Adaptive quantum teleportation

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We consider multiple teleportation in the Knill-Laflamme-Milburn (KLM) scheme. We introduce *adaptive teleportation*, i.e., such that the choice of entangled state used in the next teleportation depends on the results of the measurements performed during the previous teleportations. We show that adaptive teleportation enables an increase in the probability of faithful multiple teleportation in the KLM scheme. In particular if a qubit is to be teleported more than once then it is better to use nonmaximally entangled states than maximally entangled ones in order to achieve the highest probability of faithful teleportation.

Quantum teleportation is one of the basic primitives of quantum information theory [1]. Together with entanglement distillation, it allows sending quantum information reliably even through quantum channels of vanishing zero-way quantum capacity [2]. It can be also very useful in implementation of controlled quantum gates [3]. The latter application is very important in linear optical quantum computation, where controlled quantum gates are implemented by means of quantum teleportation [4]. If Alice wants to teleport a qubit to Bob, then she performs complete Bell measurement on a qubit to be teleported and a qubit from the maximally entangled pair which she shares with Bob. After learning the result of Alice's measurement Bob performs one of four unitary operations. However, sometimes the parties cannot perform complete Bell measurement. Such a situation naturally occurs in quantum information processing with linear optics [5]. Partially due to this fact, the first experimental demonstration of quantum teleportation succeeded only in 25% [6]. Knill *et al.* presented a protocol which enables linear optical teleportation with high probability [4]. In their protocol a qubit is encoded in superposition of vacuum and one photon state. The protocol also uses a special N-photon entangled state. The probability that quantum teleportation succeeds is equal to $1 - \frac{1}{N+1}$ and when it happens, the fidelity of the teleported qubit with the original qubit is equal to 1. Recently the protocol was generalized to polarization encoding [7]. In [8] we have found that the maximally entangled state gives the highest probability of successful single teleportation in the KLM protocol.

The situation is quite different when one considers a chain of linear optical teleportations, i.e., Alice teleports a qubit to Bob, then Bob teleports the qubit, which he received from Alice, to Charlie and so on. If we assume that each party performs KLM protocol and uses *identical* entangled state, then for a sufficiently large number of teleportations the nonmaximally entangled states give the highest probability of successful teleportation [9]. However, in order to achieve faithful teleportation one has to correct errors after the last teleportations by performing local filtering.

In this paper we show, how one can further increase the probability of faithful multiple linear optical teleportation. We introduce adaptive teleportation. In this protocol the

choice of entangled state used in the next teleportation depends on the results of measurements obtained during the previous teleportations. We show that the simple version of this protocol enables an increase in the probability of two teleportations. Moreover, the protocol has a built-in error correction, i.e., the last teleportation corrects errors. We also compare the adaptive teleportation with the multiple teleportation when one uses identical entangled states in all teleportations.

In order to illustrate the adaptive teleportation we consider three-party scenario in which Alice teleports a qubit to Bob and then Bob teleports the qubit received to Charlie (Fig. 1). We assume that each party performs the KLM protocol and uses the entangled state with N vertically polarized photons and N horizontally polarized photons. The entangled state used in each teleportation can be different. Moreover, Bob and Charlie can choose the entangled state which they use in teleportation after learning the result of Alice's measurement. Our aim will be maximization of probability of faithful multiple teleportation. The teleportation is faithful if fidelity of the teleported qubit after the last teleportation with the original qubit is equal to 1.

We assume that Alice and Bob share the entangled state of the form

$$|t_1\rangle = \sum_{i=0}^{N} c_1(i)|V\rangle^i|H\rangle^{N-i}|H\rangle^i|V\rangle^{N-i}, \qquad (1)$$

where $|V\rangle^i$ stands for $|V\rangle_1|V\rangle_2\ldots|V\rangle_i$, i.e., one vertically polarized photon in each of the subsequent modes. Similarly, $|H\rangle^{N-i}$ stands for $|H\rangle_{i+1}|H\rangle_{i+2}\ldots|H\rangle_N$, i.e., one horizontally polarized photon in each of the subsequent modes. The first N modes are held by Alice and the next N modes are held by Bob. If all amplitudes $c_1(i)$ are the same, then the state is maximally entangled. In order to teleport a qubit in the state $|\psi\rangle = \alpha|H\rangle + \beta|V\rangle$ Alice applies to the input mode and the first N modes of the entangled state the $N+1$-point quantum Fourier transform
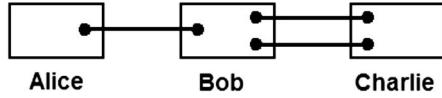
FIG. 1. Adaptive quantum teleportation. Bob and Charlie share two different entangled states. For some results of Alice's measurement Bob teleports the qubit to Charlie using the "upper" entangled state and for some results of Alice's measurement Bob teleports the qubit to Charlie using the "lower" entangled state.

$$F_N(h_k^\dagger) = \frac{1}{\sqrt{N+1}} \sum_{l_k=0}^{N} \omega^{kl_k} h_{l_k}^\dagger,$$

$$F_N(v_k^\dagger) = \frac{1}{\sqrt{N+1}} \sum_{l_k=0}^{N} \omega^{kl_k} v_{l_k}^\dagger, \qquad (2)$$

where $h_k^\dagger$ ($v_k^\dagger$) is the creation operator for a horizontally (vertically) polarized photon in mode k and $\omega = e^{i2\pi/N+1}$. Next, Alice measures the total number of horizontally and vertically polarized photons in each of these modes. If the sum of vertically polarized photons detected is m ($0 < m < N+1$), then the modified state of the qubit is found in the $(N+m)$th mode. After phase correction the state of the qubit is

$$|\psi\rangle = \frac{1}{\sqrt{p(m)}}[\alpha c_1(m)|H\rangle + \beta c_1(m-1)|V\rangle], \qquad (3)$$

where

$$p(m) = |\alpha c_1(m)|^2 + |\beta c_1(m-1)|^2 \qquad (4)$$

is probability that the sum of vertically polarized photons detected is equal to m. Next, Bob teleports a qubit which he received from Alice to Charlie. We assume that Bob and Charlie share several entangled states of the form

$$|t_{2;m}\rangle = \sum_{i=0}^{N} c_{2;m}(i)|V\rangle^i|H\rangle^{N-i}|H\rangle^i|V\rangle^{N-i}. \qquad (5)$$

The amplitudes $c_{2;m}(i)$ can be different from $c_1(i)$. Moreover, they can also be different for each entangled state shared by Bob and Charlie. The choice of the entangled state used in the teleportation depends on the result of Alice's measurement. After the choice of the entangled state, Bob applies the $N+1$-point quantum Fourier transform to the input mode and the first N modes of *chosen* entangled state and measures the total number of horizontally and vertically polarized photons in each of these modes. If the sum of vertically polarized photons detected is n ($0 < n < N+1$) then the modified state of the qubit is found in the $(N+n)$th mode. After the phase correction the state of the qubit is

$$|\psi\rangle = \frac{1}{\sqrt{p(m,n)}}[\alpha c_1(m)c_{2;m}(n)|H\rangle$$
$$+ \beta c_1(m-1)c_{2;m}(n-1)|V\rangle], \qquad (6)$$

where

$$p(m,n) = p(n|m)p(m)$$
$$= |\alpha c_1(m)c_{2;m}(n)|^2 + |\beta c_1(m-1)c_{2;m}(n-1)|^2 \quad (7)$$

is the probability that the sum of vertically polarized photons detected during the first teleportation is equal to m and the sum of vertically polarized photons detected during the second teleportation is equal to n. We require that the qubit is teleported faithfully, i.e., its state after the second teleportation is equal to

$$|\psi\rangle = \alpha|H\rangle + \beta|V\rangle, \qquad (8)$$

and hence the amplitudes $c_1(m)$ and $c_{2;m}(n)$ should satisfy the condition

$$c_1(m)c_{2;m}(n) = c_1(m-1)c_{2;m}(n-1). \qquad (9)$$

If $c_1(m) < c_1(m-1)$ [$c_1(m) > c_1(m-1)$], then the amplitudes $c_{2;m}(n)$ should form an increasing (decreasing) geometric sequence.

Now we make the following assumptions:

(1) Entangled states used in both teleportations have *even* N [10].

(2) The entangled state used in the first teleportation has the amplitudes given by the formula

$$c_1(m) = \sqrt{a_1}(\sqrt{q_1})^{N/2-|m-N/2|}, \qquad (10)$$

where $q_1 > 1$, i.e., for $m \leq N/2$ the amplitudes $c_1(m)$ form an increasing geometric sequence and for $m \geq N/2$ they form a decreasing geometric sequence. From the normalization condition $\sum_{m=0}^{N}|c_1(m)|^2 = 1$ we obtain

$$a_1 = \frac{1-q_1}{2-q_1^{N/2}(1+q_1)}. \qquad (11)$$

From Eq. (9) we obtain that the entangled state used in the second teleportation has the amplitudes given by the formula

$$c_{2;m}(n) = \sqrt{a_2}(\sqrt{q_2})^n, \qquad (12)$$

where $q_2 = 1/q_1$ if the number of vertically polarized photons m detected during the first teleportation is lower or equal to $N/2$ and $q_2 = q_1$ if the number of vertically polarized photons m detected during the first teleportation is greater than $N/2$, i.e., Bob and Charlie choose one from two entangled states which they share depending on the result of Alice's measurement. From the normalization condition $\sum_{n=0}^{N}|c_{2;m}(n)|^2 = 1$ we obtain

$$a_2 = \frac{1-q_2}{1-q_2^{N+1}}. \qquad (13)$$

In order to obtain the probability of faithful double teleportation we have to calculate the probability that both Alice and Bob register from 1 to N vertically polarized photons. If one of them registers 0 or $N+1$ vertically polarized photons, then teleportation fails. Hence, our probability is given by the following expression

N	$\left(\dfrac{N}{N+1}\right)^2$	q_{opt}	p_{opt}
2	0.444444	1.29663	0.454167
4	0.640000	1.20892	0.652198
6	0.734694	1.15822	0.746252
8	0.790123	1.12682	0.800565
10	0.826446	1.10569	0.835820

$$p = \sum_{m=1}^{N} \sum_{n=1}^{N} p(m,n) = \sum_{m=1}^{N} \sum_{n=1}^{N} |c_1(m)c_{2;m}(n)|^2. \tag{14}$$

Substituting formulas (10) and (12) together with Eqs. (11) and (13) into Eq. (14) we obtain the following expression for the probability of faithful double teleportation:

$$p = 2\frac{q - q^{N/2+1}}{2 - q^{N/2}(1+q)}\frac{q^N - 1}{q^{N+1} - 1}. \tag{15}$$

This probability should be optimized over q. In Table I we present the optimized probabilities for different N when one performs adaptive double teleportation and when one performs double teleportation with the use of maximally entangled states. We see that if the qubit is to be teleported at least two times, then it is better to perform adaptive teleportation than that with the use of maximally entangled states.

It is interesting that in this simple scenario another strategy is possible which gives the same probability of faithful double teleportation. Namely, Bob always teleports the qubit to Charlie with entangled state whose amplitudes form a decreasing geometric sequence. However, if Alice detects more than $N/2$ photons during the first teleportation Bob applies NOT gate to the teleported qubit before the second teleportation and Charlie applies NOT gate to the teleported qubit after the second teleportation. Hence, we replaced the choice of entangled state by the choice of gate.

Let us now compare the adaptive teleportation with the multiple teleportation developed in [9] when one uses identical entangled states in all teleportations and error correction is made after the last teleportation. We consider only the simplest nontrivial example when one uses the entangled states with $N=2$. We assume that entangled states have amplitudes symmetric around $i=1$ [11]. The probability of faithful multiple teleportation is given by formula [9]

$$p = \sum_{n_1=1}^{2} \cdots \sum_{n_M=1}^{2}$$
$$\times \min\{|c(n_1) \ldots c(n_M)|^2, |c(n_1 - 1) \ldots c(n_M - 1)|^2\}, \tag{16}$$

where M is the number of teleportations. Substituting $q = \left|\frac{c(1)}{c(0)}\right|^2$ and using the normalization condition $\sum_{m=0}^{2} |c(m)|^2 = 1$ we obtain

$$p = \left(\frac{1}{2+q}\right)^M \sum_{i=0}^{M} \frac{M!}{i!(M-i)!} \min\{q^{M-i}, q^i\}. \tag{17}$$

This probability should be optimized over q.

For less than six teleportations the highest probability of faithful multiple teleportation with identical entangled states is achieved for *maximally* entangled states. Hence, if we replace any two teleportations in a chain of two or more teleportations with maximally entangled states by double adaptive teleportation, then we obtain greater probability of faithful multiple teleportation. It should be noted that there is no contradiction because in adaptive teleportation one does not use identical entangled states.

On the other hand for six or more teleportations the highest probability of faithful multiple teleportation with identical entangled states is achieved for *nonmaximally* entangled states. In what follows, we show that if we replace the original nonmaximally entangled state used in the last teleportation by the adapted entangled state, then we increase the probability of faithful teleportation. Let us assume that the first sender detected n_1 vertically polarized photons, the second sender detected n_2 vertically polarized photons ..., and the $(M-1)$th sender detected n_{M-1} vertically polarized photons. Then the state of the qubit after $M-1$ teleportations is

$$|\psi\rangle = \frac{1}{\sqrt{p_{M-1}(n_1, \ldots, n_{M-1})}} \times [\alpha a(1)|H\rangle + \beta a(0)|V\rangle], \tag{18}$$

where $p_{M-1}(n_1, \ldots, n_{M-1}) = |\alpha a(1)|^2 + |\beta a(0)|^2$ is probability of this event. The coefficients $a(1)$ and $a(0)$ are given by formulas

$$a(1) = c(n_1) \ldots c(n_{M-1}),$$

$$a(0) = c(n_1 - 1) \ldots c(n_{M-1} - 1). \tag{19}$$

Because for six or more teleportations with identical entangled states, the nonmaximally entangled states give higher probability of faithful multiple teleportation, then for some results of measurements we have $a(0) \neq a(1)$. Without loss of generality we assume that $a(1)$ is *greater* than $a(0)$ and write $|a(1)|^2 = q|a(0)|^2$ $(q > 1)$. Let $b(n_M)$ be coefficients of entangled state used in the Mth teleportation. If the Mth sender detected n_M vertically polarized photons, then the state of the qubit after the Mth teleportation is

$$|\psi\rangle = \frac{1}{\sqrt{p_M(n_1, \ldots, n_M)}}$$
$$\times [\alpha b(n_M)a(1)|H\rangle + \beta b(n_M - 1)a(0)|V\rangle], \tag{20}$$

where $p_M(n_1, \ldots, n_M) = |b(n_M)\alpha a(1)|^2 + |b(n_M - 1)\beta a(0)|^2$ is probability that the first sender detected n_1 vertically polarized photons, the second sender detected n_2 vertically polarized photons ..., and the Mth sender detected n_M vertically polarized photons. If $b(n_M)a(1) = b(n_M - 1)a(0)$, then the qubit is in its original state. If $b(n_M)a(1) > b(n_M - 1)a(0)$ then the optimal strategy to return the qubit to its original state is

to perform the generalized measurement given by Kraus operators

$$E_S = \frac{b(n_M - 1)a(0)}{b(n_M)a(1)}|H\rangle\langle H| + |V\rangle\langle V|,$$

$$E_F = \sqrt{1 - \left|\frac{b(n_M - 1)a(0)}{b(n_M)a(1)}\right|^2}|H\rangle\langle H|. \qquad (21)$$

If E_S is obtained as a result of the measurement then the aim is achieved. Otherwise the state of the qubit is irreversibly destroyed. Similar measurement exists if $b(n_M)a(1) < b(n_M - 1)a(0)$. Probability that the first sender detected n_1 vertically polarized photons, the second sender detected n_2 vertically polarized photons ..., and the Mth sender detected n_M vertically polarized photons, and the error correction succeeded is

$$p_M(n_1, \ldots, n_M, S) = \min\{|b(n_M)a(1)|^2, |b(n_M - 1)a(0)|^2\}. \qquad (22)$$

In order to obtain the total probability of faithful multiple teleportation we have to sum these probabilities over the number of photons detected in each teleportation. Let us first perform summation over the number of vertically polarized photons detected in the last teleportation, i.e.,

$$p_M(n_1, \ldots, n_{M-1}, S) = \sum_{n_M=1}^{2} \min\{|b(n_M)a(1)|^2, |b(n_M - 1)a(0)|^2\}. \qquad (23)$$

If we use in the Mth teleportation a state which has amplitudes symmetric around $n_M = 1$, i.e., $b(0) = b(2)$ then the maximal probability $p_M(n_1, \ldots, n_{M-1}, S)$ is given by the formula

$$p_M^{\text{sym}}(n_1, \ldots, n_{M-1}, S) = |a(0)|^2 \frac{1+q}{2+q}. \qquad (24)$$

On the other hand, if we use in the Mth teleportation a state whose amplitudes are related by $|b(n_M - 1)|^2 = q|b(n_M)|^2$, i.e.,

they form a *decreasing* geometric sequence, then the probability $p_M(n_1, \ldots, n_{M-1}, S)$ is given by the formula

$$p_M^{\text{a}}(n_1, \ldots, n_{M-1}, S) = |a(0)|^2 \frac{1+q}{1+q^{-1}+q}. \qquad (25)$$

Because $q > 1$ we have $p_M^{\text{a}}(n_1, \ldots, n_{M-1}, S) > p_M^{\text{sym}}(n_1, \ldots, n_{M-1}, S)$.

If $a(1)$ were *smaller* than $a(0)$, then we would use in the Mth teleportation a state whose amplitudes form an *increasing* geometric sequence. The relation between $a(1)$ and $a(0)$ depends on the results of measurements obtained during first $M-1$ teleportations and hence, the choice of entangled state used in the Mth teleportation also depends on the results of these measurements, i.e., we perform adaptive teleportation. Summing $p_M(n_1, \ldots, n_{M-1}, S)$ over all possible results of measurements obtained during first $M-1$ teleportations, we obtain the probability that the state is teleported faithfully. Because $p_M^{\text{a}}(n_1, \ldots, n_{M-1}, S)$ for teleportation with an adapted entangled state is greater than the corresponding $p_M^{\text{sym}}(n_1, \ldots, n_{M-1}, S)$ for teleportation with a symmetric state [with possible exception of probabilities corresponding to the case with $a(1) = a(0)$, where they can be equal] we obtain that it is always better to perform adaptive teleportation as the last teleportation.

In conclusion, we have introduced the adaptive teleportation, in which the choice of entangled state used in the next teleportation depends on the results of measurements obtained during the previous teleportations. We have also compared the adaptive teleportation with the multiple teleportation when one uses identical entangled states in all teleportations.

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[10] Similar calculations can be made for *odd N*. However, if we assume that entangled state used in the first teleportation has amplitudes symmetric around $i = N/2$, then Bob and Charlie should choose one of three entangled states rather than one of two entangled states.

[11] It is very unlikely that entangled states which do not have amplitudes symmetric around $i = 1$ are optimal. However, we were not able to prove it rigorously.