Fractional photon-assisted tunneling for Bose-Einstein condensates in a double well

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Half-integer-photon resonances in a periodically shaken double well are investigated on the level of the N-particle quantum dynamics. Contrary to nonlinear mean-field equations, the *linear* N-particle Schrödinger equation does not contain any nonlinearity which could be the origin of such resonances. Nevertheless, analytic calculations on the N-particle level explain why such resonances can be observed even for particle numbers as low as $N=2$. These calculations also demonstrate why fractional photon resonances are not restricted to half-integer values.

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I. INTRODUCTION

Tunneling control of ultracold atoms via time-periodic shaking [1–4] of potentials is currently established as an experimental method both on the single-particle level [5] and on the level of Bose-Einstein condensates (BECs) [6]. An interesting effect is an analog of photon-assisted tunneling in periodically shaken systems of ultracold atoms. It was predicted theoretically both for the case that the driving frequency matches the potential difference between neighboring wells [3,7] and for the case that the driving frequency is resonant with the interaction energy [8]. The n-photon resonances essentially are a single-particle effect which survives interactions; one- and two-photon resonances have been observed experimentally for BECs in periodically shaken lattices [9]. The "photons" are time-periodic potential modulations in the kilohertz regime.

However, photon-assisted tunneling is not restricted to integer-photon resonances. Also half-integer Shapiro-type [10] resonances have been predicted numerically both on the mean-field (Gross-Pitaevskii) level and on the level of the multiparticle quantum dynamics (down to $N=2$ particles) [3]. While the occurrence of higher or lower harmonics in nonlinear equations is easy to understand qualitatively, it is not clear *a priori* how these resonances should occur in the linear N-particle Schrödinger equation. Thus, analytic calculations which can explain the occurrence of such resonances within the linear quantum dynamics will explain how effective nonlinearities can arise from linear dynamics even for small particle numbers. Realistic experimental values for the number of atoms in a double well can be on the order of 1000 atoms [11] for BECs and down to less than 6 atoms [12] for few-atom experiments.

Often, Floquet states [13] are useful to understand the physics of BECs in periodically driven systems [14–17]. The focus of the present paper lies on a different approach: analytic calculations on the N-particle level developed in Ref. [18] (cf. Ref. [19]). By assuming the experimentally realistic initial condition of all particles being in one well [11], the calculations are done analogously to the time-dependent perturbation theory.

The paper is organized as follows: after introducing the two-mode model for a BEC in a double well (Sec. II), we develop the technique to calculate half-integer resonances in Sec. III. A crucial test is to show that the analytic result vanishes in the limit of noninteracting particles (Sec. IV). Other fractional resonances are discussed in Sec. V.

II. MODEL: A BEC IN A DOUBLE WELL

Bose-Einstein condensates in double-well potentials are interesting both experimentally and theoretically [11,20–26]. In order to describe a BEC in a double well, we use a model originally developed in nuclear physics [27]: a multiparticle Hamiltonian in two-mode approximation [28],

$$\hat{H} = -\frac{\hbar\Omega}{2}(\hat{c}_1\hat{c}_2^\dagger + \hat{c}_1^\dagger\hat{c}_2) + \hbar\kappa(\hat{c}_1^\dagger\hat{c}_1^\dagger\hat{c}_1\hat{c}_1 + \hat{c}_2^\dagger\hat{c}_2^\dagger\hat{c}_2\hat{c}_2) + \hbar[\mu_0 + \mu_1\sin(\omega t)](\hat{c}_2^\dagger\hat{c}_2 - \hat{c}_1^\dagger\hat{c}_1), \quad (1)$$

where the operator $\hat{c}_j^{(\dagger)}$ annihilates (creates) a boson in well j, $\hbar\Omega$ is the tunneling splitting, $\hbar\mu_0$ is the tilt between well 1 and well 2, and $\hbar\mu_1$ is the driving amplitude. The interaction between a pair of particles in the same well is denoted by $2\hbar\kappa$.

The Gross-Pitaevskii dynamics can be mapped to that of a nonrigid pendulum [20]. Including the term describing the periodic shaking, the Hamiltonian function is given by

$$H_{\rm mf} = \frac{N\kappa}{\Omega}z^2 - \sqrt{1-z^2}\cos(\phi) - 2z\left(\frac{\mu_0}{\Omega} + \frac{\mu_1}{\Omega}\sin\left(\frac{\omega}{\Omega}\tau\right)\right),$$
$$\tau = t\Omega, \quad (2)$$

where ϕ and z are canonically conjugate variables. The quantity $z/2$ is the population imbalance with $z/2=0.5$ ($z/2=-0.5$) referring to the situation with all particles in well 1 (well 2). The corresponding observable on the N-particle level is given by

$$\frac{J_z(t)}{N} = \frac{\langle\Psi(t)|\hat{c}_1^\dagger\hat{c}_1 - \hat{c}_2^\dagger\hat{c}_2|\Psi(t)\rangle}{2N}. \quad (3)$$

For integer-photon-assisted tunneling, the potential difference between both wells, $2\hbar\mu_0$, has to be bridged by an integer number of photons,

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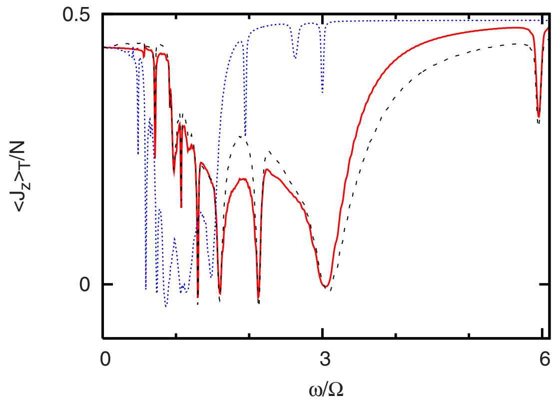

FIG. 1. (Color online) Time-averaged population imbalance $\langle J_z\rangle_T/N$ with averaging time $T=100/\Omega$, static tilt $2\mu_0/\Omega=3$, and interaction $N\kappa/\Omega=0.4$ for $N=2$ particles initially in well 1. The full (red) line with driving amplitude $2\mu_1/\omega=1.8$ (close to the maximum of the J_1-Bessel function) has, among others, a pronounced resonance at the driving frequency $\omega=3\ \Omega$ (one-photon resonance) and also a weaker resonance at $\omega=6\ \Omega$ (1/2-photon resonance). As the dashed (black) line shows, these resonances are also visible when the system is driven with an initial $\pi/2$ phase shift [$\mu_1\cos(\omega t)$ instead of $\mu_1\sin(\omega t)$ in Eq. (1)], while they are strongly suppressed with driving amplitude $2\mu_1/\omega=3.83$ (close to the first zero of J_1) illustrated by the dotted (blue) line.

$$2\hbar\mu_0 = n\hbar\omega, \quad n=1,2,\ldots. \tag{4}$$

The 1/2-integer resonance occurs for

$$2\hbar\mu_0 = \tfrac{1}{2}\hbar\omega. \tag{5}$$

For an interacting Bose gas, these resonances are furthermore shifted [3].

For some parameter regimes (especially for interactions comparable to the onset of the self-trapping transition [11]), the differences between mean-field (Gross-Pitaevskii) dynamics and the N-particle quantum dynamics can be quite remarkable [29]. However, when concentrating on the (experimentally measurable [11]) time-averaged population imbalance,

$$\frac{\langle J_z\rangle_T}{N} = \frac{1}{T}\int_0^T \frac{J_z(t)}{N}dt \tag{6}$$

for *low* interactions, the qualitative agreement between mean-field and N-particle dynamics for the occurrence of both integer and half-integer-photon-assisted tunnelings is excellent [3].

Photon-assisted tunneling is clearly visible in the experimentally measurable time-averaged population imbalance (6). Figure 1 shows integer resonances [Eq. (4)], namely, the one-photon peak with $\omega\approx 3\ \Omega$ and the two-photon peak at $\omega\approx 1.5\ \Omega$. Furthermore, there are pronounced fractional-integer resonances at $\omega\approx 6\ \Omega$, $\omega\approx 2\ \Omega$, and $\omega\approx 1.2\ \Omega$ corresponding to the 1/2-, 3/2-, and 5/2-photon peaks. While some of the resonances disappear [3] for specific choices of the driving *amplitude*, the initial phase of the periodic driving (cf. [4]) does not influence the occurrence of resonances in the situation investigated in this paper.

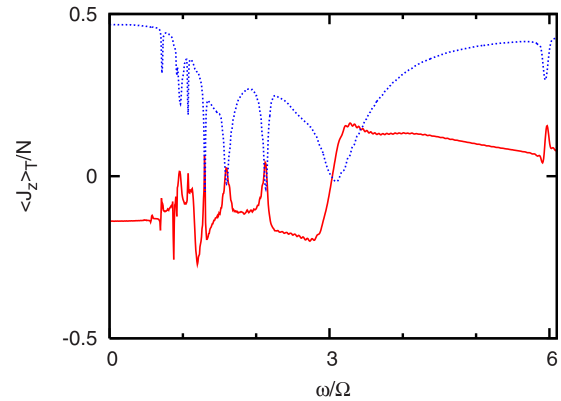

FIG. 2. (Color online) Time-averaged population imbalance $\langle J_z\rangle_T/N$ with averaging time $T=100/\Omega$, static tilt $2\mu_0/\Omega=3$, and interaction $N\kappa/\Omega=0.4$ with $N=2$ particles for two different initial states. The dotted line corresponds to the ground state of the tilted system, while the full line displays $\langle J_z\rangle_T/N$ with the ground state of the untilted system as initial state. In both cases the 1/2-photon resonance at $\omega=6\ \Omega$ appears.

Figure 2 shows that it is not essential to start with all particles in one well in order to observe photon-assisted tunneling. Both for the ground state of the untilted undriven system (for which the initial population imbalance is zero) and for the ground state of the tilted system with an initial population imbalance of $\simeq 0.467$, the main resonances of Fig. 1, where all particles were initially in well 1, can easily be identified.

Figure 3 displays the half-integer resonance for $N=2$ particles. Contrary to what was observed for both larger particle numbers and for mean field, the position of the 1/2-photon resonance does not shift with increasing energy. A first test of our analytic calculations toward the end of Sec. III will thus be to explain this feature.

III. ANALYTIC CALCULATIONS

In order to analytically describe the time evolution of the interacting system, the Fock basis $|\nu\rangle \equiv |N-\nu,\nu\rangle$ is used. The label $\nu=0,\ldots,N$ refers to a state with $N-\nu$ particles in well 1 and ν particles in well 2. Hamiltonian (1) now is the sum of two $(N+1)\times(N+1)$ matrices,

$$H = H_0(t) + H_1. \tag{7}$$

While the nondiagonal matrix H_1 is given by the tunneling terms of Eq. (1), the diagonal matrix H_0 includes both the interaction between the particles and the applied potential difference. For the solution of the Schrödinger equation,

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = [H_0(t)+H_1]|\psi(t)\rangle, \tag{8}$$

the ansatz

$$\langle\nu|\psi(t)\rangle = a_\nu(t)\exp\left[-\frac{i}{\hbar}\int_0^t \langle\nu|H_0(t')|\nu\rangle dt'\right] \tag{9}$$

turned out to be useful [18].

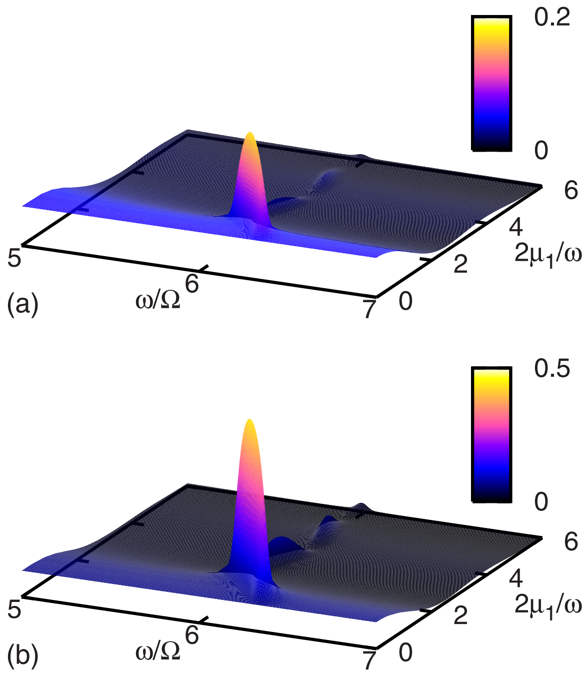

FIG. 3. (Color online) Time-averaged population imbalance for $N=2$ particles in a periodically shaken double-well potential as a function of both driving frequency ω and driving amplitude μ_1 for a static tilt of $2\mu_0/\Omega=3$. Upper panel: $N\kappa/\Omega=0.2$; lower panel: $N\kappa/\Omega=0.4$. The averaging time is $\Omega T=100$; the values plotted are shifted such that 0 corresponds to all particles having always stayed in the first well (a value of 1 would correspond to all particles being in well 2). Surprisingly, contrary to the case of $N>2$ or the mean-field case [3] for $N=2$ particles the resonance does not shift with increasing interactions.

Within this framework, a set of differential equations was derived [18] which is mathematically equivalent to the N-particle Schrödinger equation governed by Hamiltonian (1),

$$i\hbar\dot{a}_\nu(t) = \langle\nu|H_1|\nu+1\rangle h_\nu(t)a_{\nu+1}(t) + \langle\nu|H_1|\nu-1\rangle h_{\nu-1}(t)^* a_{\nu-1}(t). \tag{10}$$

In Eq. (10), the notation $a_{-1}(t)\equiv a_{N+1}(t)\equiv 0$ was used; the phase factors are given by

$$h_\nu(t) = \exp\{i[2(N-1-2\nu)\kappa t + 2\mu_0 t - 2\mu_1 \cos(\omega t)/\omega]\} \tag{11}$$

with $-\cos(\omega t)/\omega = \int_0^t \sin(\omega t')dt'$. To simplify the expression for subsequent integrals, one can use the expansion in terms of Bessel functions [30],

$$e^{iz\cos(\omega t)} = \sum_{k=-\infty}^{\infty} J_k(z) i^k e^{ik\omega t}. \tag{12}$$

Equation (10) furthermore needs

$$\langle\nu|H_1|n\rangle = -\frac{\hbar\Omega}{2}\delta_{\nu,n+1}\sqrt{N-n}\sqrt{n+1} - \frac{\hbar\Omega}{2}\delta_{\nu,n-1}\sqrt{N-n+1}\sqrt{n}, \tag{13}$$

where $\delta_{n,m}$ is the Kronecker delta (which is zero except for $n=m$ where $\delta_{n,n}=1$). The idea is to proceed along the lines of time-dependent perturbation theory [31]. Starting with a typical experimental initial condition such that all particles are in the first well [11], one has in zeroth-order perturbation theory,

$$a_0^{(0)}(t) = 1, \quad a_1^{(0)}(t) = a_2^{(0)}(t) = \cdots = 0, \tag{14}$$

where $a_\nu = \Sigma_{k=0}^{\infty} a_\nu^{(k)}$. In first-order perturbation theory, one gets

$$a_\nu^{(1)}(t) = 0 \tag{15}$$

if $\nu \neq 1$ and

$$a_1^{(1)}(t) = i\frac{\Omega}{2}\sqrt{N}\int_0^t h_0^*(t)a_0^{(0)}(t). \tag{16}$$

Using Eqs. (11) and (12) one thus has

$$a_1^{(1)}(t) = i\frac{\Omega}{2}\sqrt{N}\sum_{k=-\infty}^{\infty} i^k J_k(2\mu_1/\omega)\int_0^t \exp(i\sigma_k t')dt' \tag{17}$$

with

$$\sigma_k \equiv k\omega - 2\mu_0 - 2(N-1)\kappa. \tag{18}$$

Therefore, after solving the integral, Eq. (17) is a sum of time-periodic functions except for the special case with $\sigma_k=0$ which recovers the integer-photon resonances of Eq. (4) investigated in Refs. [3,9]. While for a double well as in Ref. [3] the population imbalance is ideal to investigate photon-assisted tunneling, the experiment [9] was performed in an optical lattice. The signatures of photon-assisted tunneling were seen in the width of the BEC after expansion in the shaken lattice. Surprisingly, a J_n^2 dependence was measured. While this might be interpreted as being an indication for transition from ballistic to diffusive transport [9], the present experiments cannot exclude other explanations. The J_n^2 dependence could either be an interaction-induced effect [32] or the result of an effective average over the precise instant within the cycle at which the current is measured [4].

As the aim of the present paper is to understand the fractional photon peaks like the interaction-induced half-integer resonances in Ref. [3], we can discard the integer-photon resonances characterized via $\sigma_k=0$ [33] and thus write

$$a_1^{(1)}(t) = i\frac{\Omega}{2}\sqrt{N}\sum_{k=-\infty}^{\infty} i^k J_k(2\mu_1/\omega)\frac{e^{i\sigma_k t}-1}{i\sigma_k}. \tag{19}$$

In second-order perturbation theory one has (see the Appendix)

$$a_2^{(2)}(t) = \left(i\frac{\Omega}{2}\right)^2 \sqrt{N}\sqrt{N-1}\sqrt{2} \times \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} i^k i^\ell J_k(2\mu_1/\omega) J_\ell(2\mu_1/\omega) \times \int_0^t \frac{\exp(i\sigma_k t') - 1}{i\sigma_k} \exp(i\tilde{\sigma}_\ell t') dt' \qquad (20)$$

with

$$\tilde{\sigma}_\ell \equiv \ell\omega - 2\mu_0 - 2(N-3)\kappa. \qquad (21)$$

Again, $\tilde{\sigma}_\ell=0$ can be discarded because it corresponds to integer-photon resonances. However, if $\tilde{\sigma}_\ell+\sigma_k=0$ then $a_2^{(2)}$ does have parts which increase linearly in time. In order to see that this indeed corresponds to a half-integer resonance, we choose $N=2$ and $\omega/2=2\mu_0$. This implies $\sigma_k=k\omega-\omega/2-2\kappa$ and $\tilde{\sigma}_\ell=\ell\omega-\omega/2+2\kappa$ and the condition

$$\tilde{\sigma}_\ell + \sigma_k = 0 \qquad (22)$$

thus becomes independent of the interaction; it results in the simple equation

$$k + \ell = 1. \qquad (23)$$

The above reasoning explains why we observe no shift of the resonance with increasing interaction in the numerics displayed in Fig. 3. The amplitude to find both particles in well 2 is given by (Appendix)

$$a_2^{(2)}(t) = [a_2^{(2)}(t)]_{\text{oscil}} - \frac{\Omega^2}{2}\sum_{k=-\infty}^{\infty} J_k(2\mu_1/\omega)J_{1-k}(2\mu_1/\omega)\frac{t}{\sigma_k} - \frac{\Omega^2}{2}\sum_{k=-\infty}^{\infty} J_k(2\mu_1/\omega)J_{1-k}(2\mu_1/\omega)\frac{e^{-i\sigma_k t}-1}{i\sigma_k}, \qquad (24)$$

where the expression $[a_2^{(2)}(t)]_{\text{oscil}}$ contains oscillatory terms which can be found in Eq. (A7). The convergence of this sum is ensured both by the scaling of σ_k on k and the behavior of Bessel functions with increasing k [30],

$$J_k(z) \sim \frac{1}{\sqrt{2\pi k}}\left(\frac{z\exp(1)}{2k}\right)^k, \quad k\to\infty, \qquad (25)$$

combined with the fact that $J_{-k}(x)=(-1)^k J_k(x)$ (for integer k). Figure 4 shows good qualitative agreement between the analytic and numeric calculations for the time-averaged probability that both particles, which initially have been in the first well, have tunneled to the second well. Already perturbation theory in the first order, in which the half-integer resonance becomes visible, correctly describes the occurrence of maxima and minima in the probability for both particles to occupy the second well.

IV. HALF-INTEGER RESONANCES DISAPPEAR IN THE LIMIT OF LOW INTERACTIONS

Despite the agreement displayed in Fig. 4, at the first glance Eq. (24) seems to contain a flaw: numerically, we observe that the half-integer resonance disappears for zero

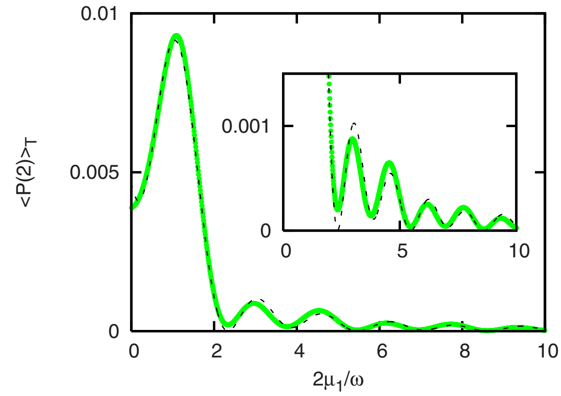

FIG. 4. (Color online) Time-averaged probability (averaged over $T\Omega=10$) that two particles have tunneled to the other well for the 1/2-photon resonance ($N=2$, $\mu_0=3\Omega/2$, $\omega=6\ \Omega$, and $\kappa=0.2\ \Omega$). Wide (green or gray) line: numerical data; dashed (black) line: perturbation theory [cf. Eq. (A8)]. The probabilities displayed here should be measurable experimentally (see Ref. [12]).

interaction. However, there seems to be a sum of nonzero terms proportional to t even for $\kappa=0$. As it is not obvious that these terms cancel, the next step will be to demonstrate that $a_2^{(2)}$ indeed approaches zero for vanishing interaction.

As shown in the Appendix, the terms proportional to t in $a_2^{(2)}$ are due to situations such that Eq. (22) is fulfilled. In the limit $\kappa\to 0$ this results again in condition (23), independent of the particle number. The part of $a_2^{(2)}$ which increases linearly in time is thus proportional to

$$A \equiv \sum_{k=-\infty}^{\infty} J_k(2\mu_1/\omega)J_{1-k}(2\mu_1/\omega)\frac{1}{\sigma_k}. \qquad (26)$$

Dividing the sum into two parts ($\Sigma_{k=1}^{\infty}\cdots+\Sigma_{\ell=-\infty}^{0}\cdots$) and then setting $1-\ell=k$, one obtains

$$A = \sum_{k=1}^{\infty} J_k(2\mu_1/\omega)J_{1-k}(2\mu_1/\omega)\left(\frac{1}{\sigma_k}+\frac{1}{\sigma_{1-k}}\right). \qquad (27)$$

In the limit $\kappa\to 0$, the position of the half-integer resonance approaches the value for $N=2$ particles. Therefore, one has $\sigma_k=k\omega-\omega/2$ and thus $\sigma_{1-k}=-\sigma_k$ which implies

$$A = 0. \qquad (28)$$

Thus, in agreement with the numerics, the half-integer-photon peak disappears with vanishing interactions.

V. FRACTIONAL INTEGER RESONANCES

Fractional integer resonances are not, however, restricted to the half-integer resonances investigated numerically in Ref. [3] and analytically in Sec. IV. For $N=3$ particles and a driving frequency such that $\omega/3=2\mu_0$, the condition

$$\sigma_k + \tilde{\sigma}_\ell + \tilde{\tilde{\sigma}}_m = 0 \qquad (29)$$

with

$$\tilde{\tilde{\sigma}}_m = m\omega - 2\mu_0 - 2(N-5)\kappa \qquad (30)$$

(throughout this section, $N=3$) is fulfilled for

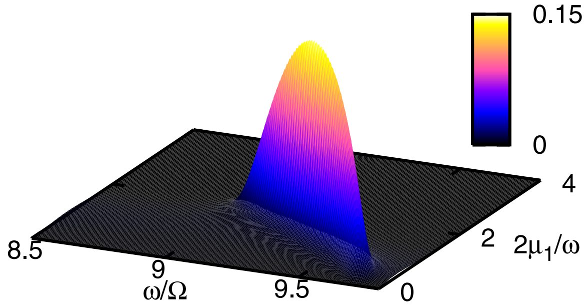

FIG. 5. (Color online) The plot illustrates the 1/3 resonance for $N=3$ particles and interaction parameter $N\kappa/\Omega=1.5$. Initially, all three particles were in the lower well; the static tilt is again given by $2\mu_0/\Omega=3$. The time-averaged probability (averaged over time $T\Omega=100$) to find all particles in the upper well as a function of the driving frequency ω/Ω and the driving amplitude $2\mu_1/\omega$ has a clear peak at $\omega\approx 9\ \Omega$.

$$k+\ell+m=1. \tag{31}$$

The amplitude to find three particles in well 2 again contains oscillatory terms; the term which becomes the leading-order term for large t can be obtained by a calculation analogously to the half-integer resonance in the Appendix,

$$[a_3^{(3)}]_{\text{linear}} = -\frac{3\Omega^3}{4}\sum_k\sum_\ell J_k(2\mu_1/\omega) \times J_\ell(2\mu_1/\omega)J_{1-\ell-k}(2\mu_1/\omega)\frac{t}{\sigma_k(\sigma_k+\tilde{\sigma}_\ell)}. \tag{32}$$

This one-third photon resonance can indeed be observed in the numerics (see Fig. 5). As this resonance only occurs in third-order perturbation theory (rather than second order for the half-integer resonances), the amplitudes would be rather small for interactions as in Fig. 3. However, choosing an also realistic value of $N\kappa/\Omega=1.5$ leads to a time-averaged population imbalance with a peak height on the same order of magnitude as in Fig. 3. In a similar manner, smaller fractions could be treated in higher order perturbation theory. As the resonances thus are a higher order effect, they will tend to decrease.

VI. CONCLUSION

Contrary to the integer-photon peaks [3], fractional-integer-photon peaks cannot be explained by simply replacing the time-dependent Hamiltonian by a time-independent Hamiltonian with renormalized tunneling frequencies. As half-integer resonances already appear for two particles in a double well, this experimentally relevant case [12] was investigated both numerically and analytically. The perturbation calculations can explain for which parameters the non-integer resonances occur. As the fractional-integer resonances are only visible for finite interactions between the particles, they allow us to investigate beyond single-particle effects for very small particle numbers. Experiments similar to Ref. [12] could thus verify fractional-integer peaks in photon-assisted tunneling and thus are useful to understand the emergence of effects similar to the nonlinearities of a mean-field approach well below the limit $N\to\infty$.

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APPENDIX: SECOND-ORDER PERTURBATION THEORY

When solving the integral

$$I_{k,\ell} \equiv \int_0^t \frac{1}{i\sigma_k}\{\exp[i(\sigma_k+\tilde{\sigma}_\ell)t'] - \exp(i\tilde{\sigma}_\ell t')\}dt' \tag{A1}$$

in Eq. (20), one can again assume $\sigma_k\neq 0$ and $\tilde{\sigma}_\ell\neq 0$ as $\sigma_k=0$ and $\tilde{\sigma}_\ell=0$ would correspond to the interphoton resonances discarded here. It then remains to distinguish cases with

$$\sigma_k+\tilde{\sigma}_\ell=0, \tag{A2}$$

which turn out to be the origin of the half-integer resonance, from those for which this equation is not fulfilled. If Eq. (A2) is fulfilled, one has

$$I_{k,\ell} = \frac{1}{i\sigma_k}\left[t+\frac{1}{i\sigma_k}[\exp(-i\sigma_k t)-1]\right], \tag{A3}$$

otherwise

$$I_{k,\ell} = \frac{1}{i\sigma_k}\left[\frac{\exp[i(\sigma_k+\tilde{\sigma}_\ell)t]-1}{i(\sigma_k+\tilde{\sigma}_\ell)} - \frac{\exp(i\tilde{\sigma}_\ell t)-1}{i\tilde{\sigma}_\ell}\right]. \tag{A4}$$

Collecting all terms given by Eq. (A3), one has the leading-order contribution,

$$[a_2^{(2)}(t)]_{\text{leading order}} = -\frac{\Omega^2}{2}\sum_{k=-\infty}^{\infty} J_k(x)J_{1-k}(x)\frac{t}{\sigma_k} - \frac{\Omega^2}{2}\sum_{k=-\infty}^{\infty} J_k(x)J_{1-k}(x)\frac{\exp(-i\sigma_k t)-1}{i\sigma_k}, \tag{A5}$$

with

$$x\equiv 2\mu_1/\omega, \tag{A6}$$

and an oscillatory part

$$[a_2^{(2)}(t)]_{\text{oscil}} = -\frac{\Omega^2}{2}\sum_{k=-\infty}^{\infty}\sum_{\ell\neq 1-k}\frac{J_k(x)J_\ell(x)i^{k+l-1}}{\sigma_k} \times\left(\frac{\exp[i(\sigma_k+\tilde{\sigma}_\ell)t]-1}{[i(\sigma_k+\tilde{\sigma}_\ell)]} - \frac{\exp(i\tilde{\sigma}_\ell t)-1}{i\tilde{\sigma}_\ell}\right). \tag{A7}$$

While Eq. (A5) includes the leading-order behavior for large times and most parameters, it vanishes in the limit $\mu_1 \to 0$. Thus to evaluate the analytic formula with the help of a computer algebra program, we include the only nonvanishing term for $\mu_1 = 0$ to obtain the data displayed in Fig. 4,

$$[a_2^{(2)}(t)]_{\mathrm{approx}} = [a_2^{(2)}(t)]_{\mathrm{leading\ order}} - \frac{\Omega^2}{2} \frac{J_0(x)J_0(x)i^{-1}}{\sigma_0} \times \left(\frac{\exp[i(\sigma_0 + \tilde{\sigma}_0)t] - 1}{[i(\sigma_0 + \tilde{\sigma}_0)]} - \frac{\exp(i\tilde{\sigma}_0 t) - 1}{i\tilde{\sigma}_0} \right). \tag{A8}$$

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