a good conductor, so that $\vec{E} \cong 0$ just outside the conductor. In addition, the condition $\vec{F} = 0$ requires that the pressure in the fluid satisfy the relation

$$\operatorname{grad} p = \operatorname{grad}\left(\rho_m \frac{\partial \epsilon}{\partial \rho_m} \frac{E^2}{8\pi}\right) - \frac{E^2}{8\pi} \operatorname{grad} \epsilon \qquad (3.7)$$

in the vicinity of S. This is the same equation that fluids satisfy in an electrostatic field. [13] Again, the relation (3.7) tells us that either $E^2 = 0$ on S, or that we must have a steady state, so that the pressure in the fluid has a chance to reach its equilibrium value (3.7). If the term $E^2 \operatorname{grad}\epsilon$ is sufficiently small to be ignored (this is true for all the experiments), then (3.7) yields for the pressure on S the value [compare with (2.18)]

$$p_s = \rho_m \left(\frac{\partial \epsilon}{\partial \rho_m}\right)\left(\frac{E^2}{8\pi}\right) + (\text{const}). \qquad (3.8)$$

If we put this result in (3.2), we find that σ_{ik} on S depends on the medium only through its dielectric constant, being given by

$$\sigma_{s,ik} = \frac{1}{4\pi}(\epsilon E_i E_k + B_i B_k) - \frac{1}{8\pi}(\epsilon E^2 + B^2)\delta_{ik}, \qquad (3.9)$$

and it is this tensor that determines the total force on V according to (3.6). Note that the total force on V has two important components, namely, the ponderomotive force exerted directly by the field on the object within V, and the force transmitted through S by the pressure (3.8) in the fluid.

The stress tensor (3.9) may be associated with the pseudomomentum $\vec{K}$ in the following way. In the absence of free charges and currents it satisfies the equation

$$\operatorname{div} \overleftrightarrow{\sigma}_s = \frac{\partial \vec{K}}{\partial t} - \frac{1}{8\pi} E^2 \nabla \epsilon , \qquad (3.10)$$

where $\operatorname{div} \overleftrightarrow{\sigma}_s$ is a vector whose components are

$$(\operatorname{div} \overleftrightarrow{\sigma}_s)_i = \sum_k \frac{\partial \sigma_{s,ik}}{\partial x_k} .$$

If we imagine that, outside S, the condition $E^2 \nabla \epsilon = 0$ continues to apply everywhere, and the dielectric contains the total field, then we obtain

$$\int_S \overleftrightarrow{\sigma}_s \cdot \hat{n}\, dS = -\frac{d}{dt}\int_{\text{outside } S} \vec{K} dV. \qquad (3.11)$$

In (3.11), the minus sign on the right-hand side appears because n has been defined as the *outward* normal on S. From (3.6) we see that if $\vec{F} = 0$ and $E^2 \nabla \epsilon = 0$ on and outside of S, then the sum of the total force exerted on the interior of S plus the rate of increase of true electromagnetic momentum within S equals a negative rate of change of pseudomomentum outside of S. The total force $\int F dV$ equals the rate of change of mechanical momentum within V, or in case the object within V is supported, equals the force exerted on the supports.

IV. DISCUSSION AND RELATION TO EXPERIMENT

The relation (1. 1) between momentum density and energy transport may be understood from the following argument. A pulse of electromagnetic waves, traveling with group velocity $\vec{v}_g$ and having energy density U, transports energy at a rate $\vec{S}=U\vec{v}_g$. Using the relativistic relation between mass and energy, $U=mc^2$, where m is the equivalent mass density, we find that $\vec{S}=m\vec{v}_g c^2$, and finally since the equivalent mass density times the group velocity is the momentum density $\vec{G}$, we find $\vec{S}=\vec{G}c^2$, which is identical with (1. 1). The same relation applies to momentum and energy transport of material particles if we use the relativistic formulas $E=mc^2$ and $\vec{p}=m\vec{v}$. However, if an excitation including both mechanical and electromagnetic momenta travels through a material medium, the same relation does *not* apply to the total transport of *free* energy [replacing $\vec{S}$ in (1. 1)] and the total momentum density [replacing $\vec{G}$ in (1. 1)]. This is clear from the example of the plane-wave pulse analyzed in Sec. II, wherein the transport of free energy and only the electromagnetic part of the total momentum density satisfied (1. 1).

When we consider experiments giving evidence of radiation pressures, it appears that they all satisfy the conditions necessary to the validity of the pseudomomentum concept. A review of early experiments is given by Jones and Richards[14] in a paper in which they report some careful work showing that the ratio of the radiation pressure, on a metallic reflector immersed in a variety of liquid dielectrics, to the radiation pressure on the same metallic reflector in air, was accurately proportional to the refractive index n of the liquid. The indices of the liquids used ranged from 1. 33 (water) to 1. 61 (carbon disulphide), and the accuracy of the measurements was estimated as about $\pm 1.2\%$. They used a tungsten lamp run at 30 W as their radiation source, and estimated that about 10^{-3} of the emitted light was used. The time constants used in the detection system were of the order of tenths of seconds. Some very recent experiments by Ashkin and Dziedzic[15] using an argon-ion laser source investigated the pressure on solid dielectric spheres immersed in liquid, and the pressure on a liquid-air interface owing to the passage of a beam of radiation. The experiments on spheres have not yet been extended to include a careful study of the dependence of the radiation pressure on the index of the liquid. The experiments on the liquid-air interface gave convincing evidence of a force on the interface outward from the liquid where the laser beam intersected the surface. The force did not depend strongly on whether the laser was directed at the interface from the air or from the liquid. These experimental results are all in accord with expectations derived from analysis of pseudomomentum changes, or correspondingly from use of the stress tensor (3. 9) in the force equation (3. 6). In Sec. III, we have discussed the conditions under which these results are to be expected, namely, that $E^2\nabla\epsilon$ be negligible in the vicinity of a surface surrounding the object of interest, and that the pressure in the fluid has a chance to reach its equilibrium value (3. 8). For an object surrounded uniformly by the dielectric, the surface can be drawn just outside the object, and then $\vec{\nabla}\epsilon=0$ on and outside the surface. The time τ that the pressure takes to reach equilibrium is of the order of the dimension of the object divided by the sound velocity, or $\tau\approx l/v_s$. If $l\approx 0.1$ cm, and $v_s\sim 10^5$ cm/sec, the time constant comes out around 10^{-6} sec. For radiation normally incident on a good metallic reflector, the equilibrium pressure in the liquid is unchanged at the surface because $E^2=0$. Thus for the Jones and Richards's experiments, even though the time constant was of the order of tenths of seconds, it did not matter; the same result would be found independent of the length of the pulse, in accord with the discussion of Sec. II. Ashkin's experiments on spheres were done with the laser operating cw, which obviously satisfies the time-constant requirement for equilibrium pressure in the liquid. The total force on the sphere results both from the ponderomotive force and the increased mechanical pressure of the liquid where the light intensity is large, and is predicted to be directly proportional to the rate of pseudomomentum change that scattering from the sphere produces.

Finally, we consider Ashkin and Dziedzic's experiments concerning the forces on liquid-air interfaces. Here they used moderately high-intensity pulses (1 kW) of about 50-nsec duration, focused to a small spot only a few wavelengths in diameter. The intensity and small spot were necessary to obtain a measurable effect. An outward motion of the liquid surface was observed, occurring mostly after the peak of the pulse had passed. This experiment is similar to the thought experiment at the end of Sec. II, but we must here consider the dynamics of the experimental situation. With a spot radius of about 10^{-4} cm, and sound velocity about 10^5 cm/sec, one may estimate that the mechanical pressure in the liquid does have enough time to approach equilibrium during the 5×10^{-8}-sec pulse. Thus an outward force is expected which results in an outward bulge on the surface where the beam passes. Again the outward force is consistent with the rate of pseudomomentum change. Note that we can draw the surface S to contain the spot of intersection of the beam with the surface. Outside S there is no field at the liquid surface, so again the condition $E^2\nabla\epsilon = 0$ is satisfied outside of S.

It is of interest to think of doing a similar experiment with a radiation pulse sufficiently short that the liquid pressure cannot equilibrate. If the pulse were 10^{-9} sec in duration, for example, might one then observe effects relating to the true electromagnetic momentum of the pulse? The answer, it would appear, is no. Figure 4 illustrates a short pulse crossing a liquid-air interface. Let us consider the various possible effects of the force (3.3). The last term $(\epsilon - 1)\partial\vec{G}/\partial t$ integrates to zero over the complete duration of the pulse, but results in a small displacement of the atoms in the direction of propagation of the pulse. This displacement Δz is approximately

$$\Delta z = (\epsilon - 1)S\tau/\rho_m c^2, \tag{4.1}$$

where S and τ are the intensity and duration of the pulse, and ρ_m is again the mass density of the fluid. Assuming an intensity of 10^{11} W/cm^2, a pulse duration of 10^{-9} sec, $\epsilon \sim 2$, and $\rho_m \sim 1$, one finds $\Delta z \sim 10^{-12}$ cm. This would surely be impossible to observe.

If we use the Clausius–Mossotti expression[12] for ϵ, the gradient terms of (3.3) give rise to inward forces around the periphery of the pulse and at the surface of the liquid. They therefore compress the liquid, and this effect should initially result in a small depression of the liquid surface. The surface pressure f due to these gradient terms, the second and third terms of (3.3), is roughly

$$f = \tfrac{1}{6}(\epsilon - 1)^2 S/c\ . \tag{4.2}$$

If such a pressure were applied over the surface of a volume of linear dimensions roughly equal to the transverse dimension w of the pulse, the resulting compression of the liquid would result in an inward motion of the surface of approximately

$$\Delta z = \tfrac{1}{36}(\epsilon - 1)^2 (S/c)wx, \tag{4.3}$$

where x is the compressibility of the liquid. Such a compression would occur during approximately the time it takes sound to travel a distance w, after which the outward impulse discussed below would take over.

For values of $w \sim 0.5$ cm, $S = 10^{10}$ W/cm^2, $\epsilon = 2$, and typical liquid compressibility in the range 10^{-10}– 10^{-11} cm^2/dyn, the value of Δz comes out to be 10^{-5} to 10^{-6} cm. Such a compression might be measured, but in any event results from electrostriction rather than anything having to do with electromagnetic momentum.

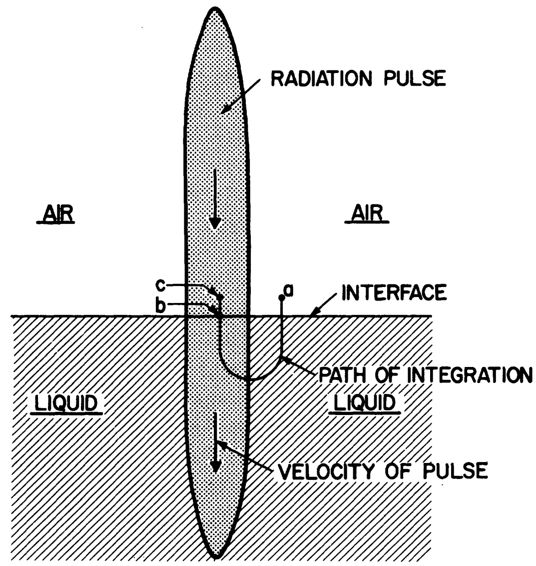

FIG. 4. Radiation pulse passing an air–liquid interface. A reflected pulse is actually present but is not illustrated.

Having dealt with the displacement and compression effects, there is a final effect resulting from the net impulse given to the liquid by the passage of the pulse. If we now think of the liquid as incompressible, it is clear that if the liquid is to move at one point the displaced liquid must go somewhere. If we chose some path, such as a-b-c in Fig. 4, the liquid can move along this path since it starts and ends outside the surface. If we integrate the force along the path, i.e., find $\int \vec{F}\,d\vec{l}$, and then integrate this result over the duration of the pulse, we see that the net impulse along any such path has a magnitude

$$(\epsilon - 1)\int \frac{E_b^2}{8\pi}\,dt \cong (\epsilon - 1)\frac{S\tau}{2c}\ ,$$

where E_b is the field at the surface at point b, and that it is directed so as to push the fluid out at the surface where the pulse passes. The motion resulting from this impulse is that observed by Ashkin and Dziedzic, and according to the above discussion, is the dominant effect to be expected in any similar experiment.

I conclude that theory and experiment are in complete accord, but that laboratory experiments designed to demonstrate the nature of the true electromagnetic momentum in dielectric media may not be feasible.

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[8]Here the gas pressure does not respond to the instantaneous square of the field strength, but rather to its average over an appropriate relaxation time. The same kind of time average is implied in all later expressions for pressure.

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[10]Reference 9, Eqs. (15.9) and (34.2).

[11]Reference 9, Eq. (56.18).

[12]One simple model of a dielectric medium results in the Clausius-Mosotti relation for the dependence of the dielectric constant on density; namely, $(\epsilon-1)/(\epsilon+2) \propto \rho_m$. From this relation it follows that $\rho_m(\partial\epsilon/\partial\rho_m) - (\epsilon-1) = (\epsilon-1)^2/3$. More generally, if one can expand $(\epsilon-1)$ in a power series in ρ_m, then the first-order term in the expression in the last sentence will cancel, giving essentially the same result.

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Calculation of the $^1P(2s2p)$ Autoionization State of He with a Pseudostate Nonresonant Continuum*

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Previous Hylleraas calculations of $\mathcal{E} = \langle QHQ \rangle$ for the lowest ^1P autoionization state of helium are here supplemented by calculations of the shift (Δ), width (Γ), and shape parameter (q) using a $(1s, 2\tilde{p})$ pseudostate nonresonant continuum function. The function is constructed to eliminate the dominant $(2s, 2p)$ configuration of the autoionization state, while at the same time containing three variationally determined radial functions. Both Δ and q are also shown to contain contributions from the discrete part of the nonresonant spectrum, although quantitatively that contribution is found to be small. Final results change previous polarized-orbital results minimally, which means that the resonance position, $\mathcal{E} = E + \Delta$, continues to be on the edge of the experimental error, and q remains somewhat outside the experimental result. Further relativistic corrections are briefly discussed, but a simple argument indicates that they are not likely to explain the differences with experiment. It is concluded that more-accurate experiments should be carried out.

I. INTRODUCTION AND FORMULAS

As has been previously emphasized,[1] the photo-"excitation" of the autoionization states of He afford a unique testing ground for precision checks of the continuum solutions of the Schrödinger equation. The basic parameters that are compared with experiment are the energy E, width Γ, and photoabsorption shape parameter q. The energy of the resonance is usually written[2] (rydberg units are used throughout)

$$E = \mathcal{E} + \Delta . \tag{1.1}$$

$\mathcal{E}$ is the result of a well-defined projection-operator variational calculation

$$\delta \frac{\langle \Phi Q H Q \Phi \rangle}{\langle \Phi Q \Phi \rangle} = 0 , \tag{1.2}$$

which we shall not discuss further except to repeat