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Helicity Modulus, Superfluidity, and Scaling in Isotropic Systems

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The ordered state of a d-dimensional isotropic system with an n-vector ($n \geq 2$) order parameter is considered. By the imposition of suitable boundary conditions it is shown how to define explicitly a helicity modulus $\Upsilon(T)$ which measures the free-energy increment associated with "twisting" the direction of the order parameter. For a Bose system the superfluid density is seen to be $\rho_s(T) = (m/\hbar)^2\Upsilon(T)$. A critical exponent υ is defined by $\Upsilon(T) \sim |T - T_c|^{\upsilon}$ as $T \to T_c$; for an ideal Bose gas and spherical model ($n \to \infty$), $\upsilon = 1$ is an exact result for all $d > 2$. The difficulties of defining a correlation length in the ordered phase are discussed. A full scaling theory of the correlations avoids these problems and may be linked to a phenomenological hydrodynamic approach, to clarify and rederive Josephson's relation $\upsilon = 2\beta - \eta\nu = 2 - \alpha - 2\nu$. This reduces to $\upsilon = (d-2)\nu$ (used by some authors with $d = 3$), *only* if one accepts d-dependent, "hyperscaling" relations such as $d\nu = 2 - \alpha$; however, both these latter relations fail for the ideal Bose gas when $d > 4$. An alternative derivation of the formula $\upsilon = 2 - \alpha - 2\nu$ is based on the scaling theory for systems with a large but finite dimension.

I. INTRODUCTION

Recent advances in the theory of phase transitions and critical phenomena have indicated the crucial role played by the symmetry of the ordered phase.[1,2] A well-known example is that a two-dimensional Ising model possesses a critical point, below which long-range order and spontaneous magnetization exist, whereas a two-dimensional Heisenberg model cannot display such spontaneous magnetization or long-range order at any nonzero temperature. In the first case, the order parameter is scalar, while in the second, the ordered phase exhibits a rotational symmetry (the order parameter being a three-component vector). Even in three-dimensional space, where both models exhibit a phase transition, the behavior depends significantly on the symmetry. This is especially so beneath the critical temperature.

In systems, like the Heisenberg model, which possess a vector order parameter $\underline{\Psi}$, a uniform or homogeneous thermodynamic phase is characterized by *both* the magnitude $\Psi = |\underline{\Psi}|$ *and* direction or (especially in the case of a two-component vector) *phase angle*[3] of $\underline{\Psi}$. We use an underline to stress the general vector character, since the space in which $\underline{\Psi}$ resides need not be the normal d-dimensional geometrical space, which the system occupies. Thus in the case of a ferromagnet with $\underline{\Psi} = \vec{M}$ (the magnetization) the two spaces do coincide,[4] but in the important example of a quantal superfluid $\underline{\Psi}(\vec{r})$ is the field operator normally denoted $\psi(\vec{r}) = \psi'(\vec{r}) + i\psi''(\vec{r})$. This is usually regarded as a complex number (or operator) but from our viewpoint it is a two vector in a completely independent "gauge" space.

This vector character of the order parameter is normally reflected in an exact *continuous symmetry* of the Hamiltonian $\mathcal{H}$ (on the phase boundary) which we then say describes an *isotropic system*. Thus, in the case of an isotropic ferromagnet $\mathcal{H}$ is invariant to spatial rotations of all the spins: In a quantal system the relevant isotropy is the gauge symmetry, i.e., invariance under the operation $\psi(\vec{r}) \Rightarrow e^{i\theta}\psi(\vec{r})$.

The basic ideas and concepts of the general theory of critical phenomena are, of course, applicable to the study of isotropic systems. However, in the ordered phase, several conceptual problems arise. For example, the isotropy implies that there is a continuous infinity of ordered phases. Furthermore, any two distinct phases may differ only infinitesimally from one another (via an infinitesimal difference of phase angle). Consequently, in the ordered phase, the decay of correlations is *not* exponentially fast; this results in an inapplicability of the usual moment-based definitions of a correlation length.[2,5]

In this paper we will discuss several aspects of the critical behavior of isotropic systems concen-

trating attention on the ordered phase. In particular, we will introduce the general concept of a *helicity modulus* $\Upsilon(T)$ and analyze its role and critical behavior. On the one hand, the helicity modulus may be considered as the analogy, for an isotropic system, of the surface tension or interfacial free energy between two phases, which do differ discretely, e.g., the "up" and "down" states in an Ising magnet. More fundamentally, however, $\Upsilon(T)$ is a measure of the response of the system to a suitable helical or "phase-twisting" field. In addition, it may be used to define a *phase coherence length*. This length, which generally diverges at the critical point, may to some extent play the role of a correlation length in the ordered phase.

Finally, for a Bose fluid, the helicity modulus is simply related to the superfluid density $\rho_s(T)$ and, indeed, provides a general, fully equilibrium definition of $\rho_s(T)$. This definition of $\rho_s(T)$, perhaps surprisingly, indicates that the superfluid density is not solely a bulk property of the system, but may rather be considered as "beyond the bulk" in the same sense that the surface tension is not normally regarded as a bulk property.

Our arguments are arranged as follows. In Sec. II we formally define the helicity modulus, and discuss its relation to the superfluid density. The results of two exact model calculations are also reported. In Sec. III we briefly discuss the problem of the decay of order in an isotropic system and consider the question of the definition of a correlation length in the ordered phase. Section IV contains a more detailed scaling analysis of the critical behavior of an isotropic system. Following the important work of Josephson,[6] this analysis allows a definite prediction to be made for the behavior of $\Upsilon(T)$ near T_c. The analysis, which can be fruitfully compared with Widom's discussion of the interfacial, or surface, tension is reviewed in Sec. V. Since the helicity modulus is a "beyond-the-bulk" property, an alternative derivation of its critical behavior may be obtained from the scaling theory[7,8] of finite-size effects. These arguments are presented in Sec. VI. Section VII contains a concluding summary.

II. DEFINITION OF HELICITY MODULUS

To define the helicity modulus, we consider an isotropic system in a uniform cylindrical domain Ω, of length $L(\Omega)$ and cross-sectional area[9] $A(\Omega)$. Normally, in order to stabilize a specific thermodynamic phase characterized, as we noted earlier, by a definite magnitude and phase angle for the order parameter, $\underline{\Psi} = \langle \underline{\Psi}(\vec{r}) \rangle$, we require some external ordering field $\underline{\zeta}$ which couples via a term $-\int_\Omega \underline{\zeta} \underline{\Psi}(\vec{r})\, d\vec{r}$ in the Hamiltonian. However, we may alternatively introduce a set of wall potentials, $\mathcal{W}_\theta$, which will establish a definite phase angle[10] θ for $\underline{\Psi}$ even in the absence of any bulk ordering field $\underline{\zeta}$. (We will assume, in view of the symmetry, that on the phase boundary we have $\underline{\zeta} = \underline{\zeta}_\sigma = 0$.) Specifically, we will require symmetry-respecting potentials on the side walls of the cylinder, Ω, but on the end walls we impose either $\mathcal{W}_{+\theta}$ or $\mathcal{W}_{-\theta}$ (with $|\theta| < \frac{1}{2}\pi$). [We may imagine the wall potentials as being generated by an infinitely strong symmetry-breaking field $\underline{\zeta}(\vec{r})$, which acts only in a thin layer at the boundaries of Ω.] The $(+,+)$ or $(-,-)$ wall combinations will yield uniform bulk phases in which $\langle \underline{\Psi}(r) \rangle$ has a constant (mean) phase angle independent of position. By symmetry the total free energies will satisfy, in an obvious notation,

$$\mathcal{F}(T; \Omega, \mathcal{W}_\theta^{++}) = \mathcal{F}(T; \Omega, \mathcal{W}_\theta^{--}) \,. \tag{2.1}$$

On the other hand, the mixed $(+,-)$ wall potentials must impose some sort of "twist" on the system: Presumably the direction of $\langle \underline{\psi}(\vec{r}) \rangle$ will now vary in a more or less uniform and continuous manner from one end of Ω to the other. Let us consider the size dependence of the resulting incremental free energy

$$\Delta\mathcal{F}(T, \Omega) = \mathcal{F}(T; \Omega, \mathcal{W}_\theta^{+-}) - \mathcal{F}(T; \Omega, \mathcal{W}_\theta^{++}). \tag{2.2}$$

When the cross-section area $A(\Omega)$ is large, we can expect approximate transverse translational symmetry so that $\Delta\mathcal{F}$ will become proportional to $A(\Omega)$. The length dependence of $\Delta\mathcal{F}$ is, however, more subtle. We can define the mean phase angle $\varphi(\vec{r})$ of $\langle \underline{\Psi}(\vec{r}) \rangle$ explicitly since the expectation $\langle \underline{\Psi} \rangle$ is simply a classical vector whatever the microscopic density or operator $\underline{\Psi}(\vec{r})$ may be; the average gradient of $\varphi(\vec{r})$ is then parallel to the axis of Ω and equal to

$$\langle\langle \nabla\phi \rangle\rangle = V_\Omega^{-1} \int [\nabla\varphi(\vec{r})]_\parallel \, d\vec{r} = 2\theta / L(\Omega). \tag{2.3}$$

Now by symmetry $\Delta\mathcal{F}$ must be an even function of $u = \langle\langle \nabla\varphi \rangle\rangle$ and must vanish when $u = 0$. On heuristic grounds we hence expect $\Delta\mathcal{F} \propto \langle\langle \nabla\varphi \rangle\rangle^2$. Furthermore, we expect the gradient of φ to induce a *local* "strain" energy (or free energy) density so that the total strain energy should be proportional to $V(\Omega)$. In summary we hence expect

$$\Delta\mathcal{F}(T, \Omega) \approx \tfrac{1}{2}\Upsilon(T) \langle\langle \nabla\varphi \rangle\rangle^2 V(\Omega) = 2\theta^2 \Upsilon(T) A(\Omega)/L(\Omega) \,, \tag{2.4}$$

where $\Upsilon(T)$ is a thermodynamic function which evidently measures the "rigidity" of an isotropic system under an imposed phase "twist." The approximate proportionality of $\Delta\mathcal{F}$ to $\langle\langle \nabla\varphi \rangle\rangle^2$ should become exact as $\langle\langle \nabla\varphi \rangle\rangle \to 0$; but for fixed θ we see from (2.3) that this corresponds to $L(\Omega) \to \infty$. Accordingly, from (2.4) and (2.2) an explicit formula

for this "*helicity modulus*" is

$$\Upsilon(T) = \lim_{A(\Omega),L(\Omega)\to\infty} [L(\Omega)/2\theta^2 A(\Omega)][\mathcal{F}(T;\Omega,\mathcal{W}_\theta^{+-}) - \mathcal{F}(T;\Omega,\mathcal{W}_\theta^{++})] . \quad (2.5)$$

Note that this definition is phrased entirely in terms of well-defined equilibrium free energies. For a Bose superfluid one may now identify $(\hbar/m)\langle\langle\nabla\varphi\rangle\rangle$, where m is the particle mass, as v_s the *velocity of superflow*. This identification is, as usual, a direct consequence of wave mechanics: If $\psi_0(\vec{r})$ is a real normalized single-particle wave function, the wave function

$$\psi(r) = e^{i\vec{q}\cdot\vec{r}}\psi_0(\vec{r}) , \quad (2.6)$$

which has a "twisting" phase angle

$$\varphi(\vec{r}) = \vec{q}\cdot\vec{r} \text{ with } \nabla\varphi = \vec{q}, \quad (2.7)$$

describes a wave packet moving with velocity

$$\vec{v}_s = \langle\psi|\vec{p}/m|\psi\rangle = -i(\hbar/m)\int \psi^*(\vec{r})\nabla\psi(\vec{r})\,d\vec{r} = (\hbar/m)\vec{q} = (\hbar/m)\langle\langle\nabla\varphi\rangle\rangle . \quad (2.8)$$

Thus, in a superfluid the increment $\Delta\mathcal{F}$ represents the additional free energy of the superflow, which may be written as a kinetic energy, viz.,

$$\Delta\mathcal{F} = \tfrac{1}{2}\rho_s(T)v_s^2 V(\Omega) , \quad (2.9)$$

where $\rho_s(T)$ is the *superfluid density* as normally defined phenomenologically.[11] By comparison we see that

$$\rho_s(T) = (m/\hbar)^2\Upsilon(T) . \quad (2.10)$$

Thus in distinction to most previous treatments,[11,12] we have constructed a microscopic definition of $\rho_s(T)$, which requires only the calculation of the standard partition function of a system under specified boundary conditions. Other definitions, in terms of particular correlation functions, add to one's insight and can be advantageous, even though, in principle, they call for the calculation of more detailed properties of the system. On the other hand, some definitions have involved the construction of special ensembles (of "fixed" superfluid velocity, etc.) which seem difficult to define explicitly in purely microscopic statistical mechanical terms.

It must, however, be stressed that the existence of a nonvanishing $\rho_s(T)$, as we have defined it here, does *not* ensure the stability of a macroscopic state of superflow with *finite* superfluid velocity v_s [of order $(\hbar/ma)$, where a is a microscopic length]. Indeed, our definition, in effect, entails the limit $v_s \to 0$. Of course, quite generally, a pure equilibrium calculation, as involved in (2.5), can say nothing about the dynamical stability or, rather metastability,[13,14] of a state of finite superflow (or finite helicity). When the order parameter $\underline{\Psi}(\vec{r})$ is a two-component vector ($n=2$), as for superfluid helium, it is certainly plausible that a state with an imposed finite twist per unit length, i.e., $\Delta\theta \propto L(\Omega)$, will be strongly metastable, since there is no continuous way for it to "untwist" unless the magnitude of $\langle\underline{\Psi}(\vec{r})\rangle$ goes to zero at some point. But this represents a large deviation from local equilibrium, corresponding essentially to the incipient nucleation of a vortex ring,[14] and hence should be very improbable. The stability of superflow on experimental time scales can thus be established only by time-dependent calculations which estimate the rate of such fluctuations and their effectiveness in "untwisting" the order parameter.

On the other hand, by considering the topology of a sphere (in contrast to that of a circle) one soon sees[15] that a three-component ($n=3$) order parameter $\langle\underline{\Psi}(\vec{r})\rangle$ can reduce its over-all helicity by continuous deformations (local changes of direction) that leave the magnitude of $\langle\underline{\Psi}\rangle$ everywhere unchanged. Thus even though (2.5) provides a definition of the helicity modulus for this case, it seems unlikely that a state of finite helicity in, say, a fully isotropic ferromagnet would be dynamically stable on any macroscopic time scale.

The basic approach we have taken here to define the helicity modulus may also be applied[2,16] to an *anisotropic* system with a scalar ($n=1$) order parameter. In that case the boundary potentials $\mathcal{W}$ cannot depend on a continuous angle θ but can only be of the form $\mathcal{W}_+$ favoring an "up" or +phase, or $\mathcal{W}_-$, favoring a "down" or −phase. The matching end-wall conditions (+,+) and (−,−) then produce a uniform system consisting of the +phase or the −phase, respectively. On the other hand, the (+,−) or (−,+) combination can no longer produce a twist; rather, in these circumstances, the system must develop a localized inhomogeneity, i.e., an *interface*, which separates the +phase at one end of the cylinder Ω from the −phase at the other end. The position of this interface will be essentially arbitrary—it will fluctuate—but the incremental free energy $\Delta\mathcal{F}$ associated with the change of boundary conditions will be constant and can be written[16] $\Sigma(T)A(\Omega)$. This then provides an analagous definition of the *interfacial free energy* (per unit area) or *surface tension*, namely,

$$\Sigma(T) = \lim_{V(\Omega),A(\Omega)\to\infty} [A(\Omega)]^{-1}[\mathcal{F}(T;\Omega,\mathcal{W}_{+-}) - \tfrac{1}{2}\mathcal{F}(T;\Omega,\mathcal{W}_{++}) - \tfrac{1}{2}\mathcal{F}(T;\Omega,\mathcal{W}_{--})] . \quad (2.11)$$

One might wonder if the possible presence of additional interfaces along the cylinder might not affect this result. In principle, configurations with 3, 5, 7, ... interfaces are included in (2.11), but in the limit $A(\Omega)\to\infty$ such configurations are thermodynamically highly improbable and so do not contribute.

A similar remark applies to fluctuations in an isotropic system which can produce extra twists of 2π in addition to the imposed twist of 2θ.

In practice the definition (2.5) may be difficult to employ owing to the inhomogeneities inevitably introduced by the presence of walls. An alternative approach[2,8] considers the difference between the free energy $\mathfrak{F}^{\tau}(T;\Omega,L)$ of a system, in a domain, Ω, with *periodic* ($\tau=0$) boundary conditions, $\underline{\Psi}(x,y,z)=\underline{\Psi}(x,y,z+L(\Omega))$, applied to the order function and that of the same[17] system, but with *antiperiodic* ($\tau=\frac{1}{2}$) boundary conditions along the axis, i.e., $\underline{\Psi}(x,y,z)=-\underline{\Psi}(x,y,z+L(\Omega))$.

The periodic boundary conditions yield a uniform system with no "twist." But the antiperiodic boundary conditions effectively force a phase twist of $2\theta=\pm\pi$ over the length L. Hence if we introduce the over-all free energy densities,

$$F^{\tau}(T;L)=\mathfrak{F}^{\tau}(T;\Omega,L)/V(\Omega)\ , \qquad (2.12)$$

and first take the limit $A(\Omega)\to\infty$, we obtain the alternative and usually more convenient definition,

$$\Upsilon(T)=\lim_{L\to\infty}(2L^2/\pi^2)[F^{1/2}(T;L)-F^{0}(T;L)]. \qquad (2.13)$$

This prescription has, in fact, been used recently[8,18,19] to calculate the helicity modulus for the spherical model[20] and for the ideal Bose gas. For these models the periodic and antiperiodic boundary conditions are applied, respectively, to the lattice spins $s(\vec{r})$ and to the single-particle wave functions $\psi_k(\vec{r})$ used to construct the underlying Fock space. Both models were studied for general dimensionality d. For $d>2$ there is a bulk critical temperature $T_{c,d}$ and, as expected, $\Upsilon(T)$ is found to vanish identically for all $T>T_{c,d}$. Beneath $T_{c,d}$, the helicity modulus for the spherical model on a hypercubical lattice (spacing a) with nearest-neighbor exchange (of strength J) is found to be[18]

$$\Upsilon(T)=(Ja^2/m^2)[M_0(T)]^2\ ,$$
$$=Ja^2[1-(T/T_{c,d})] \quad (T\leq T_{c,d})\ , \qquad (2.14)$$

where $M_0(T)$ is the spontaneous magnetization (per spin) and m is the magnetic moment per spin. The corresponding expression for the ideal Bose gas is[19]

$$\Upsilon(T)=k_B T\Lambda^2 n_0(T)/2\pi\ ,$$
$$=\rho[1-(T/T_{c,d})^{d/2}] \quad (T\leq T_{c,d})\ , \qquad (2.15)$$

where $n_0(T)$ is the density of the condensate in the bulk system, ρ is the over-all number density, and $\Lambda=(h^2/2\pi mk_BT)^{1/2}$ is the thermal de Broglie wavelength.

Both these exact model calculations yield Υ proportional to $[\Psi^0(T)]^2$, the square of the spontaneous order. [For the ideal Bose system, (2.10) reduces simply to $\rho_s(T)=n_0(T)$.] Although such a conclusion is also suggested by a naive application of the strain arguments leading to the general definition of $\Upsilon(T)$, it is *not* justified in general. However, the model calculations do correctly predict that $\Upsilon(T)$ vanishes rather rapidly as $T\to T_c$. An appropriate exponent υ may be defined through

$$\Upsilon(T)\sim|(T-T_c)/T_c|^{\upsilon} \text{ as } T\to T_c- \ . \qquad (2.16)$$

From (2.14) and (2.15) we find, for the spherical model and ideal Bose gas, that

$$\upsilon=2\beta=1\ . \qquad (2.17)$$

The only experimentally measured value of ν, namely, in helium[21] yields $\upsilon\simeq\frac{2}{3}$. This value is close to 2β if $\beta\simeq\frac{1}{3}$ as is found in most other systems; for helium, however, Ψ^0 and hence β has so far proved unobservable. A more definite and general prediction for υ is possible, however, on the basis of the scaling theory developed in Sec. IV. Before considering this we digress somewhat to discuss some related features of the behavior of ordered isotropic systems.

III. DECAY OF ORDER IN ISOTROPIC SYSTEMS

It is clear from our derivation that, in a sense, $\Upsilon(T)$ is the inverse of a susceptibility measuring a response to a suitable helical or twisting field. By standard arguments (see, e.g., Refs. 2 and 5) $\Upsilon(T)$ might then be related to the fluctuations of a *microscopic phase variable* $\phi(\vec{r})$ and to a corresponding *phase-phase correlation function* $\langle\hat{\phi}(\vec{r})\hat{\phi}(\vec{r}')\rangle$. Such an explicit microscopic characterization of Υ would be of both practical and theoretical utility in the further stydy of isotropic systems. However, a serious technical difficulty for quantum-mechanical particle or spin systems (which are of principal interest) is that it is difficult to give a completely satisfactory microscopic definition of the phase operator $\hat{\phi}(\vec{r})$ or its local gradient $\nabla\hat{\phi}(\vec{r})$. This problem has been discussed by many authors[11,12,22] and we will not enter further into it. We will, however, discuss a related problem concerning the decay of the order-parameter correlations in an ordered isotropic system.

First note[2,23] that in the absence of a symmetry breaking field, i.e., for $|\underline{\zeta}|\equiv\zeta_\sigma=0$, we should have

$$\lim_{r\to\infty}\Gamma_{\Psi\Psi}(\vec{r};T)=\lim_{r\to\infty}\langle\underline{\Psi}(0)\cdot\underline{\Psi}(\vec{r})\rangle$$
$$=\Gamma(\infty,T)=|\underline{\Psi}^0(T)|^2\ , \qquad (3.1)$$

in which $|\underline{\Psi}^0|$ is the spontaneous order. On the other hand, in the limit $|\underline{\zeta}|\to 0$ we can define the net order-order correlation function,

$$G(\vec{r};T)=\Gamma_{\Psi\Psi}(r,T)-\Gamma(\infty,T) \quad (\zeta=\zeta_\sigma+) \qquad (3.2)$$

which, of course, describes the fluctuations of $\underline{\Psi}(\vec{r})$ about the mean value $\underline{\Psi}$, which must be presumed to lie in a fixed direction (determined by $\underline{\zeta}$). Now among the fluctuations in $\underline{\Psi}(r)$ are those which result in nothing more than a small rotation of $\underline{\Psi}(\vec{r})$ over some more or less extended spatial region Ξ. Because of the isotropy of the Hamiltonian such fluctuations will be very probable: In the phenomenological language introduced above, the incremental fluctuation free energy may be written $\frac{1}{2}\Upsilon\langle\langle\nabla\varphi\rangle\rangle_{\Xi}^2 V(\Xi)$. On developing this picture either phenomenlogically (as in Sec. IV) or by microscopic calculations (see Refs. 11, 12, 22, and 24) one is led to the conclusion that $G(\vec{r};T)$ decays to zero very slowly, explicitly, as

$$G(\vec{r}, T) \sim 1/r^{d-2} \quad (r \to \infty \text{ for } T < T_c,\ d > 2). \tag{3.3}$$

[For $d \leq 2$ the fluctuations turn out to be so large as to destroy the order altogether (see below).] For particular models one can, in fact, prove rigorously an inequality of the form[25]

$$\hat{G}(\vec{k}, T) \geq c_0 a^{d-2} |\underline{\Psi}^0(T)|^2/k^2 \quad (ka \leq c_1) \tag{3.4}$$

where c_0 and c_1 are dimensionless parameters possibly depending on T. This supports (3.3) since it proves that $G(\vec{r}, T)$ cannot decay any more rapidly than $1/r^{d-2}$ whenever the spontaneous order $|\underline{\Psi}^0|$ does not vanish. In addition, this inequality can be used[26] when $d=2$ to prove that $|\underline{\Psi}^0|$ must *always* vanish, i.e., spontaneous order cannot occur in a two-dimensional isotropic system.

The difficulty with (3.3) and (3.4) comes when we attempt to define a finite correlation length $\xi(T)$ which will diverge at the critical point. Clearly, the usual definitions of ξ based on moments[5,23] of $G(\vec{r}, T)$ fail, yielding $\xi \equiv \infty$ for all $T < T_c$. A way out of this dilemma has been suggested by Halperin and Hohenberg.[27] The idea is to make $G(\vec{r})$ dimensionless by normalizing it with $\Gamma(\infty) = |\Psi^0|^2$, and then to fix the scale ξ_{iso} of the asymptotic decay by writing

$$G(\vec{r}, T) \approx |\Psi^0|^2 (\xi_{iso}/r)^{d-2} \text{ as } r \to \infty. \tag{3.5}$$

More generally this prescription may be written, up to an arbitrary numerical factor, as

$$[\xi_{iso}(T)]^{d-2} = \lim_{R\to\infty} \int_{|r|<R} G(\vec{r}, T)\, d\vec{r}/R^2 |\Psi^0|^2 . \tag{3.6}$$

This definition of ξ_{iso} is obviously special to the case in hand and makes sense only if (3.3) is rigorously correct (although it could be adapted to a different but *a priori* known power law).

Alternatively, since from (2.4), $\Upsilon(T)$ has the dimensions of (energy)/(length)$^{d-2}$, we may evidently define a "phase coherence length," or "helicity length" by

$$\Lambda^{(T)}(T) = [k_B T/\Upsilon(T)]^{1/(d-2)} \quad (d > 2) . \tag{3.7}$$

This clearly diverges to infinity as $T \to T_c$, with by definition an exponent

$$\nu^{(T)} = \upsilon/(d-2) , \tag{3.8}$$

which we distinguish from the correlation length exponent[5,23] ν. (See further below.) Recalling the identification of Υ with ρ_s, we find for a real ($d=3$) superfluid that

$$\Lambda^{(T)}(T) = (4\pi^2 k_B T/h^2)/\rho_s(T) . \tag{3.9}$$

This is essentially the definition of the 'correlation length' adopted without discussion, by Ferrel *et al.*[28] in their treatment of dynamic scaling. Both the definitions (3.6) and (3.7), of an effective correlation length in the ordered isotropic phase, are distinctly different in character from the standard definitions[5,23] of ξ which, however, still make sense as soon as the symmetry breaking field $\underline{\zeta}$ does not vanish. Hence we cannot, on *a priori* grounds, reasonably expect that $\xi(\zeta, T)$, defined for $\zeta \neq 0$, will approach $\xi_{iso}(T)$ or $\Lambda^{(T)}(T)$ in a continuous fashion, even though $G(\vec{r}, \zeta, T)$ is itself presumably continuous through $\underline{\zeta} = 0$.

If, however, we insist on the identification of $\Lambda^{(T)}(T)$ with $\xi(T)$, we may immediately conclude that the helicity modulus exponent is given by

$$\upsilon = (d-2)\nu \quad (d > 2) . \tag{3.10}$$

This is, in fact, one of the conclusions of the scaling treatment of the superfluid given by Josephson[6] *provided* one accepts the d-dependent, two-exponent or hyperscaling exponent relations[2,5,23] such as

$$d\nu = 2 - \alpha \quad \text{or} \quad 2 - \eta = d(\delta - 1)/(\delta + 1) . \tag{3.11}$$

On the other hand, (3.10) does not follow if these relations fail. We will now review Josephson's arguments in a setting somewhat more general than he originally employed.

IV. SCALING IN ISOTROPIC SYSTEMS

To explore further some of the questions raised above we now develop a more detailed scaling formulation. For simplicity we will restrict attention to the $n=2$ case where $\underline{\Psi} = (\Psi', \Psi'')$ is a two-component vector (or equivalently a complex number), as is relevant to superfluidity in a Bose system and to magnetic systems with planar isotropy (XY systems).

First we note that in the presence of an ordering field

$$\underline{\zeta} = \zeta \vec{n}_{\parallel} \quad (\zeta = |\underline{\zeta}| > 0) , \tag{4.1}$$

we may unambiguously distinguish the longitudinal and transverse (or parallel and perpendicular) components

$$\Psi_{\parallel} = \underline{\Psi} \cdot \underline{n}_{\parallel} \quad \text{and} \quad \Psi_{\perp} = \underline{\Psi} \cdot \underline{n}_{\perp} . \tag{4.2}$$

In the case where $\underline{\Psi}(r)$ is considered to be a complex quantal field $\psi(\vec{r})$, as for a Bose superfluid, we can take ζ real and then

$$\Psi_{\parallel} = \mathrm{Re}\psi = \tfrac{1}{2}(\psi^{\dagger} + \psi), \quad \Psi_{\perp} = \mathrm{Im}\psi = \tfrac{1}{2}i(\psi^{\dagger} - \psi). \tag{4.3}$$

Correspondingly, the correlation function tensor will have two nonzero (diagonal) components

$$G_{\parallel}(r, \zeta, T) = \langle \Psi_{\parallel}(0)\Psi_{\parallel}(\vec{r}) \rangle, \quad G_{\perp}(\vec{r}, \zeta, T) = \langle \Psi_{\perp}(\vec{0})\Psi_{\perp}(\vec{r}) \rangle. \tag{4.4}$$

When $\zeta = 0$ above T_c we may conclude from the isotropy of the Hamiltonian that

$$G_{\parallel}(\vec{r}, 0, T) \equiv G_{\perp}(\vec{r}, 0, T), \ (T > T_c) \sim e^{-(r/\xi_0)}/r^{(d-1)/2} \text{ as } r \to \infty , \tag{4.5}$$

where the second line is the expected asymptotic Ornstein–Zernike decay law.[5,23] The two parts of the correlation tensor should also be equal *at* the critical point but there is no need for the two components to agree below T_c in the limit $\zeta \to 0$. Indeed, various approximate arguments suggest they will be different, although in the quantum-mechanical case the possibility of a distinction when $\zeta = 0$ is not very clear on a microscopic basis.

A homogeneity or scaling hypothesis may now be introduced for both components of G by

$$G_{\parallel}(\vec{r}, \zeta, T) \approx D^{\parallel}_{\gtrless}(r|t|^{\nu}; \zeta/|t|^{\beta\delta})/r^{d-2+\eta} \quad \text{for } t = (T - T_c)/T_c \gtrless 0 , \tag{4.6}$$

as ζ and t approach zero; a similar definition for $D^{\parallel}_{<}$ with $t \leq 0$ applies. The exponents ν, β, δ, and η are defined in the standard way.[2,5,23] The pair of scaling functions $D^{\parallel}_{>}(x, y)$ and $D^{\parallel}_{<}(x, y)$ for $t > 0$ and $t < 0$, respectively, must satisfy matching conditions on the critical isotherm, i.e., in the limit $x \to 0$, $y \to \infty$ with $x^{\beta\delta}y^{\nu}$ ($\sim r^{\beta}\zeta^{\nu}$) positive and fixed These conditions (which we will not write explicitly[2]) are needed to ensure that $G_{\parallel}(\vec{r}, \zeta, T)$ is continuous and analytic *through* $T = T_c$ for all $\zeta > 0$. Furthermore, this basic requirement forces the *same* value of the correlation length exponent ν above and below T_c. One may postulate a similar form for $G_{\perp}$ in terms of a different pair of matching scaling functions $D^{\perp}_{>}(x, y)$ and $D^{\perp}_{<}(x, y)$ but the same set of exponent values. Then by virtue of (4.5) we should have

$$D^{\parallel}_{>}(x, 0) = D^{\perp}_{>}(x, 0) \sim e^{-x} x^{d-2+\eta-(d-1)/2} \quad \text{as } x \to \infty . \tag{4.7}$$

In this relation we have assumed, for simplicity, that r is measured in units chosen so that the "true correlation length" [specifying the exponential decay in (4.5)[5,23]] is given by $\xi_0(T) \sim t^{-\nu}$ for $t > 0$.

The uniqueness of the critical point behavior is ensured by

$$\lim_{x, y \to 0 \ <} D^{\parallel}_{>}(x, y) = \lim_{x, y \to 0 \ <} D^{\perp}_{>}(x, y) = D_0 . \tag{4.8}$$

This is essentially as far as general considerations can take us. To proceed further and, in particular, to examine the behavior of the helicity modulus we will, following Josephson,[6] use a phenomenological or "hydrodynamic" description of the free energy changes in a nonuniform system. Thus we suppose that $\tilde{\underline{\Psi}}(\vec{r})$ is a slowly varying coarse-grained average of $\underline{\Psi}(\vec{r})$ and postulate the existence of a total free energy functional of the form

$$\mathfrak{F}(\zeta, T; \tilde{\underline{\Psi}}(\vec{r})) = \int_{\Omega} d\vec{r}\, A(T; |\tilde{\underline{\Psi}}(\vec{r})|) - \int_{\Omega} d\vec{r}\, \zeta(\vec{r})\tilde{\Psi}_{\parallel}(\vec{r}) + \tfrac{1}{2} b_{\parallel}(T) \int_{\Omega} d\vec{r} |(\nabla\tilde{\underline{\Psi}})_{\parallel}|^2 + \tfrac{1}{2} b_{\perp}(T) \int_{\Omega} dr |(\nabla\tilde{\underline{\Psi}})_{\perp}|^2 . \tag{4.9}$$

which embodies the isotropy. Such a functional is expected to provide an asymptotically valid description in the hydrodynamic region below T_c: that is, for T fixed, less than T_c, wave numbers $ka \ll 1$ and ζ not too large. (For large ζ the inhomogeneity coefficients $b_{\parallel}$ and $b_{\perp}$ should be allowed to depend on ζ as well as on T.)

The equilibrium order parameter $\Psi_{eq}(\zeta, T)$ is to be found, as usual, by minimization of $\mathfrak{F}\{\Psi\}$. We will assume that $A(T, \Psi)$, which for consistency must just be the canonical free energy density for a uniform system,[29] has an expansion

$$A(T, \Psi) = \sigma^{-1}|\Psi - \Psi^0(T)|^{\sigma}\{A_0(T) + A_1[\Psi - \Psi^0(T)] + \cdots\}, \tag{4.10}$$

where the value of the new exponent σ will be discussed. Then, in a uniform field, the equation of state is

$$\zeta = \left(\frac{\partial A}{\partial \Psi}\right)_{eq} = A_0(T)|\Psi_{eq} - \Psi^0(T)|^{\sigma-1} + \cdots . \tag{4.11}$$

From this we see that $\Psi^0(T)$ must be the spontaneous ($\zeta \to 0$) order – as the notation indicates. The low field isotherm is thus

$$\Psi_{eq}(\zeta, T) = \Psi^0(T) + X_0(T)\zeta^{\iota} + \cdots, \quad \iota = 1/(\sigma - 1) \tag{4.12}$$

where $X_0 = A_0^{-\iota}$; the longitudinal susceptibility is hence

$$\chi_{\parallel}(\zeta, T) = \left(\frac{\partial \Psi_{\parallel}}{\partial \zeta}\right)_T \approx \iota X_0(T)/\zeta^{1-\iota}; \quad (\zeta \to 0). \tag{4.13}$$

The simplest and the traditional phenomenological assumption for the new exponents is $\sigma = 2$, $\iota = 1$.

In that case the longitudinal susceptibility $\chi_{\parallel}(\zeta)$ has a finite zero-field (or initial) value below T_c. This is generally the appropriate situation for a scalar ($n=1$) order parameter, i.e., for the case of axial anisotropy. On the other hand, the spin-wave approximation and other improved microscopic calculations,[24,30] indicate $\sigma=3$ and $\iota=\frac{1}{2}$ for three dimensions. This means that $\chi_{\parallel}(\zeta)$ diverges as $\zeta\to 0$, the $\Psi(\zeta)$ isotherm thus having an infinite initial slope *below* T_c (just as it always does *at* T_c). Indeed there is some experimental evidence for reality of this effect in near-isotropic ferromagnets.[31]

Further evidence on the values of σ and ι is provided by the spherical model,[20] which, for the present purpose, should properly be regarded as an isotropic n-vector model in the limit $n\to\infty$.[32] This yields the exact result

$$\sigma=d/(d-2) \text{ and } \iota=\tfrac{1}{2}d-1 \text{ for } 2<d\leq 4 \qquad (4.14)$$

with additional logarithmic factors $\ln\zeta$, present at $d=4$. For $d>4$ the values stick at $\sigma=2$, $\iota=1$.

For $d=3$ the formulas (4.14) agree with the spin-wave calculations and, as a matter of fact, the spin-wave approximation for general d yields the same general expression. The recent renormalization-group calculations of the $\epsilon=4-d$ and $1/n$ expansions about the spherical model by Brézin, Wallace, and Wilson[33] strongly suggest that (4.14) should apply for all $n>1$ as a rather direct reflection of the continuous rotational symmetry.

Although the low-field behavior of $\chi_{\parallel}$ is thus not completely settled there can be no doubt about the behavior of the transverse susceptibility since this follows rigorously from the observation that the free energy can depend only on $|\underline{\Psi}|$. On putting $\underline{\zeta}=(\zeta_{\parallel},\delta\zeta_{\perp})$ this leads to

$$\chi_{\perp}(\zeta,T)=\left(\frac{\partial\Psi_{\perp}}{\partial\zeta_{\perp}}\right)_T=\Psi_{eq}(\zeta,T)/\zeta\ . \qquad (4.15)$$

The divergence as $\zeta\to 0$ and $\Psi_{eq}\to\Psi^0(T)$ simply represents the freedom of rotation of the spontaneous order vector $\underline{\Psi}$ in zero field. (The expression for $\chi_{\perp}$ is also reproduced correctly by the spin-wave and related approximations.[24,30])

To study the momentum-dependent transverse susceptibility $\hat{\chi}_{\perp}(\vec{k},\zeta,T)=\hat{\chi}_{\perp}$ or, equivalently, the Fourier transform

$$\hat{G}_{\perp}(\vec{k},\zeta,T)=\int e^{i\vec{k}\cdot\vec{r}}\,G_{\perp}(\vec{r},\zeta,T)\,d\vec{r}, \qquad (4.16)$$

we calculate the increment in free energy associated with the establishment of a small transverse oscillation in $\underline{\tilde{\Psi}}(\vec{r})$, namely,

$$\underline{\tilde{\Psi}}(\vec{r})=(\Psi_{eq},0)\Rightarrow(\Psi_{eq},\sqrt{2}\,\delta\Psi_{\perp\vec{k}}\cos\vec{k}\cdot\vec{r}). \qquad (4.17)$$

From (4.9) and (4.15) with

$$(\nabla\tilde{\Psi})_{\parallel}=\nabla\tilde{\Psi}=0 \text{ and } (\nabla\tilde{\Psi})_{\perp}=\nabla\tilde{\Psi}_{\perp}=-\sqrt{2}\vec{k}\,\delta\Psi_{\perp\vec{k}}\sin(\vec{k}\cdot\vec{r}), \qquad (4.18)$$

we find

$$\Delta\mathcal{F}=\tfrac{1}{2}[(\zeta/\Psi_{eq})+b_{\perp}(T)k^2]\,V_{\Omega}(\delta\Psi_{\perp\vec{k}})^2. \qquad (4.19)$$

By the principle of equipartition[34] the mean free energy associated with such a fluctuation is $\frac{1}{2}k_BT$; in the standard way[2,5,23] we hence conclude that

$$\hat{G}_{\perp}(\vec{k},\zeta,T)\approx k_BT\hat{\chi}_{\perp}(\vec{k},\zeta,T)\approx V_{\Omega}\langle(\delta\Psi_{\perp\vec{k}})^2\rangle$$

$$\approx\frac{k_BT}{b_{\perp}(T)}\frac{1}{k^2+\kappa_{\perp}^2}\ , \qquad (4.20)$$

where

$$\kappa_{\perp}^2(\zeta,T)=1/\xi_{\perp}^2(\zeta,T)=\zeta/\Psi_{eq}(\zeta,T)b_{\perp}(T),$$
$$\approx\zeta/\Psi^0(T)b_{\perp}(T)\ . \qquad (4.21)$$

The result for $\hat{\chi}_{\perp}(\vec{k})$ reduces to the rigorous expression (4.15) when $k\to 0$. Inverting the Fourier transform for the case $d=3$ yields

$$G_{\perp}(\vec{r},\zeta,T)\approx\frac{k_BT}{b_{\perp}(T)}\frac{e^{-\kappa_{\perp}r}}{4\pi r}\quad(r\to\infty,\ t<0) \qquad (4.22)$$

which replaces the previous Ornstein–Zernike prediction (4.5), which should be valid for $T>T_c$. More generally the decay in *zero* field is predicted to be the pure power law

$$G_{\perp}(\vec{r})\approx A_d/b_{\perp}(T)r^{d-2}\ , \qquad (4.23)$$

as anticipated in Sec. III.

The expressions (4.20) and (4.21) are confirmed by microscopic calculations which should be valid away from T_c. For example, they follow from random-phase and spin-wave approximations for the isotropic Heisenberg model. More systematic approaches as developed by Stinchcombe *et al.* and Vaks, Larkin, and Pikin,[24] also contain these results in the low-temperature regime.

We may make a completely analagous calculation for the longitudinal correlation function by considering the fluctuation

$$\tilde{\underline{\Psi}}(r)=(\Psi_{eq},0)\Rightarrow(\Psi_{eq}+\sqrt{2}\,\delta\Psi_{\parallel\vec{k}}\cos\vec{k}\cdot\vec{r},0). \qquad (4.24)$$

This yields an expression for $\hat{G}_{\parallel}(\vec{k},\zeta,T)$ precisely like (4.20) but with $b_{\parallel}(T)$ replacing $b_{\perp}(T)$ and $\kappa_{\perp}^2$ replaced by

$$\kappa_{\parallel}^2(\zeta,T)=1/\xi_{\parallel}^2(\zeta,T)=\chi_{\parallel}^{-1}(\zeta,T)/b_{\parallel}(T)$$
$$\approx\zeta^{1-\iota}/\iota X_0(T)b_{\parallel}(T). \qquad (4.25)$$

It follows from this that $\hat{G}_{\parallel}(\vec{k},\zeta,T)$ reduces correctly to $\chi_{\parallel}$ when $k\to 0$. Again, in the zero-field limit $G_{\parallel}(\vec{r},0,T)$ decays only as $1/r^{d-2}$ [see (4.23)] and differs from $G_{\perp}(\vec{r},0,T)$ only insofar as $b_{\parallel}$ may differ from $b_{\perp}$.

On the other hand, microscopic calculations for

$G_{\parallel}$ are not as direct as for $G_{\perp}$, and spin-wave arguments and more systematic treatments[24] lead, for fixed $T < T_c$, to the *distinct* form[35]

$$G_{\parallel}(\vec{r}, \zeta, T) \sim [G_{\perp}(\vec{r}, \zeta, T)]^2/(\Psi^0)^2 \quad \text{as } r \to \infty, \quad (4.26)$$

where the factor $(\Psi^0)^2$ is inserted for dimensional reasons. For $d = 3$ this leads to

$$G_{\parallel}(\vec{r}, \zeta, T) \sim e^{-2\kappa_{\perp} r}/r^2 \quad (d = 3) \quad (4.27)$$

which is clearly *inconsistent* with the phenomenological results, although on integration it leads via the susceptibility/fluctuation relation to (4.13) with $\iota = \frac{1}{2}$. More generally (4.26) implies $\iota = \frac{1}{2}d - 1$ for $d < 4$, in agreement with (4.14). [For $d = 4$ it yields $\chi_{\parallel}(\zeta, T) \sim \ln\zeta^{-1}$ as $\zeta \to 0$.] These discrepancies in the predicted behavior of $G_{\parallel}(\vec{r})$ cast doubts on the validity of the phenomenological approach, even in the long-wavelength limit below T_c where it is normally expected to apply. On the other hand, to discuss the helicity modulus, to which we now turn, no appeal need be made to the behavior of the longitudinal correlation function; the phenomenological description will be needed only for the transverse fluctuations.

V. CRITICAL BEHAVIOR OF HELICITY MODULUS

To discuss the critical behavior of the helicity modulus $\Upsilon(T)$ and, in particular, to relate its exponent υ to other critical exponents we will first base the argument on the phenomenological approach expounded in Sec. IV. To this end we take $\zeta = 0$; then the components $(\nabla\tilde{\underline{\Psi}})_{\parallel}$ and $(\nabla\tilde{\underline{\Psi}})_{\perp}$ in the free-energy functional (4.9) must be taken relative to the *local* direction of $\tilde{\underline{\Psi}}(\vec{r})$. If we introduce polar coordinates or a complex-number representation, i.e.,

$$\tilde{\underline{\Psi}}(\vec{r}) = [|\tilde{\Psi}(\vec{r})|, \varphi(\vec{r})], \quad \Psi(\vec{r}) = |\tilde{\Psi}(\vec{r})| e^{i\varphi(\vec{r})}, \quad (5.1)$$

we find

$$(\nabla\tilde{\underline{\Psi}})_{\parallel} = \nabla|\tilde{\Psi}| \quad \text{and} \quad (\nabla\tilde{\underline{\Psi}})_{\perp} = \tilde{\Psi}\nabla\varphi . \quad (5.2)$$

Then if we establish a twist in the order parameter, the uniform solution

$$|\langle\tilde{\Psi}(\vec{r})\rangle| = \Psi_{eq} = \Psi^0(T), \quad \varphi(\vec{r}) = \vec{q}\cdot\vec{r}, \quad \nabla\varphi = \vec{q}, \quad (5.3)$$

will minimize the total free energy. The resulting incremental free energy is found to be

$$\Delta\mathcal{F} = \tfrac{1}{2} b_{\perp}(T)(\Psi^0)^2 q^2 V_{\Omega} . \quad (5.4)$$

On comparison with (2.4) which defines the helicity modulus, we obtain Josephson's identification[6]

$$\Upsilon(T) = [\Psi^0(T)]^2 b_{\perp}(T) = [\Psi^0(T)]^2 \lim_{k \to 0} k_B T/k^2 \hat{\chi}_{\perp}(\vec{k}, 0, T). \quad (5.5)$$

It should be noted that this does not really amount to a microscopic definition of $\Upsilon(T)$ since it is not clear, in general, how one can actually distinguish $\hat{\chi}_{\perp}$ from $\hat{\chi}_{\parallel}$, in a strictly vanishing field.

We may now compare the phenomenological expressions for $G_{\perp}(\vec{r}, \zeta, T)$ and $G_{\parallel}(\vec{r}, \zeta, T)$ valid for $r \to \infty$, small ζ, $T < T_c$, with the general scaling forms (4.6). Thus in order to reproduce (4.20) to (4.23) we must have

$$D_{\zeta}^{\perp}(x, y) \approx D_{\infty}^{\perp} x^{\eta}[1 - f_{\perp} x y^{1/2} + \cdots] \quad \text{as } x \to \infty \text{ with } yx^2 \to 0, \quad (5.6)$$

where $f_{\perp}$ is a constant. This conclusion is valid for all d. But then, for $r \to \infty$ and $\zeta r^2 \to 0$, the scaling form gives

$$G_{\perp}(\vec{r}, \zeta, T) \approx D_{\infty}^{\perp} |t|^{\eta\nu} r^{-d+2}[1 - f_{\perp} r \zeta^{1/2} |t|^{\nu - \beta\delta/2} + \cdots]. \quad (5.7)$$

Comparison with (4.22) and (4.23) leads at once to the crucial identification

$$b_{\perp}(T) \approx (k_B T/D_{\infty}^{\perp}) |t|^{-\eta\nu} \quad (t \to 0-) \quad (5.8)$$

which reveals the T dependence of $b_{\perp}$. [In the scalar ($n = 1$) case the equivalent result, namely, $b = \xi^2/\chi \sim |t|^{-\eta\nu}$ follows from the phenomenological approach *without* a direct appeal to scaling.] On substitution in (5.5) we obtain the central result

$$\Upsilon(T) \sim |\Psi^0(T)|^2 |t|^{-\eta\nu} \sim |t|^{2\beta - \eta\nu}, \quad (5.9)$$

which implies Josephson's relation for the exponent of the helicity modulus (or superfluid density), namely,

$$\upsilon = 2\beta - \eta\nu = 2\beta + \gamma - 2\nu = 2 - \alpha - 2\nu. \quad (5.10)$$

In the second part of this relation we have used the standard (d-independent) scaling relations which are implied by the scaling forms;[2,5,23] one might, in principle, write ν' and α' in place of ν and α but, as mentioned, there can be a distinction only if the scaling functions below and above T_c do *not* match properly across $T = T_c$, $\zeta > 0$.

Josephson's relation may be checked for the ($n = 2$) ideal Bose gas and for the spherical model ($n \to \infty$)[8,18,19,36] in general dimensionality $d > 2$. In all cases one has $\eta = 0$, $\beta = \frac{1}{2}$ and, as mentioned in Sec. II, $\nu = 1$ in agreement with the first line of (5.10). The second line, involving the thermodynamic and correlation exponents α, β, γ, and ν, also holds quite generally in these models.

For superfluid helium Josephson's relation demonstrates that the correlation length exponent ν can be found from observations on the specific heat, which yield $\alpha \simeq 0.00$ (corresponding to a close to logarithmic divergence),[37] combined with experiments to measure $\rho_s(T)$ which give[21] $\upsilon \simeq 0.67$. We conclude that ν is close to $\frac{2}{3}$. This method of estimating ν for helium is one of the few methods

available; another is via finite-size effects on the specific heat (see Ref. 8). The exponents β, γ, or η, however, remain unknown.

The exponent of the coefficient of $r\xi^{1/2}$ in (5.7) may be transformed via the standard exponent relations[2,23] as

$$\nu - \tfrac{1}{2}\beta\delta = \tfrac{1}{2}[2-\eta)\nu + \eta\nu - (\gamma+\beta)] = \tfrac{1}{2}(\eta\nu - \beta). \quad (5.11)$$

This agrees precisely with the phenomenological analysis [see (4.21) and (4.22)] according to which the coefficient should be proportional to $[\Psi_0 b_\perp]^{-1/2} \sim |t|^{-(\beta-\eta\nu)/2}$.

It is clearly of interest to attempt to reduce Josephson's relation (5.10) by using the hyperscaling or d-dependent exponent relations[2,23] such as (3.11). If we use $d\nu = 2-\alpha$ we find

$$\upsilon = (d-2)\nu \quad (5.12)$$

or, in the real three-dimensional world, $\upsilon = \nu$. Now this is precisely the relation (3.9) obtained by identifying the phase coherence or helicity length $\Lambda^{(T)}(T)$ with the correlation or scaling length $\xi(T)$. Conversely, the postulate $\Lambda^{(T)}(T) \sim \xi(T) \sim |t|^{-\nu}$ would lead to hyperscaling relations for isotropic systems. The fact that for helium one observes $d\nu = \frac{1}{2}d(2-\alpha-\upsilon) \simeq \frac{1}{2}3(2-\frac{2}{3}) = 2 \simeq 2-\alpha$ must be viewed as strong evidence in favor of hyperscaling in three-dimensional isotropic systems. However, the exact results[18,19,36] for the spherical model and ideal Bose gas when $d>4$, for which $\upsilon = 1$ and $\nu = \frac{1}{2}$, cast doubt on its general validity. In addition one should recall the well-known difficulties in the three-dimensional Ising model where numerical estimates[2,5,23] indicate $d\nu > 1.91$ but $2-\alpha < 1.88$.

It is instructive at this point to compare the derivation of Josephson's relation (5.10) with Widom's arguments[38] for the critical behavior of the surface tension in the scalar ($n=1$) situation (e.g., Ising ferromagnet or lattice gas). Widom's analysis leads to the very similar expressions

$$\mu = 2\beta + \gamma - \nu = 2 - \alpha - \nu$$
$$= 2\beta - \eta\nu - \nu, \quad (5.13)$$

for the surface-tension exponent μ, which is defined through

$$\Sigma(T) \sim |t|^{\mu}, \quad T \to T_c - . \quad (5.14)$$

Note that the equivalences in (5.13) again involve only the standard d-independent scaling relations. In fact, Widom employs a completely analogous phenomenological approach. However, an appeal to scaling is required only at a later stage, when it becomes essential to postulate a form for the local free-energy density *within* the two-phase region. Widom adopts a scaled form equivalent to

$$A(T, \langle\Psi\rangle) = |t|^{2-\alpha}\mathfrak{A}(\langle\Psi\rangle/|t|^{\beta}), \quad (5.15)$$

where, in the symmetric case (under $\zeta \to -\zeta$) the scaling function $\mathfrak{A}(y)$ has two equal minima corresponding to the two distinct phases with $\langle\Psi\rangle = \pm\Psi^0(T)$. Then on solving the minimization equations under boundary conditions which impose $\langle\Psi\rangle = +\Psi^0(T)$ at one end and $\langle\Psi\rangle = -\Psi^0(T)$ at the other end of the domain Ω [compare with (2.11)], an equilibrium interfacial profile $\Psi^{(\Sigma)}(\vec{r})$ is found. This is characterized by a definite interfacial thickness $\Lambda^0(T)$, which diverges as $|t| \to 0$ with an exponent $\frac{1}{2}[2-\alpha-2\beta+\eta\nu] = \nu$; in other words, one finds that $\Lambda^0(T)$ is proportional to the correlation length $\xi(T)$ [which is now definable in the standard way by moments, etc., because the correlation function $G(\vec{r})$ decays exponentially rapidly below T_c]. In parallel to (5.4) the incremental free energy associated with the interface is found to be determined by

$$\Delta\mathfrak{F} \propto \tfrac{1}{2}b(T)[\Psi^0/\Lambda^0(T)]^2\Lambda^0(T)A_\Omega$$
$$\propto A(T, \Psi^0)\Lambda^0(T)A_\Omega, \quad (5.16)$$

where A_Ω is the cross-sectional area of Ω. This expression may be understood physically by noticing that the incremental free energy comes only from a volume of dimensions $\Lambda^0(T)A_\Omega$ in which the inhomogeneity contributions, proportional to $\frac{1}{2}b(T)(\nabla\Psi)^2 \propto \frac{1}{2}b(T)(\Psi^0/\Lambda^0)^2$, and the bulk contributions, proportional to $A(T, \Psi^0)$, must be of the same magnitude.[38] Widom's relations (5.13) follow directly from (5.16).

If, at this stage, one accepts in addition one of the hyperscaling relations, say, $2-\alpha = d\nu$, one finds

$$\mu = (d-1)\nu, \quad (5.17)$$

which is the direct analog of (5.12). This result follows alternatively by defining, in parallel to (3.7), a "surface-tension length" or "phase coherence length"

$$\Lambda^{(\Sigma)}(T) = [k_B T/\Sigma(T)]^{1/(d-1)}. \quad (5.18)$$

If one then insists that this length should be proportional to the correlation length $\xi(T) \sim |t|^{-\nu}$ the relation (5.17) is immediate! Such an identification thus again forms a route to hyperscaling. For the two-dimensional Ising model the surface tension is known exactly[23] and *both* (5.13) and (5.17) are confirmed. More recently calculations of α, μ, and ν for the eight-vertex model[39-41] ($d=2$) also confirm (5.17) and the second relation in (5.13), although the exponents themselves are now functions of the interaction parameters. On the other hand, as already mentioned, the hyperscaling relations are in some doubt for three-dimensional Ising models (and also for some real systems).[23]

Returning to isotropic systems the phenomenological analysis has confirmed that there is, in fact,

no really direct way of defining $\xi(T)$ in zero field. Combination of (4.22) with Josephson's identification of $\Upsilon(T)$ leads to

$$G_\perp(\vec{r}, 0, T)/[\Psi_0(T)]^2 \approx c_d[\Lambda^{(T)}(T)/r]^{d-2} \quad (r \to \infty,\ T < T_c) \tag{5.19}$$

which relates to Hohenberg's definition of $\xi_{1\infty}(T)$ in (3.5) and (3.6). More generally the scaling relation gives

$$G_\perp(\vec{r},\ 0,\ T) \approx D_<^\perp[r/\xi(T)]/r^{d-2+\eta}\ , \tag{5.20}$$

so that even if $\eta = 0$ we would have

$$G_\perp(\vec{r}, 0, T) \approx D_\infty^\perp/r^{d-2} \text{ as } r \to \infty,\ T < T_c$$
$$\approx D_0/r^{d-2} \text{ at } T = T_c, \tag{5.21}$$

which means that the decay laws *at* and *below* T_c may well be distinct. In (5.20), $\xi(T)$ may be interpreted as the distance r at which the semimacroscopic hydrodynamic behavior, valid for $k\xi \ll 1$ or $r \gg \xi$, changes over to the distinctly microscopic or "critical" behavior appropriate for $k\xi \gg 1$ or $\xi \gg r \gg a$. (Such a matching of hydrodynamic and critical regimes, used originally in this context by Josephson,[6] has been used extensively by Ferrell *et al.*[28] and Halperin and Hohenberg[28,42] in their derivation of dynamic scaling.)

Finally, as regards the longitudinal fluctuations, we remark that the scaling hypothesis implies the limiting ($\zeta \to 0$) amplitude for $\chi_\parallel(\zeta, T)$ in (4.13) varies as

$$X_0(T) \sim |t|^{-(\iota\delta-1)\beta} \quad (t \to 0-). \tag{5.22}$$

(The exponent ι is not, of course, determined by the scaling arguments.) If the phenomenological form (4.20) is accepted also for $G_\parallel$ then, in parallel to (5.8), we find $b_\parallel(T) \sim |t|^{-\eta\nu} \sim b_\perp(T)$. Recall, however, that this form is not certain and that (4.26) might apply instead. [The amplitude (5.22) will still be given consistently, with $\iota = \frac{1}{2}d - 1$, if the hyperscaling relations are valid.]

VI. ALTERNATIVE DERIVATION OF HELICITY MODULUS EXPONENT

Section V completes our discussion of bulk scaling in an isotropic system. With regard to the helicity modulus the basic result is Josephson's relation (5.10) for the helicity modulus exponent υ. On the other hand, since, as shown in Sec. II, $\Upsilon(T)$ can be regarded as the expression of a finite-size correction to a bulk free energy, it is natural to ask if the relation for υ could not, alternatively, be derived from the scaling theory for finite-size effects developed recently.[7,8] To show that this is indeed so we make direct use of the second definition (2.13) of $\Upsilon(T)$ in terms of the free energies $F^0(T, L)$ and $F^{1/2}(T, L)$ of a system of length L with periodic ($\tau = 0$) and antiperiodic ($\tau = \frac{1}{2}$) boundary conditions, respectively. For large L we may rewrite the definition as

$$F^{1/2}(T, L) - F^0(T, L) \approx \tfrac{1}{2}(\pi/L)^2\Upsilon(T) + \cdots . \tag{6.1}$$

We now introduce the basic scaling postulates for the singular part of the free energy[7,8] in the form

$$F_s^0(T, L) \approx l^\omega X^0(l^\theta \mathring{t}), \tag{6.2}$$

$$F_s^{1/2}(T, L) \approx l^\omega X^{1/2}(l^\theta \mathring{t})\ , \tag{6.3}$$

as $\mathring{t} \to 0$ and

$$l = L/a \to \infty, \tag{6.4}$$

where a is a convenient microscopic length (such as the lattice spacing). The shifted temperature variable $\mathring{t}$ is defined by

$$\mathring{t} = t + \epsilon(l) = [T - T_c(l)]/T_c(\infty)\ , \tag{6.5}$$

where $T_c(l)$ is the finite-size critical temperature or, more generally, in case there is no sharp transition for finite L, a pseudocritical temperature.[7,8,18] The reduced shift $\epsilon(l)$ is expected to vary as[7,8]

$$\epsilon(l) = [T_c(\infty) - T_c(l)]/T_c(\infty) \approx b/l^\lambda \quad (l \to \infty), \tag{6.6}$$

where λ is a shift exponent. For the time being we will assume b and λ do not depend on the particular boundary conditions.

The exponent θ is now fixed by the postulate[7,8] that for large L the finite-size effects are determined only by the ratio $L/\xi(T)$, where $\xi(T)$ is the bulk ($L = \infty$) correlation length. This yields the relation

$$\theta = 1/\nu\ . \tag{6.7}$$

On the other hand, the exponent ω is determined by the requirement that when $L \to \infty$ the relations (6.2) and (6.3) both reproduce the correct bulk behavior, that is,

$$F_s^0(T, \infty) = F_s^{1/2}(T, \infty) = F_s(T) \sim |t|^{2-\alpha}\ , \tag{6.8}$$

where α is, as before, the specific-heat exponent.

Hence we must require,

$$X^0(x) \approx X_\infty x^{2-\alpha} \text{ as } x \to \infty \tag{6.9}$$

and

$$X^{1/2}(x) \approx X_\infty x^{2-\alpha} \text{ as } x \to \infty, \tag{6.10}$$

while

$$\omega = -(2-\alpha)\theta = -(2-\alpha)/\nu. \tag{6.11}$$

The expressions (6.9) and (6.10) represent the leading terms in the asymptotic expansions of the scaling functions about $x = \infty$ which, in turn, corresponds to $L = \infty$. Large L corrections to the bulk free energy must thus be described by higher-or-

der terms in these expansions. For example, the corresponding scaling function $X^1(x)$ for a system with two *free* surfaces should contain a term which will generate in $F^1(T, L)$ a surface free energy term proportional to $1/l$.[7,8] In the present case we are interested only in the difference of free energies entering (6.1). Accordingly, we may suppose that the higher-order terms in (6.9) and (6.10) combine to give, in leading order,

$$X^{1/2}(x) - X^0(x) \approx Y_\infty x^{-\phi} + \cdots \quad \text{as } x \to \infty, \tag{6.12}$$

where ϕ is some new exponent. On substituting in (6.2) and (6.3) we find, as $T \to T_c$,

$$F_s^{1/2}(T, L) - F_s^0(T, L) \approx Y_\infty t^{-\phi} l^{\omega - \theta\phi} + \cdots . \tag{6.13}$$

If this is compared with (6.1) we see that ϕ must be determined by

$$\omega - \theta\phi = -2 \quad \text{or} \quad \phi = 2\nu + \alpha - 2, \tag{6.14}$$

in order to produce a $1/L^2$ behavior as $L \to \infty$. On the other hand, the comparison then yields

$$\Upsilon(T) \approx (2/\pi^2) Y_\infty t^{-\phi}, \tag{6.15}$$

so that, in conclusion, the helicity-modulus exponent is

$$\upsilon = -\phi = 2 - \alpha - 2\nu. \tag{6.16}$$

This relation agrees precisely with the previous result (5.10). Note that the derivation has made no direct appeal to a phenomenological or hydrodynamic formulation or to an explicit scaling of the correlations.

As a matter of fact, however, this derivation is open to criticism since we used the assumption that the same t variable could be used for *both* periodic and antiperiodic boundary conditions. Had we been willing to make the "extended-scaling" assumptions[2,7,8] which replaces t in (6.2) and (6.3) by $t = [T - T_c(\infty)]/T_c(\infty)$ our argument would still have been valid. On the other hand, in practice, the shift amplitudes, at least, are found to depend significantly on the boundary conditions.[7,8,18] Accordingly, let us suppose that b_0 and $b_{1/2}$, and λ_0 and $\lambda_{1/2}$ are distinct, and postulate the separate expansions

$$X^0(x) \approx X_\infty x^{2-\alpha} + Y_\infty^0 x^{-\phi_0} + \cdots , \tag{6.17}$$

$$X^{1/2}(x) \approx X_\infty x^{2-\alpha} + Y_\infty^{1/2} x^{-\phi_{1/2}} + \cdots . \tag{6.18}$$

Then in place of (6.13), we obtain

$$F_s^{1/2}(T, L) - F_s^0(T, L) \approx X_\infty (2-\alpha) t^{1-\alpha} [b_{1/2} l^{-\lambda_{1/2}} - b_0 l^{-\lambda_0}] + Y_\infty^{1/2} l^{\omega - \theta\phi_{1/2}} t^{-\phi_{1/2}} - Y_\infty^0 l^{\omega - \theta\phi_0} t^{-\phi_0} + \cdots . \tag{6.19}$$

Now by comparison with (6.1) we see that λ_0 and $\lambda_{1/2}$ cannot be less than 2 (otherwise lower-order terms than $1/L^2$ would appear and dominate). If, on the other hand, we have $\lambda_0 > 2$ and $\lambda_{1/2} > 2$, then the previous conclusion remains valid since one or both of the last two terms in (6.19) must be of order l^{-2}.

The only new possibility arises if

$$\lambda_0 = 2 \quad \text{and/or} \quad \lambda_{1/2} = 2. \tag{6.20}$$

In that case the first two terms make a contribution to $\Upsilon(T)$ which varies as $|t|^{1-\alpha}$. This will be the only contribution if ϕ_0, $\phi_{1/2} > 2\nu + \alpha - 2$, in which case we find

$$\upsilon = 1 - \alpha. \tag{6.21}$$

However, if one still finds ϕ_0 or $\phi_{1/2}$ equal to $2\nu + \alpha - 2$, the helicity exponent will be

$$\upsilon = 2 - \alpha - \max\{2\nu, 1\}; \tag{6.22}$$

since 2ν normally exceeds unity this will once again reduce to the Josephson relation. We may comment in passing that for periodic boundary conditions the corrections to the bulk free energy may be expected to be exponentially small[7,8,18] in L so that $Y_\infty^0 \equiv 0$. Thus, as might reasonably be expected, the helicity modulus will be determined by the free energy of the system with antiperiodic boundary conditions.

In this last connection it is interesting to remark that Lebowitz and Onsager[43] some time ago discussed the fluctuation $\langle(\Delta \vec{P})^2\rangle$ in the total linear momentum of a system, concluding that with periodic boundary conditions $\langle(\Delta \vec{P})^2\rangle$ is proportional to the normal part of the density, $\rho_n = \rho - \rho_s$, while for a system confined by walls only the total density ρ enters.

VII. CONCLUSIONS AND SUMMARY

In the foregoing we have introduced the concept of a helicity modulus $\Upsilon(T)$ for the ordered phase of a d-dimensional isotropic system with a vector order parameter of $n = 2, 3, \ldots$ components. The helicity modulus measures the incremental free energy resulting from an imposed spatially varying "phase twist" on the order parameter. Explicitly the relation is

$$\Delta\mathcal{F} = \tfrac{1}{2}\Upsilon(T) \ll \nabla\varphi \gg^2 V, \tag{7.1}$$

where $\ll \nabla\varphi \gg$ is the mean gradient (assumed small) of the phase of the order parameter and V is the total system volume. For a Bose fluid, we saw that $\Upsilon(T)$ is simply proportional to the superfluid density, namely,

$$\rho_s(T) = (m/\hbar^2)\,\Upsilon(T)\ . \qquad (7.2)$$

In contrast to previous microscopic definitions for $\rho_s(T)$, we have given explicit formulas for the calculation of $\Upsilon(T)$, that involve only the equilibrium free energies (derived from standard partition functions) for the system with well-defined boundary conditions. The simplest expression for $\Upsilon(T)$ [Eq. (2.13)] involves *anti*periodic boundary conditions in addition to the customary periodic conditions.

As $T \to T_c$ the helicity modulus is expected to vanish continuously and an exponent υ was introduced according to

$$\Upsilon(T) \sim |(T - T_c)/T_c|^{\upsilon} = |t|^{\upsilon} \quad (T \to T_c -). \qquad (7.3)$$

Exact calculations for the ideal Bose gas and spherical model (corresponding to $n \to \infty$) give $\upsilon = 1$ in all dimensions $d > 2$. For these two models $\Upsilon(T)$ turns out to be merely proportional to the square of the spontaneous order $\Psi^0(T)$ [for the superfluid: $\rho_s(T) = n_0(T)$]. However, our formulation allows calculation for interacting systems. The most tractable models seem to be the Heisenberg and XY models at low temperatures and the weakly interacting Bose gas within the Bogoliubov type of approximation. We expect the difference between $\Upsilon(T)$ and $[\Psi^0(T)]^2$ to appear in the low-temperature behavior but confirmation of this must await the detailed analysis.

In Sec. III we indicated that owing to the slow decay of correlations in the ordered phase of an isotropic system, there is no obvious way of defining a correlation length $\xi(T)$ through the moments or exponential decay of correlations, as is done in an anisotropic ($n = 1$), or in an isotropic system above T_c, or in the presence of an ordering field. One can, indeed, define formally a phase coherence or helicity length

$$\Lambda^{(T)}(T) = [k_B T/\Upsilon(T)]^{1/(d-2)} \sim |t|^{-\nu^{(T)}} \quad (d > 2) \qquad (7.4)$$

but there are no very convincing reasons for identifying this with the correlation length or for equating its exponent $\nu^{(T)}$, which by definition satisfies

$$\upsilon = (d-2)\,\nu^{(T)}\ , \qquad (7.5)$$

to the correlation-length exponent ν. On the other hand, we showed in Sec. IV how the correlation exponent ν enters a general scaling, or homogeneity description of the correlation functions without the express need to define a correlation length *in* the ordered phase. This then clarifies Josephson's derivation of the exponent relation

$$\upsilon = 2\beta - \eta\nu = 2 - \alpha - 2\nu, \qquad (7.6)$$

which was reviewed and presented in the general context of a phenomenological approach to the fluctuations in an isotropic system. However, the inconsistencies of such a phenomenological approach with microscopic calculations of the longitudinal fluctuations were pointed out. Further theoretical exploration of this point would be worthwhile. The close conceptual analogy of Josephson's arguments with Widom's discussion of the surface-tension exponent μ was also explained.

If, but only if, the d-dependent or hyperscaling relations, such as $d\nu = 2 - \alpha$, are accepted, Josephson's relation (7.6) reduces to

$$\upsilon = (d-2)\,\nu. \qquad (7.7)$$

This would then justify the acceptance of the helicity length $\Lambda^{(T)}$ as an effective correlation length in the ordered phase, and would confirm the conjecture $\nu^{(T)} = \nu$. The experimentally observed value of $\alpha \simeq 0$ and $\upsilon \simeq \frac{2}{3}$ for superfluid helium lead, via (7.6), to $\nu \simeq \frac{2}{3}$ and hence confirm the hyperscaling relation (7.7).[44] However, this relation fails unmistakably for the ideal Bose gas in more than four dimensions and, in addition, the hyperscaling relation $d\nu = 2 - \alpha$ still seems open to serious question for the three-dimensional Ising model.

Finally, in Sec. VI we presented an alternative derivation of the original exponent relation (7.6) which made no appeal to a hydrodynamic or phenomenological approach. Rather, the argument was based directly on the definitions of Sec. II in terms of the free energies of systems with finite dimensions. A straightforward application of the recently developed scaling theory for such systems[7,8] [which embodies the idea that for large finite L the critical behavior should depend only on the scaling ratio $L/\xi(T)$] reconfirmed Josephson's expression. However, in certain special circumstances, probably not realized in practice, the relation might be replaced by $\upsilon = 1 - \alpha$.

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[1]See, for example, M. E. Fisher, J. Appl. Phys. 38, 981 (1967); *Contemporary Physics* (International Atomic Energy Association, Vienna, 1969), Vol. 27, p. 19; and in *Essays in Physics* (Academic, London, 1972), Vol. 4, p. 43.

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[3]We refer to the "phase angle" of $\underline{\Psi}$ rather than, as customary, the "phase" of $\underline{\Psi}$ to avoid double usage of the word "phase".

[4]Although in the absence of explicit terms like $\vec{S}\cdot\vec{r}$ in the Hamiltonian, the two spaces maybe treated as if distinct. Dipole-dipole interactions do, however, couple spin space and real space together.

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[10]When $\underline{\Psi}$ has two components θ denotes a single angle running from $-\pi$ to π. More generally θ will stand for all those angles needed to specify the direction of $\underline{\Psi}$.

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Critical Phenomena in Systems of Finite Thickness. III. Specific Heat of an Ideal Boson Film

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The specific heat at constant density of an ideal-Bose-fluid film of thickness $L=l\rho^{-1/3}$, but infinite lateral extent, is calculated analytically to order l^{-2}. Both hard-wall and periodic boundary conditions are considered. Good agreement with the numerical calculations of Goble and Trainor for the total specific heat under hard-wall conditions is obtained down to $l\simeq10$. In the critical region, the large-l behavior accords with the scaling theory of finite-size effects. The appropriate scaling functions and the surface specific heat are explicitly calculated.

I. INTRODUCTION AND SUMMARY

Recent analytical calculations[1] on the spherical model[2] have given considerable support to the scaling theory[3,4] of finite-size effects in the critical region. This theory appears also to predict correctly the effects of finite size and surfaces on the critical behavior of more realistic models (see Refs. 3 and 4 for a more complete discussion including various qualifications and a review of the existing calculations).

It is well known that an ideal Bose gas in three dimensions exhibits a phase transition with a sharp critical temperature[5]; the correspondence with other critical phenomena has been discussed, in detail, by Gunton and Buckingham.[6] Moreover, like the spherical model, the ideal Bose gas is mathematically tractable in all dimensions. Hence the effects of finite size and surfaces on its critical behavior may be investigated analytically,[7] and compared in detail with the scaling predictions. Although the spherical model and the ideal Bose gas are effectively equivalent in the immediate critical region[6] there are sufficient differences, both in mathematical detail and physical application, that separate discussions are illuminating. Thus the ideal Bose transition provides a model, albeit a rather crude one, of the superfluid transition in real helium; in particular, ideal Bose films are of interest in connection with real helium films. Indeed, many authors[8-11] have studied ideal Bose films and other finite geometries. For the most part, however, the previous calculations do not reveal clearly (or at all) the nature of the asymptotic behavior for large thickness L nor provide very explicit expressions for the finite-thickness thermodynamic properties. (There has also been a tendency to attempt to identify particular "condensation," "transition," or "onset" points rather than recognizing fully the absence of a sharp transition in any finite geometry, and acknowledging the consequent "rounding" and distortion of all properties.)

In this paper we aim to give a detailed discussion of the specific heat $C_V^{\tau}(T,l)$, at constant number density ρ (or constant volume) of a two-dimensional ideal boson film of thickness $L=l\rho^{-1/3}$, and infinite lateral extent. We consider both standard hard-wall boundary conditions (denoted by superscript $\tau=1$) and periodic boundary conditions applied across the film (denoted by $\tau=0$). The calculations extend, and specialize, an earlier analysis[4,7] of the shift and rounding of the transition as a function of l in a d'-dimensional ideal Bose "film" which is infinite in $d'=d-1$ dimensions but of finite thickness L in the dth dimension.

To summarize our results, recall[5,6] that the