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Correct No-Retardation Limit in Bethe-Salpeter Kernels*

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A careful discussion of the no-retardation (instantaneous) limit of the Bethe-Salpeter (BS) equation is carried out, as illustrated by one-photon exchange in the radiation gauge for positronium. Comparison is made with equations obtained by other authors, using the effective potential to order $m\alpha^4$ as a test. Detailed analysis, using a new approach to the BS equation, recently proposed by the authors, indicates that the equations do not give equivalent results to order $m\alpha^4$ and that the arguments leading to the experimentally well verified $m\alpha^4$ potentials are sometimes defective.

I. INTRODUCTION

The purpose of the present paper is to go over familiar ground with a view toward clarifying some misunderstandings which have arisen in the calculations of $m\alpha^4$ levels in the relativistic two-body bound-state problem in quantum electrodynamics. We shall restrict ourselves to positronium and to the interaction involving single-photon exchange in the radiation gauge and ignore virtual annihilation. The point we wish to make concerns the proper "instantaneous limit" (i.e., limit of negligible retardation) to be taken in the interaction kernel of the Bethe-Salpeter (BS) equation, and so our results have a wider applicability than to the illustrative example alone.

The Breit equation,[1] with the Breit interaction and one-photon virtual annihilation,[2,3] was first used to deal with the positronium problem to the order $m\alpha^4$. Although Breit was aware of the anomalous terms which arise to this order in his pseudo-empirical development,[1] and gave heuristic arguments why such terms should be dropped, the fact of the existence of these anomalies is ignored in some later work.[4,5] Since the methods used to develop the effective potentials to order $m\alpha^4$ from the interactions are somewhat heuristic themselves, the anomalous terms never appear. Little attention was paid to single-photon exchange (where these anomalies occur), since these terms were well established before the positronium calculations and the new aspect of the problem for positronium was the appearance of the virtual-annihilation term.

The same situation persisted after the introduction of the BS equation.[6] Salpeter[7] introduced the Breit interaction as a kernel in the BS equation. The unstated implication is that this procedure gives correct eigenvalues to order $m\alpha^4$. We will show in the present paper that this is not the case. Fortunately, the emphasis in this and subsequent papers by Karplus and Klein[8] and Fulton and Martin[9] is on the calculation of higher-order corrections to the energy. In order[8-10] $m\alpha^5$ and $m\alpha^6 \ln\alpha$, these calculations are not sensitive to the delicate considerations leading to $m\alpha^4$ anomalies since the approximate wave functions used in the perturbation calculations are not affected by these difficulties.

Barker and Glover,[11] using a technique developed by Chraplyvy[12] which is based on the Foldy-Wouthuysen transformation,[13] carried out a systematic development of the effective potential order by order. Unfortunately, the BS equation with instantaneous interaction kernels is not amenable to treatment by this method and they find it nec-

essary to "Hermitize" this equation, claiming that this operation does not alter the final results. As we shall show subsequently, this is not in fact correct. However, to order $m\alpha^4$, the application of the proper limiting procedure to the case of no retardation gives a result which is in agreement with those arising from the unsubstantiated modifications introduced by Barker and Glover. Incidentally, these authors reemphasize once more the disagreement between the correct effective potential to order $m\alpha^4$ and that arising from the Breit equation with the Breit interaction.

Quite aside from the need for a more systematic and clearer treatment of lower-order contributions, it becomes essential to clarify this somewhat confusing situation in order to carry out higher-order calculations, starting with order $m\alpha^6$. The recent development of the treatment of the BS equation by the present authors[14] presents us with a convenient and systematic method which is not restricted, as is the approach of Ref. 11, to Hermitian structures. In fact, the difficulties arising from the past approaches to the problem referred to above were brought into focus through the approach of Feldman, Fulton, and Townsend (FFT), and the present paper can serve as an illustration of the systematic nature and utility of that approach.

In Sec. II, we summarize and discuss briefly the various equations which have appeared in the literature for positronium, involving one-photon exchange. In Sec. III, we consider the problem of going to the limit of no retardation in detail, and derive the equation, quoted in Sec. II, which yields the correct eigenvalues to order $m\alpha^4$. Thus far we use the more standard bispinor notation. In Sec. IV, we rewrite our equations in the form given by Kummer[15] and then apply the structure-function approach of FFT. For reasons of brevity, we only consider the $^1S_0$ state of positronium and exhibit the different effective potentials our different (but heretofore assumed to be equivalent) equations of Sec. II generate to order $m\alpha^4$.

II. POSITRONIUM EQUATIONS FOR INSTANTANEOUS INTERACTIONS

In this section, in addition to introducing our notation, we review and comment on the various equations which have appeared in the literature to describe one-photon exchange in the radiation gauge in the limit of no retardation. We take the mass of free electrons and positrons to be unity ($m=1$) and take the center of mass of positronium to be at rest. The Breit equation with the Breit interaction, in the bispinor notation, is given by

$$(\mathcal{H}_0 - 2\mu)\,\varphi(\vec{r}) = U(\vec{r})\,\varphi(\vec{r})\,, \tag{2.1}$$

where

$$\mu = 1 - \tfrac{1}{2}B\,, \tag{2.2}$$

and B is the binding energy,

$$\mathcal{H}_0 = H^{(1)}(\vec{p}) + H^{(2)}(-\vec{p})\,, \tag{2.3}$$

with

$$H^{(i)}(\vec{p}) = \vec{\alpha}^{(i)}\cdot\vec{p} + \beta^{(i)}\,, \quad i = 1, 2 \tag{2.4}$$

and

$$\vec{p} = -i\vec{\nabla}\,. \tag{2.5}$$

The interaction $U(\vec{r})$ is[16]

$$U = I_C + I_B\,, \tag{2.6}$$

$$I_C(\vec{r}) = \alpha/r\,, \tag{2.7}$$

$$I_B(\vec{r}) = -(\alpha/2r)\,(\vec{\alpha}^{(1)}\cdot\vec{\alpha}^{(2)} + \vec{\alpha}^{(1)}\cdot\hat{r}\,\vec{\alpha}^{(2)}\cdot\hat{r})\,. \tag{2.8}$$

The BS equation for one-photon exchange in the rest frame is[7]

$$[\gamma^{(1)}\cdot(b+p) - 1][\gamma^{(2)}\cdot(b-p) - 1]\,\psi(x) = g_R(x)\,\psi(x)\,, \tag{2.9}$$

where p is the relative four-momentum, which in coordinate space is

$$\vec{p} = -i\vec{\nabla}\,, \quad p_0 = i\frac{\partial}{\partial t}\,, \tag{2.10}$$

and

$$b = (\mu, \vec{0})\,. \tag{2.11}$$

The interaction $g_R(x)$ is given by

$$g_R(x) = g_C(x) + g_T(x)\,, \tag{2.12}$$

$$g_C(x) = -i\delta(t)\beta^{(1)}\beta^{(2)}I_C(\vec{r})\,, \tag{2.13}$$

$$g_T(x) = -[2i\alpha/(2\pi)^3]\,\gamma_l^{(1)}\gamma_m^{(2)} \times \int d^4k\, e^{-ik\cdot x}\, k^{-2}(\delta_{lm} - \hat{k}_l\hat{k}_m), \quad l, m = 1, 2, 3 \tag{2.14}$$

and

$$\gamma_0^{(i)} = \beta^{(i)}\,, \quad \gamma_l^{(i)} = \beta^{(i)}\alpha_l^{(i)}\,. \tag{2.15}$$

In the conventional approach of Salpeter[7] and subsequent authors, one obtains an "instantantous"

interaction [i.e., one which depends on the relative time only as $\delta(t)$] by neglecting the fourth component of the photon propagator in Eq. (2.14) (i.e., one lets $k^2 \to -\vec{k}^2$). With this assumption we can obtain the well-known result[7,17] (see Sec. III for derivations of similar expressions)

$$(\mathcal{H}_0 - 2\mu)\varphi(\vec{r}) = \Lambda U(\vec{r})\varphi(\vec{r}) , \tag{2.16a}$$

or equivalently

$$2(\omega - \mu)\varphi_{++} = \Lambda_{++} U(\varphi_{++} + \varphi_{--}), \quad \varphi_{++} \equiv \Lambda_{++}\varphi, \tag{2.16b}$$

$$2(\omega + \mu)\varphi_{--} = \Lambda_{--} U(\varphi_{++} + \varphi_{--}), \quad \varphi_{--} \equiv \Lambda_{--}\varphi, \tag{2.16c}$$

$$\varphi_{+-} \equiv \Lambda_{+-}\varphi = 0 , \tag{2.16d}$$

$$\varphi_{-+} \equiv \Lambda_{-+}\varphi = 0 , \tag{2.16e}$$

where

$$\Lambda \equiv \Lambda_{++}(\vec{p}) - \Lambda_{--}(\vec{p})$$

$$\equiv \Lambda_+^{(1)}(\vec{p})\Lambda_+^{(2)}(-\vec{p}) - \Lambda_-^{(1)}(\vec{p})\Lambda_-^{(2)}(-\vec{p}) , \tag{2.17}$$

$$\Lambda_{+-}(\vec{p}) = \Lambda_+^{(1)}(\vec{p})\Lambda_-^{(2)}(-\vec{p}) ,$$

$$\Lambda_{-+}(\vec{p}) = \Lambda_-^{(1)}(\vec{p})\Lambda_+^{(2)}(-\vec{p}) , \tag{2.18}$$

and[18]

$$\Lambda_\pm^{(i)}(\vec{p}) = [\omega \pm H^{(i)}(\vec{p})]/2\omega, \quad \omega = (\vec{p}^2 + 1)^{1/2} . \tag{2.19}$$

In obtaining the above, we have made use of the identity

$$\Lambda = \tfrac{1}{2}[\Lambda_+^{(1)}(\vec{p}) - \Lambda_-^{(1)}(\vec{p}) + \Lambda_+^{(2)}(-\vec{p}) - \Lambda_-^{(2)}(-\vec{p})] = \mathcal{H}_0/2\omega, \tag{2.20}$$

which immediately yields the result that Λ commutes with $\mathcal{H}_0$.

Since the right-hand side of Eq. (2.16a) does not have a Hermitian structure, Barker and Glover[11] Hermitize it and claim that they obtain an equivalent structure, replacing Eq. (2.16a) by

$$(\mathcal{H}_0 - 2\mu)\varphi(\vec{r}) = \tfrac{1}{2}[\Lambda, U(\vec{r})]_+ \varphi(\vec{r}) . \tag{2.21a}$$

Again, using the properties of the projection operators, this is equivalent to the set

$$2(\omega - \mu)\varphi_{++} = \Lambda_{++} U(\varphi_{++} + \tfrac{1}{2}\varphi_{+-} + \tfrac{1}{2}\varphi_{-+}) , \tag{2.21b}$$

$$2(\omega + \mu)\varphi_{--} = \Lambda_{--} U(\varphi_{--} + \tfrac{1}{2}\varphi_{+-} + \tfrac{1}{2}\varphi_{-+}), \tag{2.21c}$$

and in contrast to Eqs. (2.16d) and (2.16e),

$$\varphi_{+-} = -(4\mu)^{-1}\Lambda_{+-} U(\varphi_{++} - \varphi_{--}) , \tag{2.21d}$$

$$\varphi_{-+} = -(4\mu)^{-1}\Lambda_{-+} U(\varphi_{++} - \varphi_{--}) . \tag{2.21e}$$

In fact, as we shall show in Sec. IV, Eqs. (2.16a) and (2.21a) are not equivalent. They lead to different effective interactions even in order $m\alpha^4$.

The question we then ask is which (if either) of Eqs. (2.16a) or (2.21a) correctly follows from the neglect of retardation in the BS equation [Eq. (2.9)]. A careful limiting procedure is carried out in Sec. III. For purposes of comparison, we quote the result here. Instead of Eqs. (2.16a) or (2.21a), we obtain (correct to order $m\alpha^4$) the equations

$$(\mathcal{H}_0 - 2\mu)\varphi = (\Lambda I_C + \Lambda_{++} I_B \Lambda_{++})\varphi , \tag{2.22a}$$

or equivalently

$$2(\omega - \mu)\varphi_{++} = \Lambda_{++}[I_C(\varphi_{++} + \varphi_{--}) + I_B\varphi_{++}] , \tag{2.22b}$$

$$2(\omega + \mu)\varphi_{--} = \Lambda_{--} I_C(\varphi_{++} + \varphi_{--}) , \tag{2.22c}$$

and, in addition, just as for Eq. (2.16a)

$$\varphi_{+-} = 0 , \tag{2.22d}$$

$$\varphi_{-+} = 0 . \tag{2.22e}$$

We conclude this section with a number of remarks. In the first place, it is not surprising that Eqs. (2.21a) and (2.22a) give the same $m\alpha^4$ results. We will now briefly indicate how the equivalence arises to this order. Although the functions φ_{+-} and φ_{-+} do not vanish for the set (2.21) [see Eqs. (2.21d) and (2.21e)], nevertheless, to order $m\alpha^4$, we can still neglect them in Eqs. (2.21b) and (2.21c). This is so since[19]

$$\Lambda_{++} U \Lambda_{-+} \approx \tfrac{1}{4}(1 + \beta^{(1)})(1 + \beta^{(2)}) U$$

$$\times \tfrac{1}{4}(1 - \beta^{(1)})(1 + \beta^{(2)}) = 0 . \tag{2.23}$$

Looking at Eq. (2.22a), we observe that [see, e.g., Eq. (4.31)] to order $m\alpha^4$

$$[\Lambda_{\pm\pm}, I_C] \approx 0 , \tag{2.24}$$

so that φ_{--} drops out of Eq. (2.22b) to first order, and Eq. (2.21b) is equivalent to (2.22b). The equivalence does not exist for Eq. (2.16a), since Λ_{++} and Λ_{--} do not commute, even in lowest order, with I_B, because of the presence of the $\alpha_i^{(1)}\alpha_m^{(2)}$ terms in the latter. Thus, Eq. (2.16a) is a coupled set of two equations for φ_{++} and φ_{--} even in order

$m\alpha^4$. The equivalence of Eqs. (2.21a) and (2.22a) and their nonequivalence to Eq. (2.16a) is explicitly demonstrated in Sec. IV.

Our final observation concerns the nonuniqueness of the Hermitizing process yielding Eq. (2.21a) from (2.16a). As an alternative, we define the wave function χ by the equation[7]

$$\varphi = \Lambda\chi \ . \tag{2.25}$$

Equation (2.16a) thus becomes

$$(\mathcal{K}_0 - 2\mu)\Lambda\chi = \Lambda U \Lambda\chi \ . \tag{2.26}$$

The operators of Eq. (2.26) are manifestly Hermitian, since Λ and $\mathcal{K}_0$ are Hermitian and commute; the technique of Ref. 11 can be applied. However, the analysis of Sec. IV yields, as it should, exactly the same result for the effective potential to order $m\alpha^4$ in both Eqs. (2.16a) and (2.26) and a different one for that arising from Eqs. (2.21a) and (2.22a).

III. LIMIT OF NO RETARDATION

We begin our discussion in this section by writing Eq. (2.9) as an integral equation:

$$\psi = G_0^{(1)} G_0^{(2)} g_R \psi \ , \tag{3.1}$$

where $G_0^{(i)}$ is the free-particle Green's function of particle i:

$$G_0^{(1)} = [\gamma^{(1)}(b+p)-1]^{-1}, \quad G_0^{(2)} = [\gamma^{(2)}(b-p)-1]^{-1}. \tag{3.2}$$

We will work in momentum space and define the function $\varphi(\vec{\mathrm{p}})$ by

$$\varphi(\vec{\mathrm{p}}) = \int dp_0 \psi(p) \ . \tag{3.3}$$

We exhibit the singularity structure of $G_0^{(1)} G_0^{(2)}$ in the p_0 plane by writing

$$G_0^{(1)} G_0^{(2)} = \frac{[H^{(1)}(\vec{\mathrm{p}}) + \mu + p_0][H^{(2)}(-\vec{\mathrm{p}}) + \mu - p_0]\beta^{(1)}\beta^{(2)}}{[(p_0+\mu)^2 - \omega^2 + i\epsilon][(p_0-\mu)^2 - \omega^2 + i\epsilon]}. \tag{3.4}$$

The poles of $G_0^{(1)}(p)\, G_0^{(2)}(p)$ in the p_0 plane are thus at

$$p_0 = \begin{cases} -\mu \pm (\omega - i\epsilon) \\ \mu \pm (\omega - i\epsilon) \end{cases} \tag{3.5}$$

In order to obtain an equation (accurate to order $m\alpha^4$) for the instantaneous wave function $\varphi(\vec{\mathrm{p}})$, we first write an equation equivalent to Eq. (3.1), namely,

$$\psi = G_0^{(1)} G_0^{(2)} g_C \psi + G_0^{(1)} G_0^{(2)} g_T G_0^{(1)} G_0^{(2)} g_R \psi \ . \tag{3.6}$$

If we now make the replacement

$$g_R \approx g_C \tag{3.7}$$

for the g_R appearing in Eq. (3.6), we will obtain an equation correct to order $m\alpha^4$. We work in momentum space and integrate the equation over p_0 to obtain the result

$$\begin{aligned} \varphi(\vec{\mathrm{p}}) \approx & \int dp_0 \int G_0^{(1)}(p) G_0^{(2)}(p) \beta^{(1)}\beta^{(2)} I_C(\vec{\mathrm{p}} - \vec{\mathrm{p}}')\varphi(\vec{\mathrm{p}}') d^3p' \\ & + \int dp_0 \int dp_0' \int G_0^{(1)}(p) G_0^{(2)}(p) g_T(p, p') \\ & \times G_0^{(1)}(p') G_0^{(2)}(p') \beta^{(1)}\beta^{(2)} \\ & \times I_C(\vec{\mathrm{p}}' - \vec{\mathrm{p}}'')\varphi(\vec{\mathrm{p}}'') d^3p' d^3p'' , \end{aligned} \tag{3.8}$$

where

$$I_C(\vec{\mathrm{p}}) = [-i/(2\pi)^4] \int e^{-i\vec{\mathrm{p}}\cdot\vec{\mathrm{r}}} I_C(\vec{\mathrm{r}}) d^3\gamma \tag{3.9}$$

and

$$g_T(p,p') = \frac{-2i\alpha}{(2\pi)^3} \gamma_i^{(1)} \gamma_m^{(2)} \frac{1}{k^2} (\delta_{im} - \hat{k}_i \hat{k}_m), \quad k = p - p'. \tag{3.10}$$

The first term yields

$$\left(\frac{\Lambda_{++}}{2(\omega-\mu)} + \frac{\Lambda_{--}}{2(\omega+\mu)} \right) I_C \varphi \tag{3.11}$$

after p_0 integration. The second term has both a p_0 and p_0' singularity structure, due to both photon and fermion propagators.

Let us consider the fermion propagators first. We illustrate with three typical residues where we integrate in the upper-half plane.

For $p_0 = \mu - \omega$, $p_0' = \mu - \omega'$ we have[20]

$$R_{++} \sim \frac{\Lambda_{++}}{\mu - \omega} \frac{Z_{12}}{D} \frac{\Lambda'_{++}}{\mu - \omega'} \ , \tag{3.12}$$

where

$$Z_{12} = \alpha_i^{(1)} \alpha_m^{(2)} (\delta_{im} - \hat{k}_i \hat{k}_m) \ , \tag{3.13}$$

and

$$D = (\vec{\mathrm{p}} - \vec{\mathrm{p}}')^2 - [(\mu - \omega) - (\mu - \omega')]^2 \approx (\vec{\mathrm{p}} - \vec{\mathrm{p}}')^2 \ . \tag{3.14}$$

For $p_0 = -(\mu + \omega)$, $p_0' = -(\mu + \omega')$ we have

$$R_{--} \sim \frac{\Lambda_{--}}{\mu + \omega} \frac{Z_{12}}{D} \frac{\Lambda'_{--}}{\mu + \omega'} \ . \tag{3.15}$$

For $p_0 = \mu - \omega$, $p_0' = -(\mu + \omega')$ we have

$$R_{+-} \sim \frac{\Lambda_{++}}{\mu-\omega}\frac{Z_{12}}{D'}\frac{\Lambda'_{--}}{\mu+\omega'} , \tag{3.16}$$

with

$$D' = (\vec{p}-\vec{p}')^2 - [(\mu-\omega)+(\mu+\omega')]^2 \simeq -4.$$

A similar expression, involving D', is obtained for R_{-+}. Thus the $(p_0-p_0')^2$ term cannot be neglected in the photon propagator for R_{+-} and R_{-+}. On the contrary, it provides the dominant part of the propagator and guarantees that the φ_{++} to φ_{--} coupling becomes negligible to first order owing to these terms.

The residues due to the photon propagator poles also yield results which are negligible in comparison with R_{++}. We illustrate with what should be one of the large terms: that arising from the positive energy projections of both fermion Green's-function products. We integrate over p_0' first, and examine the residues at the fermion poles. We obtain

$$R^{\gamma}_{++} \sim \int dp_0 \frac{\Lambda_{++} Z_{12} \Lambda'_{++}}{(p_0+\mu-\omega+i\epsilon)(p_0-\mu+\omega+i\epsilon)[(p_0-\mu+\omega')^2-(\vec{p}-\vec{p}')^2+i\epsilon](\mu-\omega')} . \tag{3.17}$$

The poles of the photon propagator are at

$$p_0 = \mu-\omega' \pm (|\vec{p}-\vec{p}'| - i\epsilon) , \tag{3.18}$$

and the contribution from the photon pole in the upper-half plane is the residue

$$R^{\gamma}_{++} \sim \frac{\Lambda_{++}}{D''}\frac{Z_{12}}{|\vec{p}-\vec{p}'|}\frac{\Lambda'_{++}}{\mu-\omega'} , \tag{3.19}$$

with

$$D'' = [2\mu-(\omega+\omega')-|\vec{p}-\vec{p}'|][(\omega-\omega')-|\vec{p}-\vec{p}'|]$$
$$\approx (\vec{p}-\vec{p}')^2 . \tag{3.20}$$

Comparing R_{++} with R^{γ}_{++} for small $|\vec{p}|$ and $|\vec{p}'|$, we find that

$$R_{++} \gg R^{\gamma}_{++} , \tag{3.21}$$

since

$$\frac{1}{\mu-\omega} = O\left(\frac{1}{\vec{p}^2}\right) \gg O\left(\frac{1}{|\vec{p}|}\right) = \frac{1}{|\vec{p}-\vec{p}'|} . \tag{3.22}$$

A similar argument can be made for all contributions due to poles of the photon propagator.

This somewhat tedious argument establishes the validity of Eq. (2.22a) to order $m\alpha^4$, and shows the incorrectness of going to the limit of no retardation by merely dropping the $(p_0-p_0')^2$ term in the photon propagator. The latter step leads to the incorrect Eq. (2.16a). Essentially the correct limiting procedure leads to a suppression of pair terms, leaving only φ_{++} in Eq. (2.22b).

IV. COMPARISON OF EFFECTIVE POTENTIALS IN $^1S_0$ STATE

Before calculating the effective potentials which arise from the various equations in Sec. II, it is necessary to make contact with the formalism of FFT and Ref. 15. In order to do so, we have to redefine our wave function [see Eq. (1.6) of Ref. 15] to be

$$\varphi \to \varphi C^{-1}\gamma_5 , \tag{4.1}$$

and take transposes of the $\alpha^{(2)}, \gamma^{(2)}, \beta^{(2)}$, etc., matrices. This enables us to drop the superscripts (1) and (2) and utilize the Dirac matrix trace techniques of FFT and Ref. 15. The modifications required are

$$\vec{\alpha}^{(1)}_{aa'} \to \vec{\alpha}_{aa'} , \quad \beta^{(1)}_{aa'} \to \beta_{aa'} ,$$
$$\vec{\alpha}^{(2)}_{bb'} \to -\vec{\alpha}_{b'b} , \beta^{(2)}_{bb'} \to \beta_{b'b} . \tag{4.2}$$

As an example, we have the expression

$$\mathcal{K}_0 \varphi(\vec{p}) = (\vec{\alpha}^{(1)}\cdot\vec{p}+\beta^{(1)}-\vec{\alpha}^{(2)}\cdot\vec{p}+\beta^{(2)})\varphi(\vec{p}) , \tag{4.3}$$

which goes into

$$[(\vec{\alpha}\cdot\vec{p}+\beta), \varphi(\vec{p})]_+ \tag{4.4}$$

in the new notation.

If we now restrict ourselves to the $^1S_0$ state and introduce scalar structure functions, as in FFT, the requirements of parity invariance, charge conjugation invariance, and Lorentz invariance, as well as the vectors naturally prescribed by the problem, restrict us to the most general form for φ given by[21]

$$\varphi = S+\beta V+\vec{\alpha}\cdot\vec{p}\,T . \tag{4.5}$$

We now substitute Eq. (4.5) into the various equations of Sec. II for φ or χ and take traces after multiplication by 1, β, and $\vec{\alpha}\cdot\vec{p}$ to obtain the results which follow below.

We note initially that for the φ's which appear in Eqs. (2.16a) and (2.22a) we have the property

$$\varphi_{+-}=\varphi_{-+}=0\ ,$$

which is equivalent to the condition

$$V=T\ . \tag{4.6}$$

This condition arises naturally[22] from field theoretical considerations.

Next, we observe the complete identity of the contents of Eqs. (2.16a) and (2.26). In fact, these equations reduce to coupled equations for two structure functions. If one lets

$$\chi=\overline{S}+\beta\overline{V}+\vec{\alpha}\cdot\vec{p}\,\overline{T} \tag{4.7}$$

and defines

$$\overline{V}+p^2\overline{T}=\omega S \tag{4.8}$$

and

$$\overline{S}=\omega V\ , \tag{4.9}$$

one obtains identical coupled equations for S and V from Eqs. (2.16a) and (2.26). These equations can be written in the matrix form

$$M_0\Phi=I_{(1)}\Phi\ , \tag{4.10}$$

where

$$\tilde{\Phi}=(S,V)\ , \tag{4.11}$$

$$M_0=\begin{pmatrix}-\mu & \omega^2\\ 1 & -\mu\end{pmatrix}, \tag{4.12}$$

$$I_{(1)}=\frac{1}{2\omega}\begin{pmatrix}0 & -I_C+Q\\ 3I_C & 0\end{pmatrix}, \tag{4.13}$$

$$Q=\alpha\vec{p}\cdot\frac{\vec{r}\vec{r}}{r^3}\cdot\vec{p}\ . \tag{4.14}$$

We can establish the identity

$$V=(1/\mu)[1-(3/2\omega)I_C]S\ , \tag{4.15}$$

which is the analog of Eq. (3.17) of FFT.

Elimination of V yields the analog of Eq. (3.15) of FFT:

$$\mathfrak{u}_{(1)}\varphi=0,\quad S=\varphi, \tag{4.16}$$

with

$$\mathfrak{u}_{(1)}=(\omega^2-\mu^2)-\frac{1}{2}\left(3\omega-\frac{1}{\omega}\right)I_C-\frac{1}{2\omega}Q$$
$$-\frac{3}{4\omega}I_C\frac{1}{\omega}I_C+\frac{3}{4\omega}Q\frac{1}{\omega}I_C\ . \tag{4.17}$$

The effective Hamiltonian, correct to order $m\alpha^4$ is thus[23]

$$\mathfrak{u}_{(1)}\approx H-H'\ , \tag{4.18}$$

$$H=\vec{p}^2-\frac{\alpha}{r}-\frac{\vec{p}^4}{4}-\pi\alpha\delta(\vec{r})-\frac{\alpha}{2r}\left(\vec{p}^2+\frac{r_i r_j}{r^2}p_i p_j\right)\ , \tag{4.19}$$

$$H'=\frac{\alpha^2}{r^2}\ . \tag{4.20}$$

Equation (4.19) is the correct effective Hamiltonian for $^1S_0$. Thus Eqs. (2.16a) and (2.26) yield the same anomalous result, which differs from the correct result by the spurious term $-H'$.

We turn now to the analysis of Eq. (2.22a), the equation obtained by taking the correct no-retardation limit. We again obtain two coupled equations for S and V, of the form (4.10) but with $I_{(1)}\to I_{(2)}$, where

$$I_{(2)}=\frac{1}{2\omega}\begin{pmatrix}\omega\mathfrak{F} & I_C+2R+\omega\mathfrak{F}\omega\\ I_C+\mathfrak{F} & \mathfrak{F}\omega\end{pmatrix}, \tag{4.21}$$

with

$$R=p_i(\alpha/2r)p_i \tag{4.22}$$

and

$$\mathfrak{F}=(1/4\omega)[2(\omega I_C\omega-I_C)-2R+Q](1/\omega)\ . \tag{4.23}$$

It is not difficult to show that to order $m\alpha^4$, one obtains

$$V\approx\frac{1}{\mu}\left(1-\frac{1}{2\mu}I_C-\mathfrak{F}\right)S\ . \tag{4.24}$$

One can once again eliminate V. However, this time one obtains to order $m\alpha^4$ the correct effective Hamiltonian H. This same result can be obtained more directly by using Eq. (2.22b) and neglecting φ_{---}.

The remaining Eqs. (2.1) and (2.21a) lead to three coupled equations for the structure functions S, V, T, since the Eqs. (2.22d) and (2.22e) and their consequence, Eq. (4.6), do not hold. They are of the form

$$\overline{M}_0\underline{\Phi}=I_{(n)}\underline{\Phi},\quad n=3,4 \tag{4.25}$$

with

$$\underline{\tilde{\Phi}}=(S,V,T)\ , \tag{4.26}$$

$$\overline{M}_0=\begin{pmatrix}-\mu & 1 & \vec{p}^2\\ 1 & -\mu & 0\\ \vec{p}^2 & 0 & -\mu\vec{p}^2\end{pmatrix}\ . \tag{4.27}$$

In the above,

$$I_{(3)}=\frac{1}{2}\begin{pmatrix}3I_C & 0 & 0\\ 0 & -I_C & 0\\ 0 & 0 & Q\end{pmatrix} \tag{4.28}$$

arises from the Breit equation [Eq. (2.1)] and

$$I_{(4)}=\frac{1}{4}\begin{bmatrix} 0 & 3I_C\frac{1}{\omega}-\frac{1}{\omega}I_C & 3I_C\frac{\vec{p}^2}{\omega}+\frac{1}{\omega}Q\\ \frac{3}{\omega}I_C-I_C\frac{1}{\omega} & 0 & 0\\ \frac{3\vec{p}^2}{\omega}I_C+Q\frac{1}{\omega} & 0 & 0\end{bmatrix} \tag{4.29}$$

arises from Eq. (2.21a).

If we analyze Eq. (4.25) in coordinate space, using $I_{(3)}$, eliminate S in terms of V exactly and $\vec{p}^2T$ in terms of S (and therefore of V) approximately, we obtain an effective Hamiltonian to order $m\alpha^4$ of

$$\mathcal{U}_{(3)}\approx H+H', \tag{4.30}$$

with the anomalous term H' having opposite sign to that in Eq. (4.18). Approximate elimination of V and $\vec{p}^2T$ in terms of S leads to the correct effective $m\alpha^4$ Hamiltonian H, using $I_{(4)}$.

Using the techniques of this section, it is not difficult to confirm in detail the arguments of Sec. II which lead to the establishment of the equivalence of Eqs. (2.21a) and (2.22a), and their nonequivalence to Eq. (2.16a).

Consider Eq. (2.22a) first. We obtain, using the techniques of FFT,

$$\Lambda_{++}I_C\Lambda_{--}\varphi = \Lambda_+\left(\tfrac{1}{2}I_C-\frac{1}{2\omega}I_C\frac{1}{\omega}-\frac{1}{\omega}R\frac{1}{\omega}\right)(S-\omega V), \tag{4.31}$$

which gives rise to an effective potential of order $m\alpha^6$, and is thus negligible.

Turning to Eq. (2.21a), we find

$$\varphi_{+-}+\varphi_{-+}=\omega^{-2}(-\vec{\alpha}\cdot\vec{p}+\beta\vec{p}^2)(V-T) \tag{4.32}$$

and

$$\Lambda_{++}U(\varphi_{+-}+\varphi_{-+})=-\frac{\Lambda_+}{\omega}(I_C\vec{p}^2+Q)\frac{1}{\omega^2}(V-T) \tag{4.33}$$

which again gives rise to a negligible $m\alpha^6$ effective potential. Finally, consider the Λ_{++} projection of Eq. (2.16a). It differs from the Λ_{++} projection of Eq. (2.22a) by a term $\Lambda_{++}U\Lambda_{--}\varphi$. The Coulomb interaction part of U gives an $m\alpha^6$ effective potential by Eq. (4.31). The remaining term is

$$\Lambda_{++}I_B\Lambda_{--}\phi = \Lambda_+\left(I_C+\frac{1}{\omega}I_C\frac{1}{\omega}+\frac{1}{\omega}R\frac{1}{\omega}-\frac{1}{2}\frac{1}{\omega}Q\frac{1}{\omega}\right)(S-\omega V)$$
$$\approx 2\Lambda_+I_C(S-\omega V)\approx\Lambda_+H'\varphi\,, \tag{4.34}$$

which is precisely the anomalous term our previous calculation has generated.

V. DISCUSSION

The ease with which effective potentials to order $m\alpha^4$ can be generated in a systematic fashion in Sec. IV for the various two-body equations of Sec. II underlines the utility and straightforwardness of the structure-function approach of FFT. The agreement of our Eqs. (4.30) and (4.19) with Eqs. (14) and (15) of Ref. 11 for $^1S_0$ states serves as a check of this method, since the only fully systematic method for generating effective potentials other than that of the structure-function approach of FFT is the Foldy-Wouthuysen transformation technique of Refs. 11 and 12. It is thus gratifying to see agreement between the results of the two approaches, criticize as we may the starting equation [our Eq. (2.21a)] of Ref. 11.

It should be emphasized that we have no essentially new result to offer. We do not presume to question the original formulation of the BS equation in its covariant form, which is a direct consequence of quantum field theory. Our criticism is solely directed at the way in which the no-retardation limit has been hitherto carried out for this equation.[24] Nor do we obtain a result for the $m\alpha^4$ potentials which differs from the widely accepted (and experimentally confirmed) results of many years standing. We only claim to put the derivation of these potentials on a sounder footing than has been done previously. This "tidying up" is essential if one is to carry calculations to higher orders, specifically to order $m\alpha^6$. It is true that in principle any of the equations of Sec. II could be used as a basis of perturbation theory to higher orders. They all give the correct eigenvalues to order $m\alpha^2$. In practice, it is preferable to use the equation obtained through the correct limiting procedure from the covariant BS equation. In this way, one avoids the appearance of spurious terms in various orders in α. These subsequently have to be canceled by other terms which arise later in higher-order perturbation theory. It is more than likely that they can be overlooked in an insufficiently systematic approach.

A final comment: making the potentials instantaneous as an input to the BS equation appears to present us with a paradox that the instantaneous radiation "gauge" gives us different $m\alpha^4$ energies

from the radiation gauge proper, with Eqs. (2.9) and (2.14) as starting points; it appears to contradict the gauge invariance of the eigenvalues at first sight. This is in fact not the case. The only proper interaction kernel in the radiation gauge is represented by Eqs. (2.9) and (2.14). The instantaneous radiation-gauge interaction Eqs. (2.6)–(2.8) is not a gauge at all, but is the result of approximations carried out, and carried out improperly, with Eq. (2.9) as a starting point.

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[18]The $\Lambda_\pm$ of FFT differ from those used here by factors of $\beta\omega$.

[19]Neglect of $\vec{p}$ dependence in the projection operators is justified since these terms immediately lead to energy contributions of higher order than $m\alpha^4$.

[20]In this and subsequent equations we use the fact that $\omega - \mu \approx O(\vec{p}^2)$. Thus we neglect terms $O(1)$ relative to $1/(\mu - \omega)$ and $O(\vec{p}^4)$ terms relative to $(\vec{p} - \vec{p}')^2$. The terms which we neglect contribute to order $m\alpha^5$ and higher.

[21]See Eqs. (B1)–(B5) of FFT. We have simplified the notation of FFT by dropping superscripts and subscripts on the structure functions, since there is no ambiguity if we restrict ourselves to the $^1S_0$ state with no relative energy or time dependence.

[22]See the argument leading to Eq. (C24) in FFT.

[23]The derivation is carried out analogously to that of Eq. (3.26) of FFT. Note that, to order $m\alpha^4$, $\vec{p}^2$ and α/r commute.

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