# Discrete gap solitons in waveguide arrays with alternating spacings

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We consider an array of waveguides with identical widths but alternating spacings using the discrete nonlinear Schrödinger model (tight-binding approximation). In the highly discrete (anticontinuous) limit when one of the spacings is infinite, the model reduces to an integrable chain of uncoupled dimers. From this limit, we identify the two fundamental, antisymmetric and symmetric, discrete gap solitons, which can be numerically continued to a continuum limit gap soliton at one band edge. Other composite solutions at the uncoupled limit disappear in bifurcations. Similarly to the case of waveguides with alternating widths and constant spacings, oscillatory instabilities appear for the fundamental solutions only for frequencies in the upper half of the gap. In contrast to the alternating-width case, there is no stability exchange between the two fundamental solutions: the symmetric solution is always unstable while the antisymmetric solution is always stable in the lower half of the gap. Thus, the Peierls-Nabarro barrier can vanish only in the continuum limit.

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There is a large current, theoretical and experimental, interest in the study of discrete solitons (or breathers) in many different areas [1], and in particular, in optical systems [2]. One of the most studied models is described by the discrete nonlinear Schrödinger (DNLS) equation [3]. In particular, it is the standard model to describe spatial solitons in arrays of optical waveguides with Kerr nonlinearity, assuming that the coupling between individual waveguides is sufficiently weak to allow for a tight-binding (or coupled-mode) type approximation [2,4].

A recent progress in waveguide engineering is the possibility to realize different kinds of superlattice structures by modulating the widths, as well as the distances, of the individual waveguides. In Ref. [5] it was predicted that a binary array of waveguides with alternating widths but constant separation, modeled by a DNLS equation with alternating on-site energies, should provide for the existence of *discrete gap solitons* with frequencies in the gap between the two branches of the linear dispersion relation. These solutions were analyzed in detail in Ref. [6], and they were later also observed experimentally [7]. Furthermore, modulated waveguide arrays have been suggested for controlling the switching of solitons across different nonlinear lattices [8]. More recently, the technique of modulating waveguide widths was used to experimentally study the competition between nonlinearity and linear localization in disordered and quasiperiodic systems [9].

However, the complementary approach to a binary modulated waveguide array, with constant widths but *alternating distances* (see Fig. 1), has yet been very little explored. In the tight-binding limit, such a system is modeled by a "bond-alternating" DNLS equation,

$$-i\frac{\partial \psi_n}{\partial z} = V_{n+1}\psi_{n+1} + V_n\psi_{n-1} + \gamma|\psi_n|^2\psi_n, \qquad (1)$$

where $z$ corresponds to the longitudinal direction along the waveguides and $\gamma$ is the nonlinear coefficient from the Kerr effect in each waveguide (we put $\gamma=1$ without loss of generality). The coupling constant depends on the particular site $n$ as: $V_{2j+1}\equiv V_1$, $V_{2j}\equiv V_2 (j=0,1,2,3,\ldots,N)$, with boundary conditions $V_0=V_{2N+2}=0$. In an experimental setup, the different coupling constants are related to two different separations between waveguides: the larger the separation, the smaller the coupling of energy between different sites [2].

Equation (1) is a special case of a more general model for a binary superlattice in a Kerr nonlinear medium proposed in Ref. [10] but the special properties of discrete gap solitons in the binary DNLS model with *absence of on-site modulation* (constant waveguide widths) has to our knowledge not been studied before. Some properties of the corresponding linear system were recently discussed in Ref. [11]. Also, in Ref. [12] the existence of exponentially localized gap solitons for all gap frequencies was proven rigorously, for a general class of periodically modulated DNLS equations including Eq. (1). A few papers have also discussed gap solitons in other types of bond-alternating lattices. In Ref. [13] a ferromagnetic Heisenberg spin chain with bond alternation was studied, and gap solitons were found and analyzed in the continuum limit. Reference [14] studied numerically the vibrational gap modes in an anharmonic monatomic chain with alternating force constants, modeling a row in the ⟨111⟩ direction of a diamond structure. The continuum limit of this model was also studied analytically in Ref. [15]. However, none of these works addressed any of the two issues, which are the main goals of the present paper: (i) the identification of two fundamental families of discrete gap solitons, symmetric and antisymmetric, as continuations of exact solutions at an "anticontinuous" limit of uncoupled dimers, and (ii) the study of their stability.

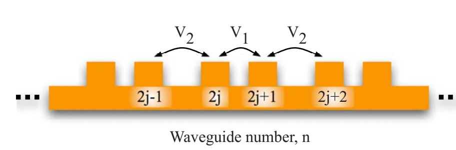


FIG. 1. (Color online) Modulated array with couplings $V_1$ and $V_2$.

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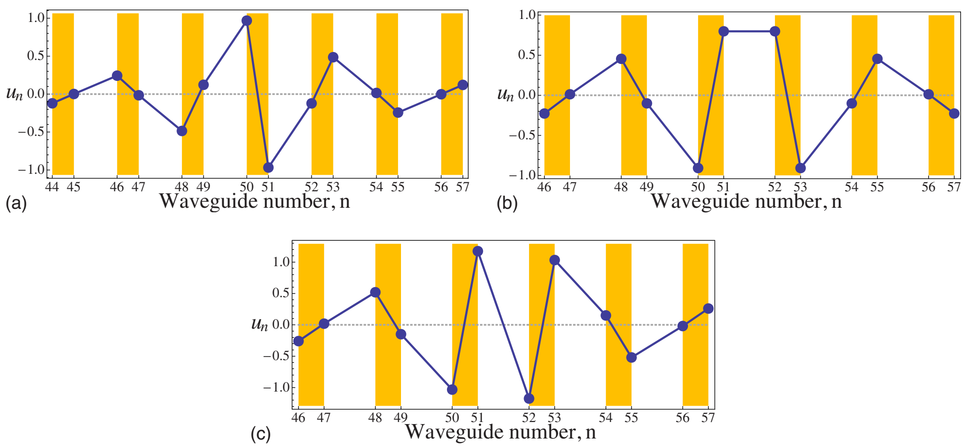


FIG. 2. (Color online) Gap modes for $\Lambda=0$: (a) *fundamental antisymmetric*, (b) *fundamental symmetric*, and (c) *twisted*. Shaded and white regions correspond to couplings $V_1$ and $V_2$, respectively. $V_1=\gamma=1$ and $V_2=0.5$.

For convenience, we take $V_1>0$ and $0\leq V_2<V_1$. (We could also put $V_1=1$ through rescalings but we keep it to make the discussion physically clearer.) In the linear limit ($\gamma=0$), extended solutions can be written as

$$\psi_n(z)=e^{ikj}e^{i\lambda z}\times\begin{cases}U_0; & n=2j,\\ U_1; & n=2j+1,\end{cases} \tag{2}$$

where $U_0$ and $U_1$ correspond to two different amplitudes in alternating sites, related by $\lambda U_1=[V_1+V_2\exp(ik)]U_0$, $k$ is related to the optical angle, and $\lambda$ is the spatial frequency. Inserting this ansatz in Eq. (1) yields the dispersion relation $\lambda=\pm\sqrt{V_1^2+V_2^2+2V_1V_2\cos k}$ [10,11,13]. Therefore, the linear spectrum of Eq. (1) consists of a lower band $[-(V_1+V_2),-(V_1-V_2)]$ and an upper band $[V_1-V_2,V_1+V_2]$, and thus a gap in the interval $]-(V_1-V_2),V_1-V_2[$ of size $\Delta\equiv2(V_1-V_2)$, which can be varied by adjusting the distance between waveguides. In the limit $V_2\to V_1$, $\Delta\to0$ and the band is the usual one for a homogeneous array, i.e., $[-2V_1,2V_1]$ [1,2].

*Discrete gap solitons* are spatially localized stationary solutions to Eq. (1) of the form $\psi_n(z)=u_ne^{i\Lambda z}$, with real and $z$-independent $u_n$ and frequency $\Lambda$ inside the gap. An appropriate anticontinuous limit for identifying exact discrete gap solitons is to let $V_2\to0$ for fixed $V_1>0$, turning the system into a chain of uncoupled DNLS dimers. As is well known, this limit is integrable [16–18], and its stationary solutions, as well as their stability, were described in [17]. The most fundamental gap soliton consists in this limit of a single dimer excited with a frequency inside the gap. With $\gamma=+1$ in Eq. (1), this solution has the frequency $\Lambda=-V_1+A^2$ and the form $u_{2j_0}=-u_{2j_0+1}\equiv A$ for some $j_0$, and thus it is spatially *antisymmetric* with respect to the center of a strong bond (i.e., the midpoint between two closely spaced waveguides). [The other two single-dimer stationary solutions [17], yielding modes that are symmetric (asymmetric) with respect to the midpoint of a small spacing, have frequencies outside the gap.] This solution can then be smoothly continued as an antisymmetric solution for nonzero $V_2$ using standard continuation techniques [6,19,20] [see Fig. 2(a)]. Likewise, for any $V_2$ it may be continued versus $\Lambda$, as illustrated in Fig. 3 (lower line) where the frequency dependence of the conserved power, $P\equiv\sum_{n=0}^{2N+1}|u_n|^2$, is shown. The continuation can be performed all the way until it either reaches a continuum gap soliton at the lower gap edge $\Lambda=-(V_1-V_2)$ if $\Lambda<0$, or becomes a discrete "outgap" soliton [6] with a nondecaying tail at the upper gap edge $\Lambda=(V_1-V_2)$ if $\Lambda>0$. This scenario is qualitatively analogous to that described for the on-site modulated DNLS model in Ref. [6]. A gap mode of this type was probably first mentioned and illustrated for a bond-alternating chain in Fig. 3 of Ref. [14] although the different kind of nonlinearity in Ref. [14] imposed an additional factor $(-1)^n$, turning the solution into a symmetric one.

On the other hand, to obtain the fundamental spatially *symmetric* discrete gap soliton of Eq. (1) at the anticontinuous limit $V_2=0$, two neighboring dimers are excited as above, with a relative phase shift of $\pi$ yielding the pattern $-u_{2j_0}=u_{2j_0+1}=u_{2j_0+2}=-u_{2j_0+3}\equiv A$. Thus, this solution is symmetric with respect to the center of a weak bond (i.e., the midpoint between two waveguides with large spacing). Also this solution can be continued to a symmetric solution for nonzero $V_2$ [see Fig. 2(b)] all through the gap reaching either a continuum gap soliton or a discrete outgap soliton at the different gap edges as discussed above [see Fig. 3 (middle line)]. For two weakly coupled DNLS dimers ($N=1$), such a solution was identified in Ref. [18] as the minimum-energy state among the set of mirror-symmetric solutions for which the four-site system is exactly solvable.

Generally, any configuration of excited dimers for $V_2=0$, in-phased or antiphased, may be continued into discrete gap modes for small nonzero $V_2$. However, in contrast to the two

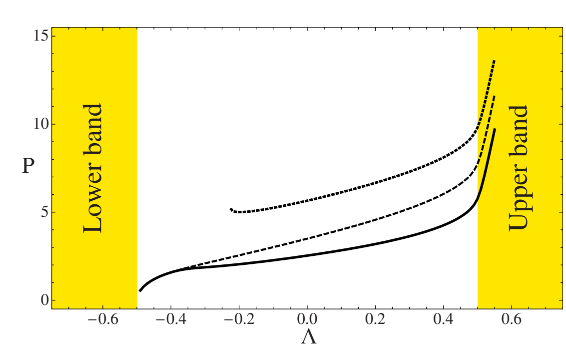


FIG. 3. (Color online) Power versus frequency. Lower, middle, and upper lines correspond to the fundamental antisymmetric, symmetric, and twisted solutions, respectively. $V_1=\gamma=1$, $V_2=0.5$.

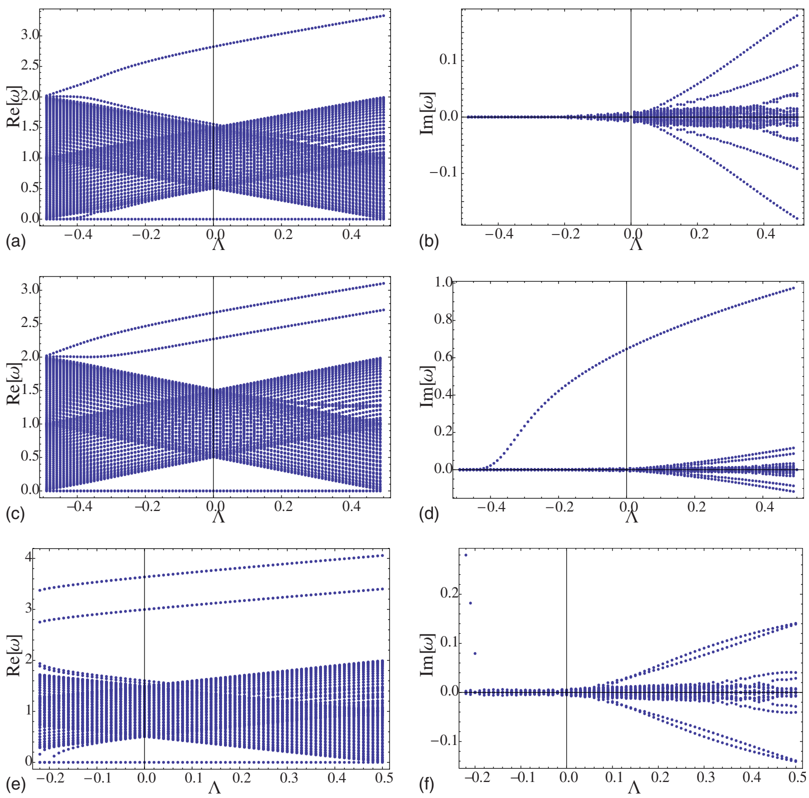

FIG. 4. (Color online) Re($\omega$) and Im($\omega$) vs $\Lambda$: *antisymmetric* (top), *symmetric* (center), and *twisted* (bottom). $V_1=1$, $V_2=0.5$, $N\simeq 50$.

fundamental modes, such composite solutions cannot be continued to continuous gap solitons but disappear in bifurcations before reaching the continuous limit. For this reason, we reserve the terminology "discrete gap soliton" for the two fundamental modes. The simplest example of a nonfundamental gap mode is obtained by in-phase excitation of two neighboring dimers at the anticontinuous limit, $-u_{2j_0}=u_{2j_0+1}=-u_{2j_0+2}=u_{2j_0+3}\equiv A$, yielding a solution antisymmetric with respect to the center of a weak bond [see Fig. 2(c)]. In analogy with the "twisted localized modes" of the nonmodulated DNLS model [21], we term this solution as "*twisted gap mode*." As seen in Fig. 3, it has a power threshold and the continuation toward the lower band stops at a minimum frequency. This can be understood since a small-amplitude solution at the band edge $\Lambda=-(V_1-V_2)$ must have $U_1=-U_0$ and a phase shift $k=\pi$ between neighboring dimers, which cannot be satisfied for a twisted mode.

Investigating the linear stability of the solutions, we introduce, for each nonlinear real stationary state $u_n$, a weak perturbation as $\psi_n(z)=[u_n+\epsilon_n(z)]\exp(i\Lambda z)$, and obtain linear evolution equations for $\epsilon_n$, which may be expressed in various ways [6,22,23]. Writing, e.g., $\epsilon_n(z)=x_n(z)+iy_n(z)=\frac{1}{2}(a_n+b_n)e^{-i\omega z}+\frac{1}{2}(a_n^*-b_n^*)e^{i\omega^* z}$ leads to standard linear eigenvalue problems, and for each real eigenfrequency $\omega>0$ one may also associate a Krein signature as $\kappa=\mathrm{sign}\Sigma_n a_n b_n$, corresponding to the sign of the Hamiltonian energy carried by the corresponding (real) eigenmode $(a_n,b_n)$ (see [6] and references therein).

As for the on-site modulated case [6], the continuous part of the spectrum (extended eigenmodes) of the linearized equations consists of two (possibly overlapping) bands for $\omega>0$: a band with $\kappa=-1$ for $\omega\in[V_1-V_2+\Lambda, V_1+V_2+\Lambda]$, and a band with $\kappa=+1$ for $\omega\in[V_1-V_2-\Lambda, V_1+V_2-\Lambda]$. Note that the $\kappa=+1$ band is the highest for $\Lambda<0$ and the $\kappa=-1$ band is highest for $\Lambda>0$, and the bands overlap when $V_2>|\Lambda|$. In particular, the bands completely overlap when $\Lambda=0$, and they partly overlap for all $\Lambda$ in the gap if $V_2>V_1/2$ (see Fig. 4). As discussed in [6,24], overlap of the bands of extended modes with opposite Krein signatures may for finite systems yield oscillatory instabilities (see Fig. 4) but their strength decreases to zero when $N\to\infty$. Thus, the only instabilities that may remain in the infinite-size limit are those associated with localized eigenmodes, which may be either unstable by themselves (imaginary $\omega$) or become unstable through resonances with other modes (extended or localized) with opposite Krein signature (complex $\omega$ with nonzero real and imaginary parts).

To get a complete picture of the linear stability, we solve the eigenvalue problems numerically as in Fig. 4, using the method from Ref. [23]. The stability regions for the antisymmetric gap soliton are illustrated in Fig. 5(a). Apart from weak finite-size instabilities, the solution is *always stable in the lower half of the gap* ($\Lambda<0$) but becomes unstable at some *nonzero instability threshold value* of $V_2/V_1$ in the *upper* half of the gap ($\Lambda>0$). This threshold apparently goes to zero as $\Lambda\to 0$ although the instabilities for small $\Lambda$ become very weak.

Close to the anticontinuous limit (small $V_2$), we may deduce the linear stability from perturbative calculations of localized eigenmode frequencies as in Ref. [6]. For example, the simplest symmetric linear eigenmode of the antisymmetric gap soliton, involving only oscillations $x_{j_0}=x_{j_0+1}$, $y_{j_0}=y_{j_0+1}$ at the limit $V_2=0$, has frequency

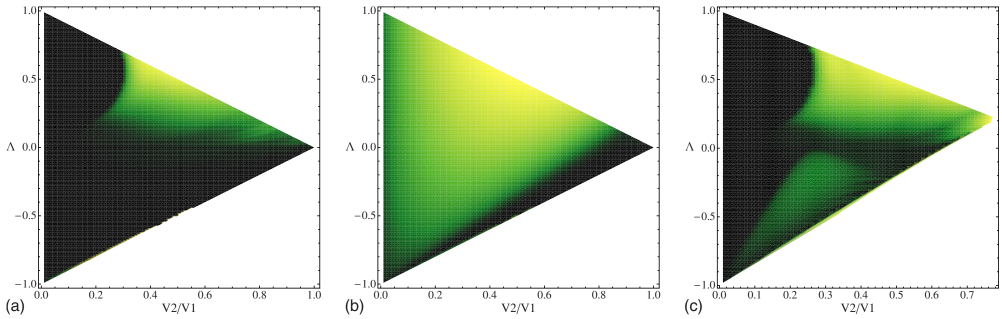


FIG. 5. (Color online) Stability zones of the (a) fundamental antisymmetric, (b) fundamental symmetric, and (c) twisted discrete gap modes. Black parts are stable while the brighter parts correspond to stronger instabilities.

$$\omega = 2V_1\sqrt{\frac{\Lambda}{V_1}+2} + \mathcal{O}(V_2^2)$$

and Krein signature $\kappa = +1$. Thus, the frequency of this mode is real and *above both bands* of extended modes, and thus can cause no instabilities for small $V_2$. In fact, numerically we find that it always remains above both bands [cf. Fig. 4(a)]. Since not only the frequency itself but also all its harmonics are outside the continuous spectrum, this linear eigenmode should correspond to an exact quasiperiodic (two-frequency) "breather" solution, similarly to what was found for the on-site modulated DNLS in Ref. [25] (such breathers with *oscillating intensity* were also observed experimentally for another type of waveguide array with a multiband structure [26]). For the simplest nontrivial ($\omega \neq 0$) *antisymmetric* eigenmodes of the antisymmetric discrete gap soliton, with $x_{j_0} = -x_{j_0+1}$, $y_{j_0} = -y_{j_0+1}$ nonzero for $V_2 = 0$, the perturbative expressions for the frequency are more complicated. However, it can be seen that there are two such eigenmodes with opposite Krein signatures, whose eigenfrequencies for small $V_2$ are always real [these two modes are visible in Fig. 4(a)], but when $\Lambda > 0$ may cause oscillatory instabilities when colliding with the band of opposite Krein signature for some nonzero $V_2$, as seen in Fig. 4(b).

The corresponding stability results for the symmetric discrete gap soliton are illustrated in Figs. 4(c), 4(d), and 5(b). In addition to the oscillatory instabilities, this mode always has a purely imaginary eigenvalue with an antisymmetric eigenmode, corresponding to a *translational instability*. Thus, in contrast to the on-site modulated case [6], there is *no stability exchange* between the two fundamental discrete gap solitons, and therefore, the corresponding Peierls-Nabarro barrier [27] should be always nonzero. For the twisted gap mode, the results presented in Figs. 4(e), 4(f), and 5(c) show that there are considerable regions of stability, primarily for small $V_2$, while it generally becomes unstable for larger $V_2$.

In conclusion, we have analyzed properties of discrete gap modes in an array of waveguides with constant width but alternating spacings. We showed how the fundamental gap solitons appeared from the limit of uncoupled DNLS dimers, and described their stability for general parameter values. We hope that our results may stimulate further experimental activity in this area.

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