Elastic Electron-Atom Scattering at Intermediate Energies*

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We analyze elastic electron-atom scattering at intermediate and high energies by combining the Born series and eikonal series to get a consistent picture of the scattering amplitude through order k_i^{-2}. Our eikonal-Born-series approach is compared with the usual Glauber approximation, which is shown to be seriously deficient. We also discuss the Schwinger variational principle, exchange effects, and forward dispersion relations. Applications are made to the elastic scattering of electrons by atomic hydrogen and helium. The agreement between our results and the experimental data is very good.

I. INTRODUCTION

The nonrelativistic elastic scattering of electrons by atoms is one of the most fundamental problems in atomic physics. In this paper, we present a theoretical approach to such processes based on a detailed study of the Born and Glauber[1] multiple scattering series. Our method applies essentially to the region of intermediate and high energy. Although we shall formulate our approach for the general case of electron-atom scattering, we shall illustrate it by considering the elastic scattering of electrons by atomic hydrogen and helium. These processes have recently attracted a considerable amount of theoretical interest,[2-18] while experimental data have also become available,[19-23] particularly in helium,[19-22] where absolute measurements of angular distributions have been made.

After recalling some basic equations, we present in Sec. II a detailed comparison of the Born series and the Glauber multiple scattering series.[1] First, the second Born term is evaluated by using an average excitation energy, and its analytic behavior is discussed. Then the Glauber series is analyzed in detail. As in the case of potential scattering,[24] it is shown that important contributions are missing from the Glauber series. We conclude Sec. II with a brief discussion of the exchange amplitude. Section III is devoted to the application of our method to the analysis of elastic electron-hydrogen and electron-helium scattering. Our results are compared with other theoretical approaches and with the experimental data. We find good agreement between experiment and our theoretical values. We also find at all energies a very significant difference between our results and those of the Glauber approximation. We conclude Sec. III by using the Schwinger variational principle[25] to estimate higher-order corrections to our eikonal-Born-series results.

Finally, in Sec. IV we discuss the important question of forward dispersion relations and show that our results for the real part of the forward elastic scattering amplitude are in good agreement with the dispersion-relation calculation of Bransden and McDowell.[9]

II. THEORY

A. Basic Equations

Let us consider the elastic scattering of an electron by a neutral atom of atomic number Z. We assume that the collision is nonrelativistic. Since we are interested in the intermediate and high-energy region, we shall first neglect the effects of the Pauli principle between the incident and target electrons; corrections due to exchange effects will be considered separately below. The initial and final wave vectors of the electron will be denoted by $\vec{k}_i$ and $\vec{k}_f$, respectively, with $|\vec{k}_i| = |\vec{k}_f| = k_i$. The nucleus of the atom being the origin of our coordinate system, we shall denote the coordinate of the projectile electron by $\vec{r}_0$, while the positions of the atomic electrons will be labeled by $\vec{r}_i$ $(i = 1, \ldots, Z)$. Atomic units will be used throughout this paper.

The free motion of the two colliding particles is described by the Hamiltonian

$$H_0 = K + h, \tag{2.1}$$

where K is the kinetic-energy operator

$$K = -\tfrac{1}{2}\nabla_{\vec{r}_0}^2 \tag{2.2}$$

and h is the internal Hamiltonian of the target. Moreover, we have

$$h|n\rangle = \omega_n|n\rangle, \tag{2.3}$$

where $|n\rangle$ denotes an eigenstate of the target Hamiltonian and ω_n is the corresponding internal ener-

gy. We shall denote by $|0\rangle$ the initial (and final) eigenket of the target.

The full Hamiltonian of the system is such that

$$H = H_0 + V, \tag{2.4}$$

where V, the interaction potential between the projectile and the target, is simply

$$V = \sum_{i=1}^{Z} \frac{1}{r_{i0}} - \frac{Z}{r_0}, \tag{2.5}$$

with $r_{i0} = |\vec{r}_i - \vec{r}_0|$.

The scattering amplitude for elastic scattering is then given by the expressions

$$f = -(2\pi)^2 \langle \Phi_f | V | \Psi_i^+ \rangle \tag{2.6a}$$

or

$$f = -(2\pi)^2 \langle \Psi_f^- | V | \Phi_i \rangle, \tag{2.6b}$$

where Φ_i and Φ_f are eigenstates of H_0 (i.e., free waves), while Ψ_i^+ and Ψ_f^- denote the full scattering wave functions satisfying the Lippmann-Schwinger equations

$$\Psi_i^+ = \Phi_i + G_0^+ V \Psi_i^+ \tag{2.7a}$$

and

$$\Psi_f^- = \Phi_f + G_0^- V \Psi_f^-. \tag{2.7b}$$

In these equations, the Green's operators $G_0^{\pm}$ are given by

$$G_0^{\pm} = (E - H_0 \pm i\epsilon)^{-1}. \tag{2.8}$$

Since we are considering only nonrelativistic collisions and we neglect the Pauli principle between the incident and target electrons, we may ignore the spin of the projectile; the indices i and f therefore label the momentum of the projectile together with the internal quantum numbers of the target. In what follows we shall write the asymptotic states $|\Phi_i\rangle$ and $|\Phi_f\rangle$ more explicitly as $|0, \vec{k}_i\rangle$ and $|0, \vec{k}_f\rangle$, respectively. More generally, an eigenstate $|\Phi_c\rangle$ of H_0 will be written as $|n, \vec{q}\rangle$. The normalization condition which we adopt is such that

$$\langle n', \vec{q}' | n, \vec{q} \rangle = \delta_{nn'} \delta(\vec{q} - \vec{q}'). \tag{2.9}$$

Hence, in the coordinate representation

$$\Phi_c(\vec{r}_0, \xi) = (2\pi)^{-3/2} e^{i\vec{q}\cdot\vec{r}_0} \psi_n(\xi), \tag{2.10}$$

where ξ denotes the collection of target coordinates.

B. Born and Eikonal Series

Let us return to the Lippmann-Schwinger equations (2.7) and solve them by iteration, starting from the free waves Φ_i and Φ_f, respectively. We obtain in this way

$$\Psi_i^+ = \Phi_i + G_0^+ V \Phi_i + G_0^+ V G_0^+ V \Phi_i + \cdots \tag{2.11a}$$

and

$$\Psi_f^- = \Phi_f + G_0^- V \Phi_f + G_0^- V G_0^- V \Phi_f + \cdots. \tag{2.11b}$$

By substitution into Eqs. (2.6), we see that the Born series for the scattering amplitude is given by

$$f = \sum_{n=1}^{\infty} \bar{f}_{Bn}, \tag{2.12}$$

where

$$\bar{f}_{Bn} = -(2\pi)^2 \langle \Phi_f | V G_0^+ V \cdots G_0^+ V | \Phi_i \rangle. \tag{2.13}$$

In this last expression the potential V appears n times, and the Green's operator G_0^+ appears $n-1$ times. We shall also define f_{Bn} to be the sum of the first n terms of Eq. (2.12). That is,

$$f_{Bn} = \sum_{j=1}^{n} \bar{f}_{Bj}. \tag{2.14}$$

We now consider the "many-body" Glauber[1] approximation to the scattering amplitude, namely,

$$f_G = \frac{k_i}{2\pi i} \int d^2 b_0 \, e^{i\vec{K}\cdot\vec{b}_0} \langle 0 | e^{i\chi_G(k_i, \vec{b}_0, \ldots, \vec{b}_Z)} - 1 | 0 \rangle, \tag{2.15}$$

where

$$\vec{K} = \vec{k}_i - \vec{k}_f \tag{2.16}$$

is the momentum transfer. In writing Eq. (2.15) we have also decomposed the vector $\vec{r}_0$ as

$$\vec{r}_0 = \vec{b}_0 + z_0 \hat{z}, \tag{2.17}$$

and we have chosen the z axis to be perpendicular to the momentum transfer. The implications of this choice, first proposed by Glauber,[1] have been discussed in detail by Byron, Joachain, and Mund[24] in the case of potential scattering. We shall return to this point below.

The Glauber phase-shift function χ_G which appears in Eq. (2.15) is given by

$$\chi_G(k_i, \vec{b}_0, \ldots, \vec{b}_Z) = -\frac{1}{k_i} \int_{-\infty}^{\infty} V(\vec{b}_0, z_0, \vec{r}_1, \ldots, \vec{r}_Z) \, dz_0, \tag{2.18}$$

where we have decomposed the vectors $\vec{r}_i$ as

$$\vec{r}_i = \vec{b}_i + z_i \hat{z} \quad (i = 1, 2, \ldots, Z). \tag{2.19}$$

We have used the fact that for any potential V which depends only on r_0 and r_{i0}—in particular, for the interaction given by Eq. (2.5)—the Glauber phase-shift function does not depend on z_i.

By analogy with the Born series [Eq. (2.12)], we now define the Glauber multiple scattering series

$$f_G = \sum_{n=1}^{\infty} \bar{f}_{Gn}, \tag{2.20}$$

where

$$\bar{f}_{Gn} = \frac{k_i}{2\pi i}\frac{i^n}{n!}\int d^2b_0\, e^{i\vec{K}\cdot\vec{b}_0}\langle 0|\chi_G^n(k_i,\vec{b}_0,\ldots,\vec{b}_Z)|0\rangle . \tag{2.21}$$

We shall also denote by f_{Gn} the sum of the first n terms in Eq. (2.20). Thus,

$$f_{Gn} = \sum_{j=1}^{n}\bar{f}_{Gj} . \tag{2.22}$$

Before we analyze in detail the Born series of Eq. (2.12) and the Glauber series of Eq. (2.20), we first recall that with the choice of z axis which we have adopted it is straightforward to show that $f_{B1} = f_{G1}$ for all scattering angles. Moreover, for the interaction potential given by Eq. (2.5) the phase-shift function χ_G can be readily evaluated; it is given by

$$\chi_G = \frac{1}{k_i}\sum_{i=1}^{Z}\ln\left(1 - 2\frac{b_i}{b_0}\cos(\varphi_i - \varphi_0) + \frac{b_i^2}{b_0^2}\right), \tag{2.23}$$

where φ_i is the azimuthal angle of $\vec{b}_i$ in the xy plane, and $b_i = |\vec{b}_i|$. Since χ_G depends only on the differences $\varphi_i - \varphi_0$, it is clear that the only φ_0 dependence in Eqs. (2.15) and (2.21) is contained in $e^{i\vec{K}\cdot\vec{b}_0}$. Thus, we may perform the φ_0 integration and write

$$f_G = \frac{k_i}{i}\int db_0\, b_0 J_0(Kb_0)\langle 0|e^{i\chi_G} - 1|0\rangle \tag{2.24}$$

and

$$\bar{f}_{Gn} = \frac{k_i}{i}\frac{i^n}{n!}\int db_0\, b_0 J_0(Kb_0)\langle 0|\chi_G^n|0\rangle . \tag{2.25}$$

It is now obvious from Eq. (2.25) that the terms of the Glauber multiple scattering series [Eq. (2.22)] are alternately purely real (odd order) and purely imaginary (even order). This is in contrast to the Born series of Eq. (2.12), where already the second-order term $\bar{f}_{B2}$ contains both a real and an imaginary part.

C. Second Born Term

Let us now analyze in more detail the term $\bar{f}_{B2}$. If we neglect exchange and write the states $|\Phi_i\rangle$ and $|\Phi_f\rangle$ as $|0,\vec{k}_i\rangle$ and $|0,\vec{k}_f\rangle$, respectively, where zero denotes the ground state, then we have

$$\bar{f}_{B2} = 8\pi^2\int d^3q\sum_n \frac{\langle 0,\vec{k}_f|V|n,\vec{q}\rangle\langle n,\vec{q}|V|0,\vec{k}_i\rangle}{q^2 - k_i^2 + 2(\omega_n - \omega_0) - i\epsilon} , \tag{2.26}$$

where the interaction potential V is given by Eq. (2.5). This expression has already been analyzed by several authors,[2,3,6,14,15,18] so that we shall only give the essential steps of our calculations.

First of all, because $|n,\vec{q}\rangle = |n\rangle|\vec{q}\rangle$, with the normalization specified by Eq. (2.9), the integration on the plane-wave parts of the matrix elements in Eq. (2.26) can be done immediately. We obtain

$$\bar{f}_{B2} = \frac{2}{\pi^2}\int d\vec{q}\sum_\mu \frac{1}{K_i^2 K_f^2}\left(\langle 0|\sum_{m=1}^{Z}(e^{-i\vec{K}_f\cdot\vec{r}_m} - 1)|\mu\rangle\langle\mu|\sum_{n=1}^{Z}(e^{i\vec{K}_i\cdot\vec{r}_n} - 1)|0\rangle\right)\Big/[q^2 - k_i^2 + 2(\omega_\mu - \omega_0) - i\epsilon], \tag{2.27}$$

where $\vec{K}_i = \vec{k}_i - \vec{q}$ and $\vec{K}_f = \vec{k}_f - \vec{q}$.

The basic approximation which we shall make in this section is to replace the quantity $\omega_\mu - \omega_0$ in the denominator of Eq. (2.27) by an average energy difference Δ. By thus removing the μ dependence of the denominator, we see that the sum on intermediate states in Eq. (2.27) can be done by closure, giving the simplified second Born approximation (SB2)

$$\bar{f}_{SB2} = \frac{2}{\pi^2}\int d^3q\,\frac{1}{K_i^2 K_f^2}\,\frac{1}{q^2 - p_i^2 - i\epsilon} \times\langle 0|\left(\sum_{m=1}^{Z}(e^{-i\vec{K}_f\cdot\vec{r}_m} - 1)\right)\left(\sum_{n=1}^{Z}(e^{i\vec{K}_i\cdot\vec{r}_n} - 1)\right)|0\rangle, \tag{2.28}$$

where $p_i^2 = k_i^2 - 2\Delta$. If we write $|0\rangle$ as an antisymmetrized sum of products of orthonormal orbitals, where the orbitals are denoted by $\phi_1, \phi_2, \ldots, \phi_Z$, then it is straightforward to reduce this expression to

$$\bar{f}_{SB2} = \frac{2}{\pi^2}\int d^3q\,\frac{1}{K_i^2 K_f^2}\,\frac{1}{q^2 - p_i^2 - i\epsilon} \times\Bigg(\sum_{n=1}^{Z}\langle\phi_n|e^{i\vec{K}\cdot\vec{r}} - e^{i\vec{K}_i\cdot\vec{r}} - e^{-i\vec{K}_f\cdot\vec{r}} + 1|\phi_n\rangle + \sum_{n,m=1}^{Z}\langle\phi_n|e^{i\vec{K}_i\cdot\vec{r}} - 1|\phi_n\rangle\langle\phi_m|e^{-i\vec{K}_f\cdot\vec{r}} - 1|\phi_m\rangle - \sum_{n,m=1}^{Z}\langle\phi_m|e^{i\vec{K}_i\cdot\vec{r}} - 1|\phi_n\rangle\langle\phi_n|e^{-i\vec{K}_f\cdot\vec{r}} - 1|\phi_m\rangle\Bigg) . \tag{2.29}$$

Note that in the case $\vec{k}_i = \vec{k}_f$ the singularity coming from the term $K_i^{-2}K_f^{-2} = K_i^{-4}$ is canceled by the term in large parentheses, which vanishes at least as fast as K_i^2 in this case. The matrix elements in Eq. (2.29) can be easily evaluated for any collection of orbitals $\{\phi_n\}$ which can be written as a sum of terms of the form $r^l e^{-ar}$. Then the integration on $\vec{q}$ can be reduced to a single integral on a Feynman parameter t by the usual Feynman techniques.[26]

For the special case of elastic scattering from the ground state of hydrogen, the last two sums in Eq. (2.29) do not contribute, and we have

$$\bar{f}_{SB2} = \frac{2}{\pi^2}\int d^3q \frac{1}{K_i^2 K_f^2}\frac{1}{q^2-p_i^2-i\epsilon}\langle\phi_{1s}^{H}|e^{i\vec{K}\cdot\vec{r}}-e^{i\vec{K}_i\cdot\vec{r}}-e^{-i\vec{K}_f\cdot\vec{r}}+1|\phi_{1s}^{H}\rangle, \tag{2.30}$$

where $\phi_{1s}^{H}=e^{-r}/\sqrt{\pi}$ is the hydrogen ground-state orbital. Similarly, for the case of elastic scattering from the ground state of helium, the final sum in Eq. (2.29) does not contribute (because of the orthogonality of the two spin orbitals in helium), and we find

$$\begin{aligned}\bar{f}_{SB2} = \frac{2}{\pi^2}\int d^3q \frac{1}{K_i^2 K_f^2}\frac{1}{q^2-p_i^2-i\epsilon}\,(&2\langle\phi_{1s}^{He}|e^{i\vec{K}\cdot\vec{r}}-e^{i\vec{K}_i\cdot\vec{r}}-e^{-i\vec{K}_f\cdot\vec{r}}+1|\phi_{1s}^{He}\rangle\\ &+2\langle\phi_{1s}^{He}|e^{i\vec{K}_i\cdot\vec{r}}-1|\phi_{1s}^{He}\rangle\langle\phi_{1s}^{He}|e^{-i\vec{K}_f\cdot\vec{r}}-1|\phi_{1s}^{He}\rangle),\end{aligned} \tag{2.31}$$

where ϕ_{1s}^{He} is the ground-state single-particle orbital for helium.

For atomic hydrogen and helium targets an obvious improvement can be made on Eqs. (2.30) and (2.31). Indeed, it is possible in these cases to include exactly a certain number of low-lying target states in the summation on μ appearing in Eq. (2.27), and to perform the summation on the other states by closure.[6,14,15,18] Since the ground state lies far below the cluster of other bound states near the continuum, it is sensible to deal with the intermediate state $\mu=0$ explicitly in Eq. (2.27). This will result in only a trivial modification from a computational point of view. For atomic hydrogen, we find that the simplified second Born amplitude $\bar{f}_{SB2}$ given by Eq. (2.30) is replaced by

$$\begin{aligned}\bar{f}_{B2} = &\frac{2}{\pi^2}\int d^3q \frac{1}{K_i^2 K_f^2}\frac{1}{q^2-p_i^2-i\epsilon}\langle\phi_{1s}^{H}|e^{i\vec{K}\cdot\vec{r}}-e^{i\vec{K}_i\cdot\vec{r}}-e^{-i\vec{K}_f\cdot\vec{r}}+1|\phi_{1s}^{H}\rangle\\ &+\frac{2}{\pi^2}\int d^3q \frac{1}{K_i^2 K_f^2}\frac{1}{q^2-k_i^2-i\epsilon}\langle\phi_{1s}^{H}|e^{-i\vec{K}_f\cdot\vec{r}}-1|\phi_{1s}^{H}\rangle\langle\phi_{1s}^{H}|e^{i\vec{K}_i\cdot\vec{r}}-1|\phi_{1s}^{H}\rangle\\ &-\frac{2}{\pi^2}\int d^3q \frac{1}{K_i^2 K_f^2}\frac{1}{q^2-p_i^2-i\epsilon}\langle\phi_{1s}^{H}|e^{-i\vec{K}_f\cdot\vec{r}}-1|\phi_{1s}^{H}\rangle\langle\phi_{1s}^{H}|e^{i\vec{K}_i\cdot\vec{r}}-1|\phi_{1s}^{H}\rangle,\end{aligned} \tag{2.32}$$

while for helium the expression (2.31) is replaced by

$$\begin{aligned}\bar{f}_{B2} = &\frac{4}{\pi^2}\int d^3q \frac{1}{K_i^2 K_f^2}\frac{1}{q^2-p_i^2-i\epsilon}\langle\phi_{1s}^{He}|e^{i\vec{K}\cdot\vec{r}}-e^{i\vec{K}_i\cdot\vec{r}}-e^{-i\vec{K}_f\cdot\vec{r}}+1|\phi_{1s}^{He}\rangle\\ &+\frac{8}{\pi^2}\int d^3q \frac{1}{K_i^2 K_f^2}\frac{1}{q^2-k_i^2-i\epsilon}\langle\phi_{1s}^{He}|e^{-i\vec{K}_f\cdot\vec{r}}-1|\phi_{1s}^{He}\rangle\langle\phi_{1s}^{He}|e^{i\vec{K}_i\cdot\vec{r}}-1|\phi_{1s}^{He}\rangle\\ &-\frac{4}{\pi^2}\int d^3q \frac{1}{K_i^2 K_f^2}\frac{1}{q^2-p_i^2-i\epsilon}\langle\phi_{1s}^{He}|e^{-i\vec{K}_f\cdot\vec{r}}-1|\phi_{1s}^{He}\rangle\langle\phi_{1s}^{He}|e^{i\vec{K}_i\cdot\vec{r}}-1|\phi_{1s}^{He}\rangle.\end{aligned} \tag{2.33}$$

Let us now evaluate Eq. (2.30) for an orbital of the form

$$\phi_{1s}^{H,Z^*} = (Z^{*3}/\pi)^{1/2}e^{-Z^*r}. \tag{2.34}$$

The physical value of the important first term in Eq. (2.32) is then obtained by setting $Z^*=1$, and the analogous first term in Eq. (2.33) is a linear combination of such terms with various values of Z^*, since we write the Hartree-Fock orbital ϕ_{1s}^{He} as[27]

$$\phi_{1s}^{He} = (4\pi)^{-1/2}(Ae^{-\alpha r}+Be^{-\beta r}), \tag{2.35}$$

with $A=2.60505$, $B=2.08144$ and $\alpha=1.41$, $\beta=2.61$.

The additional terms necessary to evaluate Eqs. (2.32) and (2.33) are straightforward, since they are essentially second Born terms for scattering by a simple potential obtained by averaging over the ground-state wave functions. Such potentials decay exponentially, and hence the corresponding second Born terms exhibit familiar potential scattering behavior. The real part varies like k_i^{-2} for k_i large, and the imaginary part varies like k_i^{-1} for k_i large. In all cases these extra terms were evaluated by reducing them to an integral on one Feynman parameter, which was done numerically.

Using Eq. (2.34) in Eq. (2.30), one finds after straightforward calculation with Feynman methods that $f_{SB2}^{Z^*}$ is given by

$$\bar{f}^{Z^*}_{SB2} = 4\left(I - \alpha^2 \frac{dI}{d\alpha^2}\right)_{\alpha = 2Z^*} - 2K^2 \frac{K^2 + 8Z^{*2}}{(K^2 + 4Z^{*2})^2} I(\alpha = 0), \tag{2.36}$$

where we have introduced an obvious notation for $\bar{f}_{SB2}$ evaluated with an orbital of the type given in Eq. (2.34), and where

$$\mathrm{Re}I(\alpha) = \int_0^1 \frac{\alpha^2 t + 2\Delta}{[(\alpha^2 + K^2)t - K^2 t^2]^{1/2}\{(\alpha^4 - 4p_i^2 K^2)t^2 + [4p_i^2(\alpha^2 + K^2) + 4\Delta\alpha^2]t + 4\Delta^2\}} dt \tag{2.37a}$$

and

$$\mathrm{Im}I(\alpha) = 2p_i \int_0^1 \frac{dt}{(\alpha^4 - 4p_i^2 K^2)t^2 + [4p_i^2(\alpha^2 + K^2) + 4\Delta\alpha^2]t + 4\Delta^2}. \tag{2.37b}$$

Both of these expressions can be evaluated exactly, but the results are too cumbersome to be useful. In all results presented in subsequent sections, these equations were evaluated by numerical integration. However, if k_i is large, simplifications occur, and one finds

$$\mathrm{Re}I(\alpha) \simeq \frac{\pi}{2k_i(\alpha^2 + K^2)} + \frac{\alpha^2(\alpha^2 + K^2) + 2\Delta K^2 - 2\Delta\alpha^2}{2k_i^2\,\alpha(\alpha^2 + K^2)^2} \tag{2.38a}$$

and

$$\mathrm{Im}I(\alpha) \simeq \frac{1}{k_i(\alpha^2 + K^2)} \ln\frac{k_i(\alpha^2 + K^2)}{\Delta\alpha}, \tag{2.38b}$$

where we have assumed $\alpha \gtrsim 1$. For the case $\alpha = 0$, Eqs. (2.37a) and (2.37b) are straightforward to evaluate. One finds

$$\mathrm{Re}I(0) = \frac{\pi}{K(p_i^2 K^2 + 4\Delta^2)^{1/2}} \tag{2.39a}$$

and

$$\mathrm{Im}I(0) = \frac{2}{K(p_i^2 K^2 + 4\Delta^2)^{1/2}} \ln\frac{p_i K + (p_i^2 K^2 + 4\Delta^2)^{1/2}}{2\Delta\alpha}. \tag{2.39b}$$

Using Eqs. (2.38a), (2.38b), (2.39a), and (2.39b) in Eq. (2.36), we find in the limit of large k_i

$$\mathrm{Re}\bar{f}^{Z^*}_{SB2} = 2\pi \frac{K^2 + 8Z^{*2}}{(K^2 + 4Z^{*2})^2}\left(\frac{1}{k_i} - \frac{K}{(k_i^2 K^2 + 4\Delta^2)^{1/2}}\right) + \frac{4Z^{*2} + 6\Delta}{2Z^* k_i^2 (K^2 + 4Z^{*2})} + \frac{16Z^{*3}}{k_i^2 (K^2 + 4Z^{*2})^2} - \frac{128\Delta Z^{*3}}{k_i^2 (K^2 + 4Z^{*2})^3} \tag{2.40a}$$

and

$$\mathrm{Im}\bar{f}^{Z^*}_{SB2} = 4\frac{K^2 + 8Z^{*2}}{(K^2 + 4Z^{*2})^2}\left(\frac{1}{k_i}\ln\frac{k_i(K^2 + 4Z^{*2})}{2\Delta Z^*} - \frac{K}{(k_i^2 K^2 + 4\Delta^2)^{1/2}}\ln\frac{k_i K + (k_i^2 K^2 + 4\Delta^2)^{1/2}}{2\Delta}\right) + \frac{2(K^2 - 4Z^{*2})}{k_i (K^2 + 4Z^{*2})^2}. \tag{2.40b}$$

We notice in Eqs. (2.40a) and (2.40b) that there are two distinct angular regions of interest. When $K \lesssim 2\Delta/k_i$, i.e., when $\theta \lesssim 2\Delta/k_i^2$, $\mathrm{Im}\bar{f}^{Z^*}_{SB2}$ varies with energy like $k_i^{-1}\ln k_i$, and $\mathrm{Re}\bar{f}^{Z^*}_{SB2}$ varies like k_i^{-1}. For scattering angles $\theta \gtrsim 2\Delta/k_i^2$, $\mathrm{Im}\bar{f}^{Z^*}_{SB2}$ varies with energy like k_i^{-1} and $\mathrm{Re}\bar{f}^{Z^*}_{SB2}$ varies like k_i^{-2}. The leading corrections to Eqs. (2.40a) and (2.40b) are of order k_i^{-3} and $k_i^{-3}\ln k_i$, respectively.

We may remark in passing that the zero-angle results

$$\mathrm{Re}\bar{f}^{Z^*}_{SB2}(\theta = 0) = \pi/k_i Z^{*2} \tag{2.41a}$$

and

$$\mathrm{Im}\bar{f}^{Z^*}_{SB2}(\theta = 0) = (2/k_i Z^{*2})\ln k_i \tag{2.41b}$$

satisfy a forward dispersion relation,[18]

$$\mathrm{Re}\bar{f}^{Z^*}_{SB2}(\theta = 0, E) = \frac{1}{\pi} P \int_0^\infty \frac{\mathrm{Im}\bar{f}^{Z^*}_{SB2}(\theta = 0, E')}{E' - E} dE', \tag{2.42}$$

where $E = \frac{1}{2}k_i^2$, as may be verified by direct integration.

D. Eikonal Series

In order to facilitate the comparisons with experiment which we shall present in Sec. II E, we shall discuss in some detail the eikonal series for the elastic scattering of electrons by hydrogen and helium. Our starting point is the quantity $\bar{f}_{Gn}$, defined by Eq. (2.25). Let us write this equation in more detail for these two special cases.

For hydrogen, we have, using Eq. (2.23),

$$\bar{f}_{Gn} = \frac{1}{n!}\left(\frac{i}{k_i}\right)^{n-1} \int J_0(Kb_0)|\phi^{\mathrm{H}}_{1s}(b_1, z_1)|^2 \times \ln^n\left(1 - 2\frac{b_1}{b_0}\cos\phi_1 + \frac{b_1^2}{b_0^2}\right) b_0 db_0 b_1 db_1 d\phi_1 dz_1, \tag{2.43}$$

where ϕ^{H}_{1s} is given by Eq. (2.34) with $Z^* = 1$. If we define the quantity $A^{\mathrm{H}}_n(b_0)$ by

$$A_n^{\mathrm{H}}(b_0) = \int |\phi_{1s}^{\mathrm{H}}(b_1, z_1)|^2 \times \ln^n\left(1 - 2\frac{b_1}{b_0}\cos\phi_1 + \frac{b_1^2}{b_0^2}\right) b_1 db_1 d\phi_1 dz_1 , \tag{2.44}$$

then Eq. (2.43) can be written simply as

$$\bar{f}_{Gn} = \frac{1}{n!}\left(\frac{i}{k_i}\right)^{n-1} \int J_0(Kb_0) A_n^{\mathrm{H}}(b_0)\, b_0 db_0 . \tag{2.45}$$

For helium, the situation is qualitatively very similar to that of hydrogen. If we use a product of Hartree-Fock orbitals $\phi_{1s}^{\mathrm{He}}(\vec{r}_1)$ and $\phi_{1s}^{\mathrm{He}}(\vec{r}_2)$ as given by Eq. (2.35) to represent the ground state of helium, then it is straightforward to show that for helium

$$\bar{f}_{Gn} = \frac{1}{n!}\left(\frac{i}{k_i}\right)^{n-1} \int J_0(Kb_0) \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} \times A_{n-m}^{\mathrm{He}}(b_0) A_m^{\mathrm{He}}(b_0)\, b_0 db_0 , \tag{2.46}$$

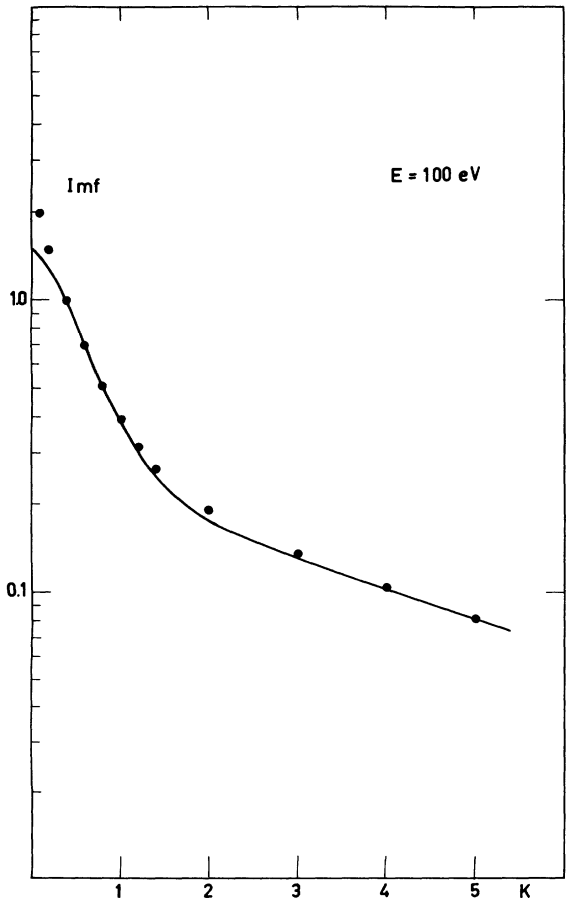

FIG. 1. Solid curve shows $\mathrm{Im}\bar{f}_{\mathrm{SB2}}^{Z^*}$ [Eq. (2.30)] for $Z^* = 1$ as a function of momentum transfer K. Dots are the values of $\mathrm{Im}\bar{f}_{G2}$ for electron-hydrogen elastic scattering.

where

$$A_n^{\mathrm{He}}(b_0) = \int |\varphi_{1s}^{\mathrm{He}}(b_1, z_1)|^2 \times \ln^n\left(1 - 2\frac{b_1}{b_0}\cos\varphi_1 + \frac{b_1^2}{b_0^2}\right) b_1 db_1 d\varphi_1 dz_1 . \tag{2.47}$$

Apart from the choice of orbital, Eqs. (2.44) and (2.47) are identical. The terms in Eq. (2.46) which differ from the hydrogenic result of Eq. (2.45) have a simple physical origin. A disconnected term like $A_n^{\mathrm{He}}(b_0)$ [note that $A_0^{\mathrm{He}}(b_0) = 1$] arises from n successive interactions of the projectile with a particular electron in the target; it is clearly completely equivalent to $A_n^{\mathrm{H}}(b_0)$, apart from the choice of orbital. A connected term like $A_{n-m}^{\mathrm{He}}(b_0) \times A_m^{\mathrm{He}}(b_0)$ results from a situation in which the projectile interacts m times with one helium electron and $n-m$ times with the other helium electron. The coefficient in front of the term $A_{n-m}^{\mathrm{He}}(b_0) A_m^{\mathrm{He}}(b_0)$ corresponds to the number of possible orderings of the interactions. These terms are obviously not present in the case of hydrogen; in helium, however, they are very important.

The functions $A_n^{\mathrm{H}}(b_0)$ and $A_n^{\mathrm{He}}(b_0)$ are the key quantities in the evaluation of the eikonal series. For any orbital of the form $r_1^l e^{-\alpha r_1}$ in Eq. (2.44) or Eq. (2.47), the z_1 integration can be done analytically in terms of Bessel functions of the second kind, so Eqs. (2.74) and (2.47) involve only a two-dimensional integral which is straightforward to evaluate numerically. The $A_n(b_0)$ have characteristic rates of decrease for large values of b_0; $A_1(b_0)$ decreases exponentially with increasing b_0, whereas all higher $A_n(b_0)$ decrease like an inverse power of b_0 as b_0 becomes large:

$$\begin{aligned} A_n(b_0) &\to a_n/b_0^n \quad (n \text{ even},\ n>0) \\ &\to a_n/b_0^{n+1} \quad (n \text{ odd},\ n>1). \end{aligned} \tag{2.48}$$

The very slow falloff of $A_2(b_0)$ gives rise to a divergence of $\bar{f}_{G2}$ at $K=0$, as may be seen from Eq. (2.45). This corresponds to the fact that for $K=0$ $\bar{f}_{B2}$ diverges logarithmically as Δ, the average energy of excitation in intermediate states, tends to zero. This is apparent from Eq. (2.40b). Such a result is expected, since the derivation of the many-body Glauber result of Eq. (2.15) assumes essentially that $\Delta = 0$. Note, however, that the higher $A_n(b_0)$ ($n \geq 3$) are such that the integrals in Eqs. (2.45) and (2.46) converge at the upper limit of integration, even when $K=0$.

We show in Fig. 1 a comparison of $\mathrm{Im}\bar{f}_{\mathrm{SB2}}^{Z^*}|_{Z^*=1}$ with $\mathrm{Im}\bar{f}_{G2}$ for the case of elastic electron scattering in hydrogen. We see that for very small momentum transfers there is a substantial dif-

ference between $\bar{f}_{SB2}$ and $\bar{f}_{G2}$, but otherwise the two agree very well, even at momentum transfers corresponding to $\theta = 180°$.[28] This remarkable result has also been noted by the authors in potential scattering.[24] Since the forward divergence occurs in $\bar{f}_{G2}$ only because of an insufficiently rapid falloff of $A_2(b_0)$, it is likely that the results for $\bar{f}_{G3}$, $\bar{f}_{G4}$, etc., will agree with the corresponding Born terms $\mathrm{Re}\bar{f}_{B3}$, $\mathrm{Im}f_{B4}$, etc., at least for angles satisfying the usual eikonal criterion[1] $\theta \lesssim (2/k_i)^{1/2}$. In fact, the range of validity may be much greater. The authors have found in a study of the eikonal method in potential scattering that for many types of potential $\mathrm{Re}\bar{f}_{E3} = \mathrm{Re}\bar{f}_{B3}$, $\mathrm{Im}\bar{f}_{E4} = \mathrm{Im}\bar{f}_{B4}$, etc., in the high momentum limit for *all* scattering angles.

In concluding this section, we should note that in order to include the leading $1/k_i^2$ corrections to the first Born differential cross section, one needs $\mathrm{Re}\bar{f}_{B3}$ in addition to $\bar{f}_{B1}$ and $\bar{f}_{B2}$ because $\mathrm{Re}\bar{f}_{B3}$ is of order $1/k_i^2$ and will give contributions of this order to the differential cross section. Since $\bar{f}_{B3}$ is formidably difficult to evaluate, on the basis of our foregoing discussion we shall use $\mathrm{Re}\bar{f}_{G3}$ in place of $\mathrm{Re}\bar{f}_{B3}$. The reason that we use $\bar{f}_{B2}$ instead of the simpler $\bar{f}_{G2}$ is also obvious from our analysis. In addition to the unphysical divergence of $\mathrm{Im}\bar{f}_{G2}$ at very small angles, $\bar{f}_{B2}$ contains a real part which is completely missing from $\bar{f}_{G2}$. This term is numerically very important, particularly at small angles where, as we have seen above, $\mathrm{Re}\bar{f}_{B2}$ is of order $1/k_i$ and hence gives the dominant correction to the first Born differential cross section.

E. Exchange Corrections

As we have seen above, if we wish to make consistently the leading corrections to the first Born amplitude, we must include all terms of order k_i^{-2}. This leads to the conclusion that the leading term in the exchange amplitude must also be included, since it is well known to be of order k_i^{-2}. Fortunately, higher exchange terms may be neglected at this level of accuracy, although they would become important if we were to go beyond order k_i^{-2}. The first Born exchange amplitude is given in the case of electron-hydrogen elastic scattering by

$$\bar{g}_{B1}^{\mathrm{H}} = -\frac{1}{2\pi}\int e^{-i\vec{k}_f\cdot\vec{r}_1}\phi_{1s}^{\mathrm{H}}(\vec{r}_0)\left(\frac{1}{r_{01}}-\frac{1}{r_0}\right) \times e^{i\vec{k}_i\cdot\vec{r}_0}\phi_{1s}^{\mathrm{H}}(\vec{r}_1)\,d\vec{r}_0\,d\vec{r}_1\,, \tag{2.49}$$

and by

$$\bar{g}_{B1}^{\mathrm{He}} = -\frac{1}{2\pi}\int e^{-i\vec{k}_f\cdot\vec{r}_1}\phi_{1s}^{\mathrm{He}}(\vec{r}_0)\phi_{1s}^{\mathrm{He}}(\vec{r}_2)\left(\frac{1}{r_{01}}+\frac{1}{r_{02}}-\frac{2}{r_0}\right) \times e^{i\vec{k}_i\cdot\vec{r}_0}\phi_{1s}^{\mathrm{He}}(\vec{r}_1)\,\phi_{1s}^{\mathrm{He}}(\vec{r}_2)\,d\vec{r}_0\,d\vec{r}_1\,d\vec{r}_2 \tag{2.50}$$

in the case of helium, where we use Hartree-Fock orbitals. We should remark that although we have written the exchange amplitude in "prior" form, the same results for $\bar{g}_{B1}^{\mathrm{He}}$ are obtained if one uses the "post" form.

In order to have a consistent expression for the exchange amplitude it is necessary to look in somewhat more detail at Eqs. (2.49) and (2.50). The dominant contribution to both of these equations comes from the $1/r_{01}$ term in the interaction potential; this is sensible, since it represents the interaction between the incident and ejected electrons. This dominant correction falls off like k_i^{-2} for large k_i. The contribution from $1/r_0$ falls off like k_2^{-6}, as is readily seen by inspection of Eqs. (2.49) and (2.50). Consistent with our present level of approximation we may neglect this term. In fact, there are more compelling reasons for neglecting it; one can show that the next term in the Born series will contain a part which will cancel this term, leaving a contribution of still higher order in k_i. Similarly, the contribution to $\bar{g}_{B1}^{\mathrm{He}}$ from the term $1/r_{02}$ in Eq. (2.50) will be neglected, since its value at the energies of interest is very small, being only of order k_i^{-10}. Thus, Eqs. (2.49) and (2.50) simplify, respectively, to

$$\bar{g}_{B1}^{\mathrm{H}} \simeq -\frac{1}{2\pi}\int e^{-i\vec{k}_f\cdot\vec{r}_1}\phi_{1s}^{\mathrm{H}}(\vec{r}_0)\frac{1}{r_{01}} \times e^{-\vec{k}_i\cdot\vec{r}_0}\phi_{1s}^{\mathrm{H}}(\vec{r}_1)\,d\vec{r}_0\,d\vec{r}_1\,, \tag{2.51}$$

$$\bar{g}_{B1}^{\mathrm{He}} \simeq -\frac{1}{2\pi}\int e^{-i\vec{k}_f\cdot\vec{r}_1}\phi_{1s}^{\mathrm{He}}(\vec{r}_0)\frac{1}{r_{01}} \times e^{i\vec{k}_i\cdot\vec{r}_0}\phi_{1s}^{\mathrm{He}}(\vec{r}_1)\,d\vec{r}_0\,d\vec{r}_1\,. \tag{2.52}$$

Finally, if we transform these two equations into momentum space and retain only the leading term in the large-k_i expansion, we arrive at the Ochkur approximation[29]

$$\bar{g}_{B1}^{\mathrm{H}} \simeq g_{\mathrm{Och}}^{\mathrm{H}} = -\frac{2}{k_i^2}\int e^{i\vec{K}\cdot\vec{r}}[\phi_{1s}^{\mathrm{H}}(\vec{r})]^2\,d\vec{r}, \tag{2.53}$$

$$\bar{g}_{B1}^{\mathrm{He}} \simeq g_{\mathrm{Och}}^{\mathrm{He}} = -\frac{2}{k_i^2}\int e^{i\vec{K}\cdot\vec{r}}[\phi_{1s}^{\mathrm{He}}(\vec{r})]^2\,d\vec{r}, \tag{2.54}$$

where, as usual, K is the momentum transfer. In Eqs. (2.53) and (2.54) the k_i^{-2} dependence of this leading contribution to the exchange amplitude is explicit. Using Eq. (2.34) with $Z^* = 1$ for ϕ_{1s}^{H} and Eq. (2.35) for ϕ_{1s}^{He}, we have

$$\bar{g}_{B1}^{\mathrm{H}} \simeq g_{\mathrm{Och}}^{\mathrm{H}} = -\frac{32}{k_i^2(K^2+4)^2}\,, \tag{2.55}$$

$$\bar{g}_{B1}^{He} \simeq g_{Och}^{He} = -\frac{8}{k_i^2}\left(\frac{\alpha A^2}{(K^2+4\alpha^2)^2} + \frac{2\gamma AB}{(K^2+4\gamma^2)^2} + \frac{\beta B^2}{(K^2+4\beta^2)^2}\right), \quad (2.56)$$

where $A = 2.60505$, $B = 2.08144$, $\alpha = 1.41$, $\beta = 2.61$, and $\gamma = \alpha + \beta = 2.01$.

With the inclusion of exchange, the full scattering amplitudes for hydrogen will be given by

$$f_{sing} = f + g, \quad f_{trip} = f - g, \quad (2.57)$$

and then

$$\frac{d\sigma}{d\Omega} = \tfrac{1}{4}|f_s|^2 + \tfrac{3}{4}|f_t|^2. \quad (2.58)$$

To our order of approximation, we will use $g \simeq g_{Och}^{H}$. For helium there is just one amplitude, $f - g$, and

$$\frac{d\sigma}{d\Omega} = |f - g|^2, \quad (2.59)$$

where again $g \simeq g_{Och}^{He}$.

III. APPLICATIONS TO ELASTIC SCATTERING BY ATOMIC HYDROGEN AND HELIUM

In Figs. 2–11 we present our results concerning differential cross sections for elastic electron scattering by atomic hydrogen and helium. We include for comparison on each graph the Glauber cross section and the first Born cross section, in addition to the existing experimental results. In most of these figures we have multiplied the various differential cross sections by $\sin\theta$, so that the curves will be more readable. Thus, the area under the curves in such figures is directly proportional to the total cross section for elastic scattering, apart from contributions from larger angles. Our eikonal-Born-series results are obtained by writing the direct amplitude f_d as

$$\mathrm{Re}\, f_d = \bar{f}_{B1} + \mathrm{Re}\,\bar{f}_{B2} + \mathrm{Re}\,\bar{f}_{G3}, \quad (3.1a)$$

$$\mathrm{Im}\, f_d = \mathrm{Im}\,\bar{f}_{B2}. \quad (3.1b)$$

As discussed in Sec. II E, we include for consistency the Ochkur approximation to the exchange amplitude.

Figures 2 and 3 show elastic electron-hydrogen scattering at 100 and 200 eV, respectively. The most striking aspect of these results is that at a rather high energy such as 100 eV, the difference between the Glauber and the eikonal-Born-series results is very large—50% or greater at all angles shown. Even at 200 eV, the difference between the two is always greater than 25% over the range shown, which clearly comprises most of the contribution to elastic scattering. These differences are due primarily to the term $\mathrm{Re}\,\bar{f}_{B2}$, which is completely missing from the Glauber amplitude. At small angles, this term actually gives the leading correction to the differential cross sections, and at larger angles it gives a correction equal in importance to $\mathrm{Re}\,\bar{f}_{G3}$ and $\mathrm{Im}\,\bar{f}_{B2}$.

It is interesting that in both cases at angles greater than 20° the first Born approximation is closer to the eikonal-Born-series result than is the eikonal result, whereas the eikonal result is closer at small angles. If one performs the integral over all angles, one finds that at these energies the total cross sections given by the first

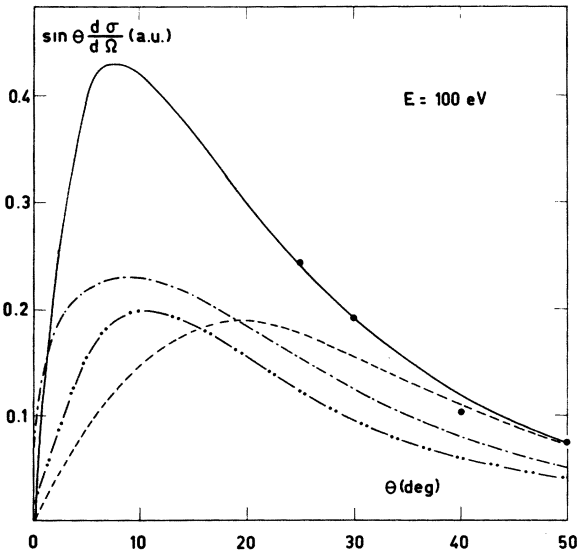

FIG. 2. Quantity $\sin\theta\, d\sigma/d\Omega$ (in a.u.) vs θ for elastic electron scattering by atomic hydrogen at 100 eV. Solid curve gives eikonal-Born-series result, dashed curve gives first Born approximation, dash-dotted curve gives Glauber approximation, and dashed-double-dotted curve refers to our eikonal-Born-series results for positron scattering. Circles are experimental results of Ref. 23, normalized to Born-series result at 30°.

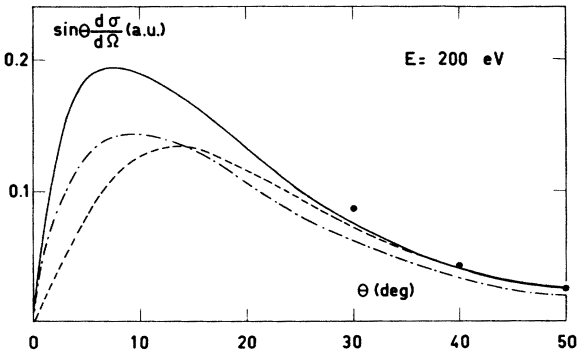

FIG. 3. Curves are same as in Fig. 2, but at 200 eV. Circles are experimental results of Ref. 23, normalized to Born-series result at 40°.

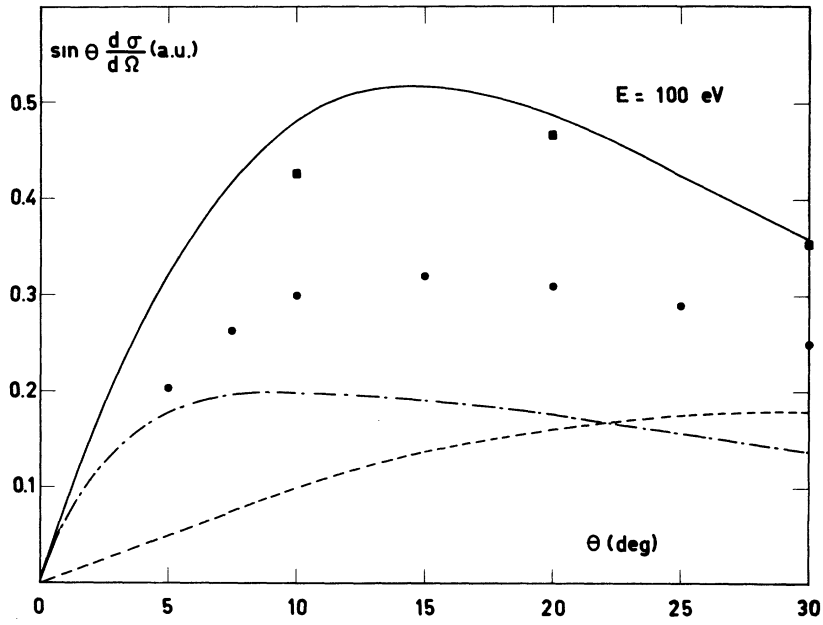

FIG. 4. Quantity $\sin\theta\, d\sigma/d\Omega$ (in a.u.) vs θ for elastic electron scattering for helium at 100 eV. Solid curve gives eikonal-Born-series results, dashed curve gives first Born approximation, and dash-dotted curve refers to Glauber approximation. Circles are experimental results of Ref. 19 as normalized by Ref. 21. Squares are experimental results of Ref. 22.

Born and Glauber results are very nearly equal. From such an agreement, one might be tempted to conclude that the first Born (or Glauber) approximation gives a good account of elastic scattering at energies above 100 eV. We see that this is emphatically not the case. The total elastic cross section given by our Born-series result will be much greater than the first Born (or Glauber) result at both 100 and 200 eV.

The experimental results of Teubner *et al.*[23] are also shown in Figs. 2 and 3. Since these are not absolute measurements, we have normalized them to our eikonal-Born-series results at a single point. We see that the agreement is good (well within the experimental uncertainty), although the agreement with the Glauber result would be equally good if we had normalized the experimental points differently.[5] This results from the fact that over the angular range in which we have experimental data, the eikonal-Born-series differential cross section and the Glauber differential cross section differ by a very nearly constant factor. It is clear that measurements at smaller angles would be necessary in order to distinguish between the two theories. Finally, we also note from Fig. 2 the very large difference between electron scattering and positron scattering in our theory. These differences are missing from the first Born and Glauber approximation predictions.

Now let us turn to elastic electron-helium scattering. Figures 4–11 show the results that we have obtained in the range of incident energies

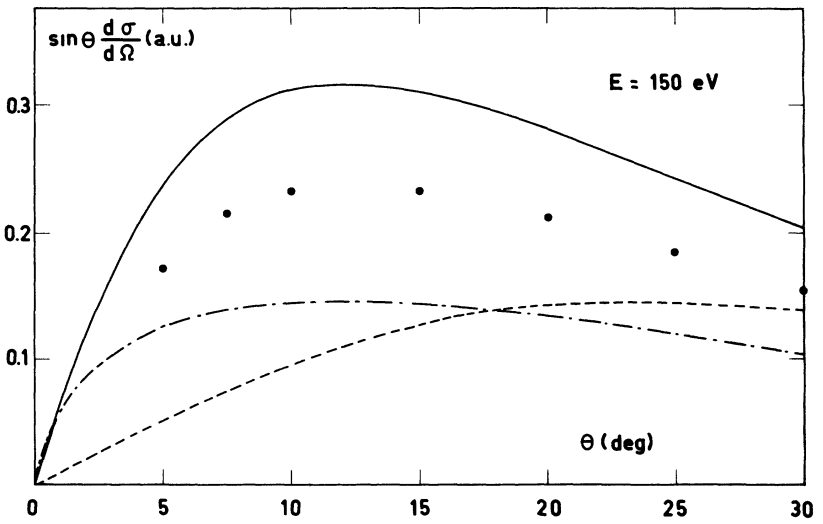

FIG. 5. Same as Fig. 4, but at 150 eV. There are no experimental values from Ref. 22 at this energy.

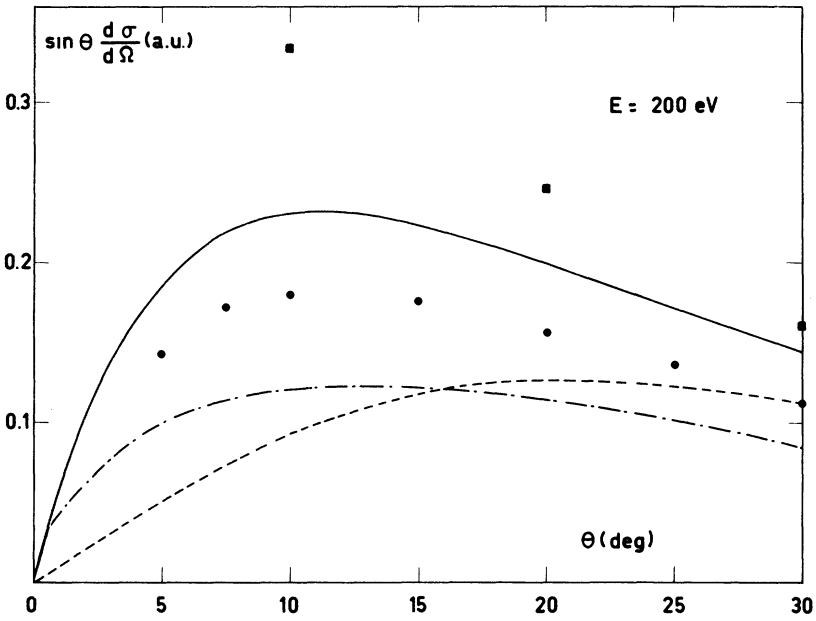

FIG. 6. Same as Fig. 4, but at 200 eV.

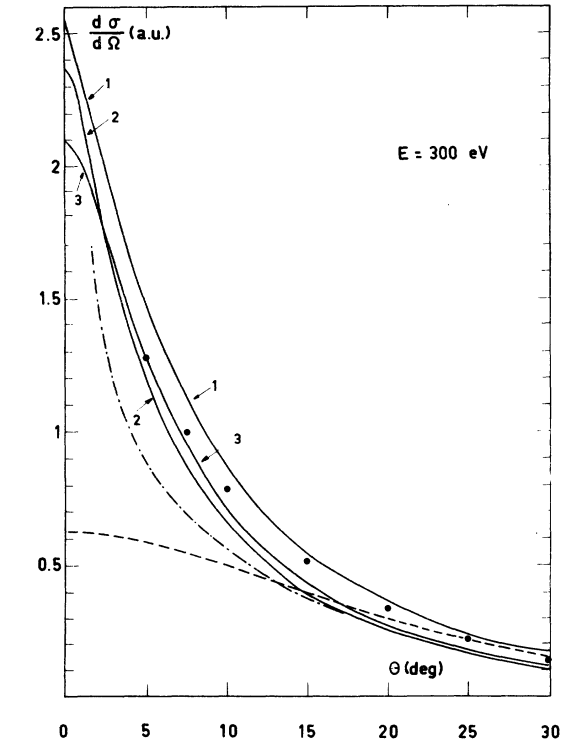

FIG. 7. Differential cross section (in a.u.) for elastic electron scattering by helium at 300 eV. Curve 1: the eikonal-Born-series results; curve 2: modified close-coupling calculations of Ref. 16; curve 3: eikonal optical-model results of Ref. 13. Dashed curve gives first Born approximation and dash-dotted curve refers to Glauber approximation. Circles are experimental data of Ref. 19, as normalized by Ref. 21.

100–500 eV. The comparison between the eikonal-Born-series and Glauber differential cross sections[30] is qualitatively the same as for electron-hydrogen scattering, although at the lowest energies in question the discrepancy between them is even larger than in the hydrogen case. The existing experimental results are also given in Figs. 4–11. At energies of 100, 150, 200, 300, and 400 eV (Figs. 4–9), the circles denote the results of Vriens *et al.*[19] as normalized by the measurements of Chamberlain *et al.*[21] at 5°. At energies of 100, 200, and 400 eV (Figs. 4, 6, and 9), the squares denote the results of Crooks and Rudd.[22] In Figs. 10 and 11, the circles denote the experimental values obtained by Bromberg[20] at 500 eV. All of these helium measurements are absolute, so we do not have the freedom to normalize experimental points to theoretical curves.

It is interesting to note that in cases where two sets of experimental data are available, our eikonal-Born-series results lie between the two, except at 100 eV. In every case, the Glauber results fall below all experimental data. It is difficult to assess the accuracy of such experiments because of the ever-present possibility of systematic errors. The statistical error quoted by Chamberlain *et al.*[21] is about 5%; the results of Refs. 19 and 21 tend to fall below our theoretical Born-series curves by about 10–20% at all energies in question. At the lowest energies, such a discrepancy could be explained by higher terms missing from our Born-series result, but at the highest energies we feel that this is unlikely for reasons that will be discussed below. Similarly, at energies where we have the experimental data

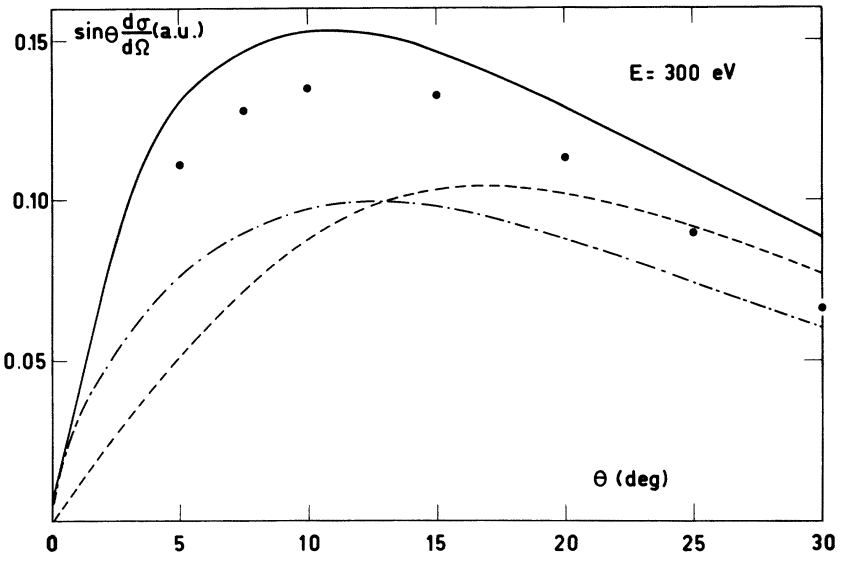

FIG. 8. Same as Fig. 4, but at 300 eV. There are no experimental values from Ref. 22 at this energy.

of Crooks and Rudd,[22] our eikonal-Born-series curve lies systematically below their data points, except at 100 eV. Clearly, more experimental work is necessary before the situation at energies below 400 eV can be discussed definitively.

Before we discuss the situation at 500 eV, let us comment briefly on Fig. 7, where we compare our calculations (curve 1) not only with the first Born and Glauber results, but also with some other recent theoretical calculations. Curve 2 shows the results obtained by Berrington, Bransden, and Coleman[16] by using a modification of the close-coupling method which allows for the static coupling between the $1\,^1S$, $2\,^1S$, and $2\,^1P$ helium target states, together with a second-order pseudopotential in the elastic channel. Curve 3 corresponds to the eikonal optical-model results of Joachain and Mittleman,[13] normalized at $\theta = 5°$ to the experimental results.[19,21] It is worth pointing out that our curve lies consistently *above* the experimental results of Refs. 19 and 21, but *below* those obtained by interpolating between the experimental data of Crooks and Rudd[22] at 200 and 400 eV. Not shown in Fig. 7 are the results of second Born calculations of Holt *et al.*[14] and of Woollings and McDowell,[18] which agree more poorly with experiment, particularly at larger angles.

Let us now examine in detail the results at 500 eV. In Fig. 10 the small-angle behavior is emphasized, while the situation at larger angles is more clearly illustrated by Fig. 11. We note that the experimental data of Bromberg[20] agree extraordinarily well with our eikonal-Born-series results. Bromberg quotes a statistical error of about 3%; our results agree with his at all points to within this error. On the basis of the analysis

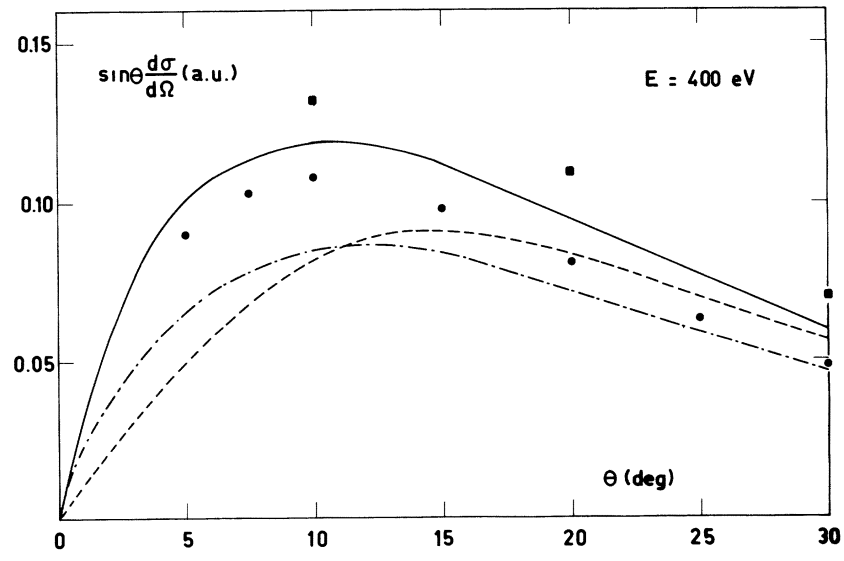

FIG. 9. Same as Fig. 4, but at 400 eV.

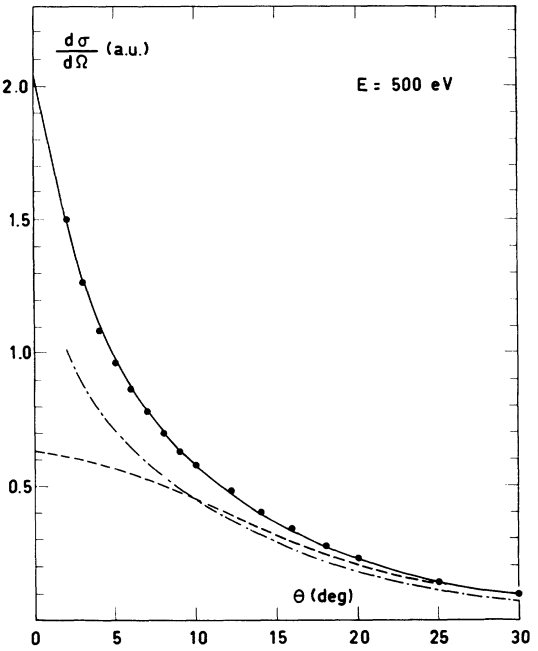

FIG. 10. Differential cross section (in a.u.) for elastic electron scattering from helium at 500 eV. Solid curve gives our eikonal-Born-series result, dashed curve gives first Born approximation, and dash-dotted curve refers to Glauber approximation. Circles are experimental data of Ref. 20.

given below, we feel that our Born-series results might be off by perhaps as much as 5% at 500 eV, although this probably is a bit conservative. It is interesting to note that even at this high energy our results are significantly sensitive to each term included. The Glauber result shown in Fig. 11 shows that the omission of $\mathrm{Re}\bar{f}_{B2}$ and $\bar{g}_{B1}$ leads to an error of about 20%. If we had included $\bar{f}_{B2}$ but omitted $\mathrm{Re}\bar{f}_{G3}$, the Glauber contribution, then our results would have been too large by 10–20%. Finally, had we neglected $\bar{g}_{B1}$, our results would have been too small by about 8–10%.

In order to give a more precise picture of the importance of the various terms contributing to the Born series, we show in Tables I and II the contributions to the real and imaginary parts of the scattering amplitude for $E = 300$ eV. Table I gives the terms appropriate to the real part of the amplitude. Notice how both $\mathrm{Re}\bar{f}_{B2}$ and $\mathrm{Re}\bar{f}_{G3}$ play important roles in correcting the first Born approximation. The term $\mathrm{Re}\bar{f}_{G5}$ gives an idea of the order of magnitude of possible higher-order corrections to $\mathrm{Re}f_d$; we see that its effect will be rather small at 300 eV, although as one goes to lower energies (remember that $\mathrm{Re}\bar{f}_{G3}$ and $\mathrm{Re}\bar{f}_{G5}$ scale with energy as E^{-1} and E^{-2}, respectively) these higher corrections become substantial. Note that the Ochkur approximation to the exchange amplitude (which is purely real) is by no means insignificant at these energies and angles.

In Table II similar results are given for the contributions to $\mathrm{Im}f_d$. Also, $\mathrm{Im}\bar{f}_{G2}$ is included for comparison with $\mathrm{Im}\bar{f}_{B2}$; we see that the two agree rather well. It is clear that the term $\mathrm{Im}\bar{f}_{G4}$ is a fairly substantial fraction of $\mathrm{Im}\bar{f}_{B2}$, so we may expect that terms coming from higher orders could play a substantial role in correcting $\mathrm{Im}f_d$. We should emphasize that in the context of this work it would not be consistent to include $\mathrm{Im}\bar{f}_{G4}$ in $\mathrm{Im}f_d$ because we expect that the term $\mathrm{Im}\bar{f}_{B3}$, which is entirely missing from the Glauber series, will be of the same order of magnitude as $\mathrm{Im}\bar{f}_{G4}$. This is why it was important to obtain $\mathrm{Re}\bar{f}_{B2}$ to correct $\mathrm{Re}\bar{f}_{G3}$, so that a consistent result through order k_i^{-2} could be obtained. In a similar way, the missing term $\mathrm{Re}\bar{f}_{B4}$ would presumably make a significant correction to the term $\mathrm{Re}\bar{f}_{G5}$ shown in Table I. If one were to throw caution to the winds and ask what would happen to the differential cross section if we added $\mathrm{Im}\bar{f}_{G4}$ and $\mathrm{Re}\bar{f}_{G5}$ to f_d, the answer is that at 300 eV this would change the differential cross section by no more than 3% at any angle. Even at 100 eV, such a change would have no more than a 5% effect on the differential cross section.

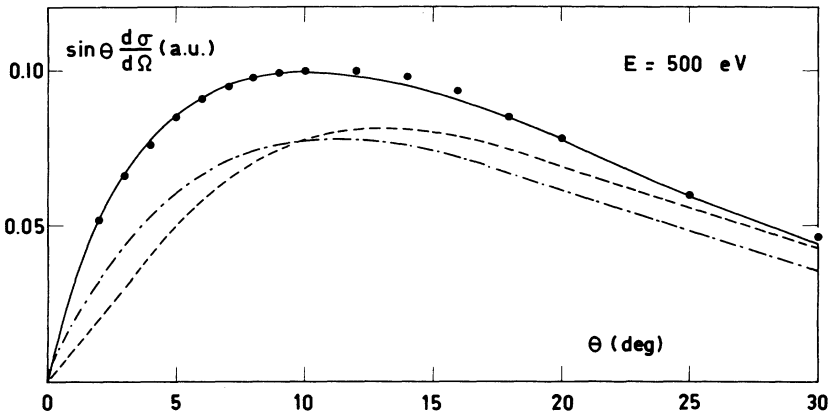

FIG. 11. Curves are same as Fig. 4, but at 500 eV. Circles are experimental data of Ref. 20.

TABLE I. Real part of the scattering amplitude (in a.u.) for electron-helium elastic scattering at an incident electron energy of 300 eV. The quantity $\mathrm{Re}f_d = \bar{f}_{B1} + \mathrm{Re}\bar{f}_{B2} + \mathrm{Re}\bar{f}_{G3}$. The average excitation energy used in calculating $\mathrm{Re}\bar{f}_{B2}$ is $\Delta = 1.3$ a.u.

θ (deg)	$\bar{f}_{B1}=\bar{f}_{G1}$	$\mathrm{Re}\bar{f}_{B2}$	$\mathrm{Re}\bar{f}_{G3}$	$\mathrm{Re}\bar{f}_{G5}$	$g^{\mathrm{He}}_{\mathrm{Och}}$	$\mathrm{Re}f_d$
0	0.792	0.586	−0.078	0.016	−0.091	1.300
5	0.770	0.259	−0.086	0.016	−0.088	0.943
10	0.712	0.124	−0.096	0.016	−0.080	0.740
15	0.632	0.078	−0.102	0.017	−0.069	0.608
20	0.546	0.060	−0.104	0.017	−0.058	0.502
25	0.465	0.052	−0.102	0.017	−0.047	0.415
30	0.393	0.046	−0.098	0.017	−0.038	0.341

TABLE II. Imaginary part of the scattering amplitude (in a.u.) for electron-helium elastic scattering at an incident electron energy of 300 eV. The average excitation energy used in calculating $\mathrm{Im}\bar{f}_{B2}$ is $\Delta = 1.3$ a.u.

θ (deg)	$\mathrm{Im}f_d = \mathrm{Im}\bar{f}_{B2}$	$\mathrm{Im}\bar{f}_{G2}$	$\mathrm{Im}\bar{f}_{G4}$
0	0.800	∞	−0.057
5	0.661	0.682	−0.055
10	0.463	0.448	−0.051
15	0.332	0.329	−0.047
20	0.255	0.260	−0.044
25	0.210	0.218	−0.042
30	0.182	0.191	−0.040

Let us now discuss possible sources of error in our results for f_d. The question of wave-function accuracy is always a difficult one in many-electron systems. We have used a Hartree-Fock wave function for the ground state of helium given by Eq. (2.35). Table III shows a comparison of first Born differential cross sections calculated using the wave function of Eq. (2.35) with those obtained[31] by using a correlated 20-term Hylleraas wave function.[32] The difference between the two is most severe at 0°, but even here it never exceeds 1%. We feel that wave-function errors of this magnitude are also present in our calculation of $\bar{f}_{B2}$.

Also, the choice of Δ will effect f_d, as may be seen by comparing the values of $\mathrm{Im}\bar{f}_{B2}$ and $\mathrm{Im}\bar{f}_{G2}$ in Table II. Since the Glauber method uses $\Delta = 0$, this is a severe comparison, but even here the magnitude of the discrepancy is never more than about 5%, except, of course, at 0°. At very small angles, $\mathrm{Im}\bar{f}_{B2}$ depends logarithmically on Δ [see Eq. (2.40a)], while at large angles it is strongly dominated by the intermediate state $|\mu\rangle = |0\rangle$, which we include exactly. For $\mathrm{Re}\bar{f}_{B2}$ at small angles there is almost no dependence on Δ [see Eq. (2.40b)], and again at large angles the intermediate state $|\mu\rangle = |0\rangle$ dominates. The value of Δ actually used, $\Delta = 1.3$ a.u., was chosen to make $(4\pi/k)\,\mathrm{Im}\bar{f}_{B2}(\theta = 0)$ agree with the result of Inokuti *et al.*[33] for the Bethe-Born value of the *total* cross section in electron-helium scattering at 500 eV. We feel that it is unlikely that the errors associated with the choice of Δ exceed a few percent of the differential cross section.

TABLE III. Comparison of first Born differential cross sections for elastic electron-helium scattering using a Hartree-Fock wave function and a correlated wave function of the Hylleraas type.

		First Born differential cross section	
E (eV)	θ (deg)	BJ Hartree-Fock wave function[a]	Correlated 20-term Hylleraas wave function[b]
100	0	0.627	0.634
	10	0.584	0.589
	20	0.476	0.480
	30	0.354	0.356
200	0	0.627	0.634
	10	0.543	0.548
	20	0.372	0.375
	30	0.225	0.226
400	0	0.627	0.634
	10	0.475	0.479
	20	0.244	0.246
	30	0.112	0.112

[a] Byron and Joachain (Ref. 27).

[b] The results quoted in this column are obtained by using the values of the atomic form factor of the helium atom calculated by Kim and Inokuti (Ref. 31) using the 20-term Hylleraas wave function of Hart and Herzberg (Ref. 32).

TABLE IV. Comparison of Glauber and Schwinger-variational-principle scattering amplitudes for electron-helium elastic scattering at 500 eV.

θ (deg)	$\mathrm{Re}\bar{f}_{G3}$	$\mathrm{Re}\bar{f}_{S3}$	$\mathrm{Im}\bar{f}_{S3}$	$\mathrm{Im}\bar{f}_{G4}$	$\mathrm{Im}\bar{f}_{S4}$	$\mathrm{Re}\bar{f}_{S4}$	$\mathrm{Re}\bar{f}_{G5}$
5	−0.053	−0.283	0.146	−0.025	−0.156	−0.135	0.006
10	−0.060	−0.126	0.045	−0.023	−0.052	−0.029	0.006
15	−0.062	−0.073	0.026	−0.021	−0.025	−0.014	0.006
20	−0.061	−0.059	0.023	−0.019	−0.020	−0.012	0.006
25	−0.057	−0.049	0.020	−0.019	−0.017	−0.011	0.006
30	−0.054	−0.048	0.019	−0.018	−0.019	−0.012	0.006

Finally, we turn to the most serious probable error in our eikonal-Born-series results, namely, the fact that we include only terms through order k_i^{-2} (i.e., through $\mathrm{Re}\bar{f}_{B3}$) in our series. We have remarked earlier that if we take the Glauber results $\mathrm{Im}\bar{f}_{G4}$ and $\mathrm{Re}\bar{f}_{G5}$ to give the corrections through order k_i^{-4}, then even at 100 eV the resulting changes in the differential cross section are only of the order of a few percent. However, having seen the importance of $\mathrm{Re}\bar{f}_{B2}$ in correcting $\mathrm{Re}\bar{f}_{G3}$, we should not be too quick to accept these Glauber values.

Another handle on the higher corrections can be obtained by using the fractional form of the Schwinger variational principle.[25] If we use simple plane waves for the trial functions in the Schwinger principle, then, as is well known, we obtain the approximate amplitude

$$f_S = \bar{f}_{B1}(1 - \bar{f}_{B2}/\bar{f}_{B1})^{-1}. \tag{3.2}$$

In the region in which the Born series is convergent, we may write

$$f_S = \bar{f}_{B1} + \bar{f}_{B2} + \bar{f}_{B2}^2/f_{B1} + \ldots, \tag{3.3}$$

so that, using our familiar notation, we have

$$\bar{f}_{Sn} = (\bar{f}_{B2})^{n-1}/(\bar{f}_{B1})^{n-2} \tag{3.4}$$

for $n \geq 1$. In Table IV we list the values of $\mathrm{Re}\bar{f}_{S3}$, $\mathrm{Im}\bar{f}_{S3}$, $\mathrm{Re}\bar{f}_{S4}$, and $\mathrm{Im}\bar{f}_{S4}$ obtained by using our values for $\bar{f}_{B1}$ and $\bar{f}_{B2}$ at $E = 500$ eV. In addition, we give the values of $\mathrm{Re}\bar{f}_{G3}$ and $\mathrm{Im}\bar{f}_{G4}$ for comparison. It will be seen that the agreement between $\mathrm{Re}\bar{f}_{S3}$ and $\mathrm{Re}\bar{f}_{G3}$ is quite good at angles greater than about 15°, and there is similar agreement between $\mathrm{Im}\bar{f}_{S4}$ and $\mathrm{Im}\bar{f}_{G4}$. The breakdown of the agreement coincides with the onset of the anomalous small-angle behavior of $\bar{f}_{B2}$ owing to the long range of the Coulomb potential. At very small angles, a better trial function in the Schwinger principle is probably called for. It is clear that by adding $\mathrm{Im}\bar{f}_{S3}$ to $\mathrm{Im}\bar{f}_{G4}$ we drastically change the simple Glauber picture of the leading correction to $\mathrm{Im}\bar{f}_{B2}$, since at angles greater than 15° $\mathrm{Im}\bar{f}_{S3}$ nearly cancels $\mathrm{Im}\bar{f}_{G4}$. Similarly, if we add $\mathrm{Re}\bar{f}_{S4}$ to $\mathrm{Re}\bar{f}_{G5}$ to get the k_i^{-4} correction to our result for $\mathrm{Re}f_d$, we see that in the same angular region we obtain a term of different sign than we would have gotten by using just $\mathrm{Re}\bar{f}_{G5}$.

Unfortunately, we cannot rely on this procedure for all angles, a restriction that becomes more important as we move to lower energies, where the angle at which the anomalous "small-angle" behavior sets in becomes larger and larger. For example, at 100 eV the Schwinger results are not particularly good, even at 30°. Thus, the values of $\mathrm{Im}\bar{f}_{G4}$ and $\mathrm{Re}\bar{f}_{G5}$ can probably only serve as estimates of magnitudes of possible errors in our results for f_d, not as realistic higher-order corrections. A glance at Tables I and II shows that the uncertainties arising from higher-order terms (as estimated by $\mathrm{Im}\bar{f}_{G4}$ and $\mathrm{Re}\bar{f}_{G5}$) can be quite considerable at low energies, and possibly as large as 5–10% at the highest energies considered. However, on a more optimistic note, we remark that if we use the Schwinger corrections $\mathrm{Im}\bar{f}_{S3}$ and $\mathrm{Re}\bar{f}_{S4}$ to supplement our values of $\mathrm{Im}\bar{f}_{G4}$ and $\mathrm{Re}\bar{f}_{G5}$, then at 400 eV the resulting change in $d\sigma/d\Omega$ at 20°, 25°, and 30° is of the order of 3–4%. Similarly, at 200 eV and 30°, the corrections to $d\sigma/d\Omega$ obtained in this way are about 8%. Thus, a reasonable assessment concerning percent errors due to omission of higher terms is that such errors might reach perhaps 20% at the lowest energy considered and are probably less than 5% at the highest energy. It is clearly very difficult to make a precise analysis of this problem.

IV. FORWARD DISPERSION RELATIONS

In comparing with experiment in Sec. III, the point $\theta = 0$ is of necessity missing from the experimental results. On the other hand, the amplitudes vary rapidly in the vicinity of $\theta = 0$, so this point is a sensitive test for any theory. However, as is well known, the point $\theta = 0$ is accessible if experiments on total cross sections are combined with dispersion relations via an equation such as

TABLE V. Comparison of the real part of the forward elastic amplitude as given by dispersion relations with the same quantity as calculated in this paper. The energy E is in eV.

E	$\mathrm{Re}[f_d(\theta=0)-g_{\mathrm{Och}}^{\mathrm{He}}(\theta=0)]$	$\mathrm{Re}\bar{f}_{G5}(\theta=0)$	$\mathrm{Re}f(\theta=0)$
100	1.91	0.14	1.91
150	1.67	0.06	1.81
200	1.54	0.04	1.71
300	1.39	0.02	1.48
400	1.30	0.01	1.36
500	1.24	0.01	1.29

$$\mathrm{Re}f(\theta=0,E)=\bar{f}_{B1}(\theta=0,E)-\bar{g}_{B1}(\theta=0,E)+\frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}f(\theta=0,E')}{E'-E}\,dE', \quad (4.1)$$

where $\mathrm{Im}f(\theta=0,E)$ is given by the optical theorem as

$$\mathrm{Im}f(\theta=0,E)=(k/4\pi)\,\sigma_{\mathrm{tot}}(E). \quad (4.2)$$

One must, of course, *assume* that forward dispersion relations are valid for elastic scattering from a composite system with Coulomb forces. No proof of this assumption exists, although it is very plausible.[34] Combining Eqs. (4.1) and (4.2) and using $E=\frac{1}{2}k^2$, we obtain

$$\mathrm{Re}f(\theta=0,k)=\bar{f}_{B1}(\theta=0,k)-\bar{g}_{B1}(\theta=0,k)+\frac{1}{2\pi^2}P\int_0^\infty \frac{q^2\sigma_{\mathrm{tot}}(q)}{q^2-k^2}\,dq. \quad (4.3)$$

Thus, if one has values of $\sigma_{\mathrm{tot}}(q)$ from $q=0$ to $q=\infty$, then $\mathrm{Re}f(\theta=0,k)$ can be obtained for any k.

A calculation for helium has been carried out by Bransden and McDowell[9] using the Bethe-Born values of Kim and Inokuti[31] for σ_{inel} and the Born values of Kennedy[35] for σ_{el}, when $k\geq 4.5$ a.u. ($\sigma_{\mathrm{tot}}=\sigma_{\mathrm{el}}+\sigma_{\mathrm{inel}}$), the elastic-scattering results of Golden and Bandel[36] below $k=1.2$, and a three-term interpolation formula between $k=1.2$ and $k=4.5$. Fortunately, the results for $\mathrm{Re}f(\theta=0,k)$ do not depend very sensitively on the region where the interpolation is done, at least for energies between 100 and 500 eV. Bransden and McDowell[9] estimate that their results for $\mathrm{Re}f(\theta=0,k)$ in this region should be accurate to about 10–15%.

In Table V we show our results at various energies and compare with the dispersion-theory results of Bransden and McDowell.[9] In column 1 we list $\mathrm{Re}[f_d(\theta=0)-g_{\mathrm{Och}}^{\mathrm{He}}(\theta=0)]$; in column 2 we show $\mathrm{Re}\bar{f}_{G5}(\theta=0,k)$ to give an idea of the order of magnitude of corrections to be expected from higher-order Born terms; and in column 3 we give $f(\theta=0,k)$, as determined by Bransden and McDowell. If we take 10% as a reasonable estimate of the uncertainty in their results, we see that at all energies we are in agreement with dispersion relations, although we do seem to fall systematically on the low side of these calculations. It would obviously be desirable to have reliable measurements of total cross sections in helium to see if this discrepancy persists when more precise dispersion-relation results, coming from improved values of σ_{tot} between the first inelastic threshold and the Born region, are obtained.[37]

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