Spectrum conditions for symmetric extendible states

Geir Ove Myhr* and Norbert Lütkenhaus

Institut für Theoretische Physik I and Max-Planck Institute for the Science of Light, Universität Erlangen-Nürnberg, Staudtstraße 7/B2, 91058 Erlangen, Germany and Department of Physics and Astronomy and Institute for Quantum Computing, University of Waterloo, 200 University Avenue W., Waterloo, Ontario, Canada N2L 3G1

(Received 23 February 2009; published 9 June 2009)

We analyze bipartite quantum states that admit a symmetric extension. Any such state can be decomposed into a convex combination of states that allow a *pure* symmetric extension. A necessary condition for a state to admit a pure symmetric extension is that the spectra of the local and global density matrices are equal. This condition is also sufficient for two qubits but not for any larger systems. Using this condition, we present a conjectured necessary and sufficient condition for a two-qubit state to admit symmetric extension, which we prove in some special cases. The results from symmetric extension carry over to degradable and antidegradable channels and we use this to prove that all degradable channels with qubit output have a qubit environment.

DOI: 10.1103/PhysRevA.79.062307 PACS number(s): 03.67.Mn, 03.65.Ta

I. INTRODUCTION

Different bipartite quantum states can be useful for different tasks, and one of the goals of quantum information theory is to find out which properties are required from a state for it to be a useful resource for a given task. Some mathematical properties of the states can tell something about what you can or cannot do with it. For example, if the partial transpose of a state is a positive semidefinite operator it is not possible to distill entanglement from that state no matter how many copies one has available [1]. Similarly, for a state ρ_{AB}, if the operator $I_A \otimes \rho_B - \rho_{AB}$ is not positive semidefinite, it *is* possible to distill entanglement from many copies [2]. For distilling secret key, the only known precondition is that the state must be entangled [3], i.e., it is not possible to express it as a convex combination of pure product states.

One can consider the tasks of distilling entanglement or secret key using classical communication in one direction. In this work, we will consider communication from a party named Alice in possession of system A to a party named Bob in possession of system B. If a state admits a *symmetric extension* to two copies of B none of these tasks will be possible due to the monogamy of entanglement and secret key. The focus in this work is on characterizing the states that admit a symmetric extension.

The bipartite quantum states we consider live on the system AB with the two subsystems A and B. The corresponding Hilbert spaces are $\mathcal{H}_A$, $\mathcal{H}_B$ and $\mathcal{H}_{AB}=\mathcal{H}_A \otimes \mathcal{H}_B$. To the system AB we add another system B', which is a copy of B, and an isometry between B and B', so that for an operator on or vector in $\mathcal{H}_B$, there is a corresponding one in $\mathcal{H}_{B'}$. The extended system is ABB' with Hilbert space $\mathcal{H}_{ABB'}=\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_{B'}$.

Because of the isometry, we can define the swap operator $P_{BB'}$ as the unitary operator that interchanges states on the two systems B and B'. In terms of corresponding orthogonal bases $P_{BB'}=\Sigma_{ij}|ij\rangle\langle ji|$. The swap is a Hermitian operator since it is unitary and $P_{BB'}^2=I$. We say that a state $\rho_{ABB'}$ is *symmetric* if $\rho_{ABB'}=P_{BB'}\rho_{ABB'}P_{BB'}^{\dagger}$. For the main part of this paper, we ignore whether a state has support only on the symmetric subspace [states that satisfy $\rho_{ABB'}=\pi_+\rho_{ABB'}\pi_+^{\dagger}$ for $\pi_{\pm} := (I \pm P_{BB'})/2$] or only on the antisymmetric subspace (states that satisfy $\rho_{ABB'}=\pi_-\rho_{ABB'}\pi_-^{\dagger}$) or both, and in general a symmetric state will have support on both (but see Appendix A).

Finally, we say that a bipartite state ρ_{AB} has a symmetric extension (or is *symmetric extendible*) if there exists a tripartite state $\sigma_{ABB'}$ such that $\mathrm{tr}_{B'}[\sigma_{ABB'}]=\rho_{AB}$ and $\sigma_{ABB'}=P_{BB'}\sigma_{ABB'}P_{BB'}^{\dagger}$, i.e., $\sigma_{ABB'}$ is symmetric.

The intuition behind this is that when ρ_{AB} has a symmetric extension, there may be another system B' around so that the state on AB' is exactly the same as on AB, $\rho_{AB}=\rho_{AB'}$. The system B' would be part of the environment or—in the context of cryptographic applications—in the hands of an eavesdropper. This joint state $\rho_{ABB'}$ needs not be symmetric in order to satisfy $\rho_{AB}=\rho_{AB'}$, but if such a $\rho_{ABB'}$ exists, the state $\sigma_{ABB'}=(\rho_{ABB'}+P_{BB'}\rho_{ABB'}P_{BB'}^{\dagger})/2$ exists, is symmetric, and has the same reductions to AB and AB'. We can therefore impose the symmetry to the extended state for free, and since this puts more constraints on $\sigma_{ABB'}$, it simplifies the problem somewhat.

This relates the symmetric extension to the monogamy of entanglement. In its basic form, the monogamy states that if Alice is maximally entangled with Bob, then she cannot be entangled with anyone else. This extends to any pure entangled state: if there are three particles, A, B, and C, and the reduced state on AB is pure and entangled, A or B cannot be entangled with particle C. Once mixed states are introduced, it is possible for one system to be entangled with more than one other system, but in this case the *amount* of entanglement may be limited. For instance, some entanglement parameters E satisfy monogamy inequalities such as $E(\rho_{AB})+E(\rho_{AC}) \leq E(\rho_{A(BC)})$ [4–6], where the parentheses mean that BC is treated as a single system. The symmetric extension characterizes the monogamy of mixed-state entanglement in the following sense: if Alice and Bob share the entangled quantum state ρ_{AB}, Alice's system can be entangled with a

*gomyhr@iqc.ca

third party Charlie *in the exact same way* (i.e., $\rho_{AB}=\rho_{AC}$) if and only if ρ_{AB} has a symmetric extension.

In general, one can consider extensions to n_A copies of system A and n_B copies of system B and this is called a (n_A,n_B)-symmetric extension. This has been used to derive algorithms for deciding whether a state is entangled or separable [7]. Questions such as whether a state admits symmetric extensions can also be formulated as quantum marginal problems [8–12]. Asking if a state ρ_{AB} has a $(1,N)$-symmetric extension is just a special case of the marginal problem of deciding if there exists a state on the $N+1$ systems $A,B_1,\ldots,B_N$ with given reduced states ρ_{AB_j}. This becomes a symmetric extension when one demands that all ρ_{AB_j} are equal to the given ρ_{AB} which is to be extended. If one such state exists it can always be symmetrized to give a state that is invariant under any permutations of the systems B_j.

Since we are interested in the one-way communication aspect, we will only be considering (1,2)-symmetric extensions. In this setting, any state ρ_{AB} where $\rho_A:=\mathrm{tr}_B[\rho_{AB}]$ is maximally mixed corresponds to a channel through the Choi-Jamiołkowski isomorphism. Those states that are also symmetric extendible correspond to *antidegradable* channels [13].

The reason that symmetric extension is interesting in a one-way classical communications setting is that no matter what operations Alice and Bob perform, the state will keep a symmetric extension if communication from Bob to Alice is not allowed.

Lemma 1 (*Nowakowski and Horodecki* [14]). Let Λ be a (not necessarily trace-preserving) quantum operation that can be realized with local operations assisted by one-way classical communication (1-LOCC), i.e., it is of the form

$$\Lambda(\rho)=\sum_{ij}(I\otimes B_{ij})(A_i\otimes I)\rho(A_i\otimes I)^{\dagger}(I\otimes B_{ij})^{\dagger}, \qquad (1)$$

where $\Sigma_i A_i^{\dagger}A_i\leq I$ and $\Sigma_j B_{ij}^{\dagger}B_{ij}=I$ for all i since Bob cannot communicate the outcome of a probabilistic operation back to Alice. If ρ_{AB} admits a symmetric extension, then so does $\Lambda(\rho_{AB})$.

An interesting special case is when Alice performs an invertible filter operation and Bob performs a unitary. Then the operation can be reversed with nonzero probability, so the output state admits a symmetric extension if *and only if* the input state admits one.

Knowing when a state admits a symmetric extension can also be useful in the analysis of two-way distillation protocols for entanglement or secret key. A two-way protocol consists of a finite number of one-way rounds going in alternating directions. Before the last round, the state cannot have a symmetric extension to two copies of the receiving party's system if the protocol is to succeed [15].

This paper is organized as follows. In Sec. II, we show that any state with symmetric extension can be written as a convex combination of states with pure symmetric extension. In Sec. III, we give a necessary condition for a state to have a pure symmetric extension. This condition is proved to be sufficient for two qubits in Sec. IV, and Sec. V shows that this is not true for any higher dimension. In Sec. VI, we give a conjectured necessary and sufficient condition for a two-qubit state, which we prove in some special cases. The techniques from the previous sections are applied to antidegradable and degradable channels in Sec. VII.

II. DECOMPOSITION INTO PURE-EXTENDIBLE STATES

Separable quantum states are those states that can be written as convex combinations of product states $\rho_A\otimes\rho_B$ and they can even be decomposed further into convex combinations of pure product states. That is,

$$\rho_{\mathrm{sep}}=\sum_j p_j|\psi_j\rangle\langle\psi_j|\otimes|\phi_j\rangle\langle\phi_j|. \qquad (2)$$

Although it can be difficult to determine whether or not a given state can be written on this form or not—and if it can, to find some $|\psi_j\rangle$ and $|\phi_j\rangle$ explicitly—the fact that all separable states can be written like this allows us to prove properties of separable states in general.

One may ask if there is an analog to this for states that allow for a symmetric extension. Clearly, it is not true that any ρ_{AB} that has a symmetric extension can be decomposed into pure states with the same property. This is because the only pure states that have a symmetric extension are the pure product states, and their convex hull is the set of separable states. But it turns out that if we consider the *extended* states—the $\rho_{ABB'}$ that are invariant under exchange of B and B'—they can be written as convex combinations of pure states with the same property. In fact, the pure states in the spectral decomposition can be chosen to have this property.

Lemma 2. A tripartite state $\rho_{ABB'}$ which is invariant under exchange of B and B', $\rho_{ABB'}=P_{BB'}\rho_{ABB'}P_{BB'}^{\dagger}$, can be written in the spectral decomposition

$$\rho_{ABB'}=\sum_j\lambda_j|\phi_j\rangle\langle\phi_j| \qquad (3)$$

in such a way that $|\phi_j\rangle\langle\phi_j|=P_{BB'}|\phi_j\rangle\langle\phi_j|P_{BB'}^{\dagger}$, i.e., $P_{BB'}|\phi_j\rangle=\pm|\phi_j\rangle$.

Proof. Since $\rho_{ABB'}=P_{BB'}\rho_{ABB'}P_{BB'}^{\dagger}$, $\rho_{ABB'}P_{BB'}=P_{BB'}\rho_{ABB'}$, so $\rho_{ABB'}$ and $P_{BB'}$ are commuting diagonalizable operators and therefore have a common set of eigenvectors. Since $P_{BB'}^2=I$, $P_{BB'}$ has eigenvalues ±1 and all its eigenvectors therefore satisfy $P_{BB'}|\phi\rangle=\pm|\phi\rangle$. ■

The above lemma applies to the extended state $\rho_{ABB'}$, but our main interest is for bipartite states ρ_{AB} that *admit* a symmetric extension. By tracing out the B' system we get the following.

Corollary 3. A bipartite quantum state ρ_{AB} admits a symmetric extension if and only if it can be written as a convex combination,

$$\rho_{AB}=\sum_j p_j\rho_{AB}^j,\quad 0\leq p_j\leq 1,\quad \sum_j p_j=1, \qquad (4)$$

of states ρ_{AB}^j which allow a pure symmetric extension.

Hence, all the extremal states in the convex set of symmetric extendible states are extendible to pure states. We will call those states *pure extendible*. In Sec. III, we give a simple necessary condition for a state to be pure extendible, and in the following sections we show that it is sufficient if and only if it is a state on two qubits.

III. SPECTRUM CONDITION FOR PURE-EXTENDIBLE STATES

Let $\vec{\lambda}(\rho)$ denote the vector of nonzero eigenvalues of ρ in nonincreasing order.

Theorem 4. Let ρ_{AB} be a state that allows a pure symmetric extension to $|\psi\rangle\langle\psi|_{ABB'}$. Then

$$\vec{\lambda}(\rho_{AB}) = \vec{\lambda}(\rho_B). \tag{5}$$

Proof. Using the Schmidt decomposition with the splitting $AB|B'$, we can write the extended state as

$$|\psi\rangle_{ABB'} = \sum_j \sqrt{\lambda_j}|\phi_j\rangle_{AB}|j\rangle_{B'}. \tag{6}$$

The reduced density matrices of this state are

$$\rho_{AB} = \sum_j \lambda_j|\phi_j\rangle\langle\phi_j|, \quad \rho_{B'} = \sum \lambda_j|j\rangle\langle j|, \tag{7}$$

i.e., the spectra of ρ_{AB} and $\rho_{B'}$ are equal. By symmetry between B and B', $\rho_B = \rho_{B'}$ so $\vec{\lambda}(\rho_{AB}) = \vec{\lambda}(\rho_B)$. ■

In general, we do not expect all states that satisfy condition (5) to have a pure symmetric extension. The following corollary provides a test that can rule out a pure symmetric extension.

Corollary 5. For any state ρ_{AB} that has a pure symmetric extension and any operator M on $\mathcal{H}_A$, the (un-normalized) state

$$\tilde{\rho}_{AB} = (M \otimes I_B)\rho_{AB}(M \otimes I_B)^\dagger \tag{8}$$

satisfies condition (5).

Proof. Let $|\psi\rangle_{ABB'} = \pm P_{BB'}|\psi\rangle_{ABB'}$ be the pure symmetric extension of ρ_{AB}. The filter M acts only on $\mathcal{H}_A$, so it commutes with $P_{BB'}$. Therefore $M|\psi\rangle_{ABB'} = \pm MP_{BB'}|\psi\rangle_{ABB'} = \pm P_{BB'}M|\psi\rangle_{ABB'}$, so $M|\psi\rangle_{ABB'}$ is a symmetric extension of its reduced state $\tilde{\rho}_{AB}$. Because of theorem 4, $\tilde{\rho}_{AB}$ then satisfies Eq. (5). ■

This condition is useful since if given a state that is not pure extendible but satisfies condition (5), applying a random filter on system A will usually break the condition and reveal that it is not pure extendible.

IV. SUFFICIENCY FOR TWO QUBITS

In this section it is shown that if ρ_{AB} is a two-qubit state and satisfies $\vec{\lambda}(\rho_{AB}) = \vec{\lambda}(\rho_B)$, then there exists a pure state $|\psi\rangle_{ABB'}$ such that $|\psi\rangle_{ABB'} = P_{BB'}|\psi\rangle_{ABB'}$ and $\mathrm{tr}_{B'}[|\psi\rangle\langle\psi|_{ABB'}] = \rho_{AB}$. We first start by giving an equivalent condition to the spectrum condition.

Lemma 6. Given a bipartite state ρ_{AB}, then $\vec{\lambda}(\rho_{AB}) = \vec{\lambda}(\rho_B)$ if and only if there exists a pure tripartite state $|\psi\rangle_{ABB'}$ with reductions ρ_{AB}, ρ_B, and $\rho_{B'}$ where $\rho_B = \rho_{B'}$.

Proof. Assume $\vec{\lambda}(\rho_{AB}) = \vec{\lambda}(\rho_B) = (\lambda_j)$. We can write the states in the spectral decomposition, $\rho_{AB} = \Sigma_j \lambda_j|\varphi_j\rangle\langle\varphi_j|$, $\rho_B = \Sigma_j \lambda_j|b_j\rangle\langle b_j|$. Then, a purification of ρ_{AB} is

$$|\psi\rangle_{ABB'} = \sum_j \sqrt{\lambda_j}|\varphi_j\rangle_{AB}|b_j\rangle_{B'}. \tag{9}$$

Tracing out the AB system we get $\rho_{B'} = \Sigma_j \lambda_j|b_j\rangle\langle b_j| = \rho_B$.

Conversely, assume that there exists a pure (not necessarily symmetric) extension of ρ_{AB}, $|\psi\rangle_{ABB'}$ with the reduced states $\rho_B = \rho_{B'}$. In the spectral decomposition, $\rho_B = \rho_{B'} = \Sigma_j \lambda_j|b_j\rangle\langle b_j|$. A purification of $\rho_{B'}$ to ABB' is Eq. (9), and the spectrum of ρ_{AB} is (λ_j), just like ρ_B. ■

Theorem 7. For a two-qubit state, $\vec{\lambda}(\rho_{AB}) = \vec{\lambda}(\rho_B)$ is a necessary and sufficient condition for it to have a pure symmetric extension.

Proof. The condition is necessary for any dimension and this is already dealt with in Sec. III. Here, we only prove sufficiency for two qubits. By lemma 6, the condition implies that there exists a purification $|\psi\rangle_{ABB'}$ of ρ_{AB} which is such that $\rho_B = \rho_{B'}$. We will prove that for such a pure state, there is always a unitary operator on the B' system alone that will make it symmetric between B and B'.

First, we prove the special case when ρ_B is completely mixed. Then $\rho_{BB'} = \mathrm{tr}_A[|\psi\rangle\langle\psi|_{ABB'}]$ is a state with maximally mixed subsystems. For such a state, there exist local unitaries $U_B, V_{B'}$ such that $(U_B \otimes V_{B'})\rho_{BB'}(U_B \otimes V_{B'})^\dagger$ is Bell-diagonal [16]. Moreover, since A is a qubit, $\rho_{BB'}$ is of rank 2 and we have

$$(U_B \otimes V_{B'})\rho_{BB'}(U_B \otimes V_{B'})^\dagger = p|\psi_1\rangle\langle\psi_1| + (1-p)|\psi_2\rangle\langle\psi_2|, \tag{10}$$

with $|\psi_1\rangle$ and $|\psi_2\rangle$ being two of the four Bell-diagonal states $|\Phi^\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$, $|\Psi^\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$. Since the Bell basis can be permuted arbitrarily with local unitaries [17], we can choose U_B and $V_{B'}$ such that $|\psi_1\rangle = |\Phi^+\rangle$ and $|\psi_2\rangle = |\Phi^-\rangle$, so that we avoid the antisymmetric state $|\Psi^-\rangle$. The state in Eq. (10) can now be purified to $\sqrt{p}|0\rangle_A|\Phi^+\rangle_{BB'} + \sqrt{1-p}|1\rangle_A|\Phi^-\rangle_{BB'}$. Since all purifications of a state are equivalent up to a local unitary on the purifying system—in this case A—this is related to the pure state that we started out with as

$$(T_A \otimes U_B \otimes V_{B'})|\psi\rangle_{ABB'} = \sqrt{p}|0\rangle_A \otimes |\Phi^+\rangle_{BB'} + \sqrt{1-p}|1\rangle_A \otimes |\Phi^-\rangle_{BB'}, \tag{11}$$

where T_A is the unitary operator on A that relates this purification to the one where A is left unchanged. We now perform the unitary $T_A^\dagger \otimes U_B^\dagger \otimes U_{B'}^\dagger$ on the state, and a unitary of this form will not change the symmetry between B and B'. This gives

$$(I_A \otimes I_B \otimes U_B^\dagger V_{B'})|\psi\rangle_{ABB'} = (T_A^\dagger \otimes U_B^\dagger \otimes U_{B'}^\dagger) \times (\sqrt{p}|0\rangle_A \otimes |\Phi^+\rangle_{BB'} + \sqrt{1-p}|1\rangle_A \otimes |\Phi^-\rangle_{BB'}). \tag{12}$$

From this we can conclude that performing the unitary $U^\dagger V$ on system B' will take the starting state $|\psi\rangle_{ABB'}$ to a symmetric one, so the state ρ_{AB} has a symmetric extension.

We now consider the generic case when the reduced state ρ_B is *not* maximally mixed. In this case, the two nondegenerate eigenvectors of ρ_B provide a preferred basis for B and the corresponding basis in B' is an eigenbasis for $\rho_{B'}$. By choosing the bases in this way, we make sure that $\rho_B=\rho_{B'}$ are diagonal.

An arbitrary state vector of the system ABB' can be written as $a|000\rangle+b|001\rangle+c|010\rangle+d|011\rangle+e|100\rangle+f|101\rangle+g|110\rangle+h|111\rangle$, where $a,\ldots,h$ are complex numbers whose absolute square sum to 1. It is symmetric under permutation of B and B' if $b=c$ and $f=g$. In Appendix B we show that imposing that the reduced states ρ_B and $\rho_{B'}$ are equal, diagonal, and are not maximally mixed implies that the amplitudes satisfy

$$|b|=|c|, \quad |f|=|g|, \tag{13}$$

and

$$|c||g|(\exp[i(\phi_b-\phi_c)]-\exp[i(\phi_f-\phi_g)])=0, \tag{14}$$

where $b=|b|\exp[i\phi_b]$ and similarly for c, f, and g. So while the absolute values of the relevant amplitudes are equal, the complex phases might be off. This can be corrected with a phase gate on B' as follows. If $b=c=0$, the unitary operator on B' is

$$U_{B'}=|0\rangle\langle 0|+\exp[-i(\phi_f-\phi_g)]|1\rangle\langle 1| \tag{15}$$

and if $f=g=0$ it is

$$U_{B'}=|0\rangle\langle 0|+\exp[-i(\phi_b-\phi_c)]|1\rangle\langle 1|. \tag{16}$$

If none of the relevant amplitudes are zero, Eq. (14) implies that the two expressions are equal, so the same unitary operator will correct both amplitude relations.

Hence, for two-qubit states ρ_{AB} that satisfy spectrum condition (5), we have shown that there exists a pure state vector $|\psi\rangle_{ABB'}$ which is symmetric, $|\psi\rangle_{ABB'}=P_{BB'}|\psi\rangle_{ABB'}$. ■

This theorem, together with corollary 3, fully characterizes the set of two-qubit states with symmetric extension. It is the convex hull of the set of states that satisfies condition (5). Not all the states that satisfy Eq. (5) are extremal, however. While any pure-extendible state that is itself pure (i.e., a product state) is extremal for both the set of states and the subset of extendible states, there are some mixed pure-extendible states that are not extremal. The following proposition characterizes the mixed nonextremal pure-extendible states of two qubits.

Proposition 8. For a two-qubit mixed pure-extendible state ρ_{AB} the following are equivalent:

(i) ρ_{AB} can be written as a convex combination of other pure-extendible states,

(ii) ρ_{AB} is separable, and

(iii) ρ_{AB} is of the form

$$\rho_{AB}=\lambda|\psi_0 0\rangle\langle\psi_0 0|+(1-\lambda)|\psi_1 1\rangle\langle\psi_1 1|, \tag{17}$$

where $\langle 0|1\rangle=0$, $\langle\psi_0|\psi_1\rangle$ is arbitrary, and $0<\lambda<1$.

Proof. $3\Rightarrow 2$ is trivial as the state in Eq. (17) is a convex combination of two product states. $2\Rightarrow 1$ is also trivial since any mixed separable state can be decomposed into a convex combination of pure product states $\rho_{AB}=\sum_j p_j|\psi_j\phi_j\rangle\langle\psi_j\phi_j|$ and the product states have the pure symmetric extension $|\psi_j\rangle_A|\phi_j\rangle_B|\phi_j\rangle_{B'}$.

The only nontrivial part is $1\Rightarrow 3$. For this part, assume that ρ_{AB} can be written as a convex combination of other pure-extendible states,

$$\rho_{AB}=\sum_j p_j\rho_{AB}^j, \tag{18}$$

where ρ_{AB} and all ρ_{AB}^j satisfy spectrum condition (5). Tracing out A gives

$$\rho_B=\sum_j p_j\rho_B^j. \tag{19}$$

Since ρ_{AB} has support on a two-dimensional subspace, the support of the ρ_{AB}^j must be on that same subspace. We can parametrize the states on AB by Pauli operators $I_S,\Sigma_x,\Sigma_y,\Sigma_z$ on this two-dimensional subspace,

$$\rho_{AB}^j=\tfrac{1}{2}(I_S+X_j\Sigma_x+Y_j\Sigma_y+Z_j\Sigma_z)=\tfrac{1}{2}(I_S+\vec{R}\cdot\vec{\Sigma}). \tag{20}$$

Note that the I_S here is not the identity on the four-dimensional Hilbert space of the system AB but a projector to the two-dimensional support of ρ_{AB}. The reduced states on system B can be written as

$$\rho_B^j=\tfrac{1}{2}(I_B+x_j\sigma_x+y_j\sigma_y+z_j\sigma_z)=\tfrac{1}{2}(I_B+\vec{r}\cdot\vec{\sigma}), \tag{21}$$

where $\sigma_x,\sigma_y,\sigma_z$ are the Pauli operators on the qubit B. Similarly, we can write $\rho_{AB}^j=(I_S+\vec{R}_j\cdot\vec{\Sigma})/2$ and $\rho_B^j=(I_B+\vec{r}_j\cdot\vec{\sigma})/2$. In this representation, Eqs. (18) and (19) become $\vec{R}=\sum_j p_j\vec{R}_j$ and $\vec{r}=\sum_j p_j\vec{r}_j$.

The eigenvalues of ρ_{AB} and ρ_B are determined by the length of the vectors $\vec{R}$ and $\vec{r}$,

$$\vec{\lambda}(\rho_{AB})=\tfrac{1}{2}(1+|\vec{R}|,1-|\vec{R}|), \tag{22}$$

$$\vec{\lambda}(\rho_B)=\tfrac{1}{2}(1+|\vec{r}|,1-|\vec{r}|). \tag{23}$$

The ρ_{AB}^j and ρ_{AB} are pure extendible, so they satisfy condition (5). In terms of the above parametrization, this means that $|\vec{R}_j|=|\vec{r}_j|$ and $|\vec{R}|=|\vec{r}|$.

Since tracing out a part of a quantum system never can increase the trace distance between the states [18], we have

$$\|\rho_{AB}^j-\rho_{AB}^k\|_1\geq\|\rho_B^j-\rho_B^k\|_1. \tag{24}$$

The trace distance can be written in terms of $\vec{R}_j$ and $\vec{r}_j$ as $\|\rho_{AB}^j-\rho_{AB}^k\|_1=|\vec{R}_j-\vec{R}_k|$ and $\|\rho_B^j-\rho_B^k\|_1=|\vec{r}_j-\vec{r}_k|$. From $|\vec{R}_j-\vec{R}_k|^2\geq|\vec{r}_j-\vec{r}_k|^2$ we get

$$|\vec{R}_j|^2-2\vec{R}_j\cdot\vec{R}_k+|\vec{R}_k|^2\geq|\vec{r}_j|^2-2\vec{r}_j\cdot\vec{r}_k+|\vec{r}_k|^2,$$

and since $|\vec{R}_j|=|\vec{r}_j|$, this gives

$$\vec{R}_j\cdot\vec{R}_k\leq\vec{r}_j\cdot\vec{r}_k. \tag{25}$$

Now we can use $|\vec{R}|=|\vec{r}|$ and Eq. (25) to show that when ρ_{AB} is a pure-extendible state, the trace distance between the ρ_{AB}^j does not decrease when system A is traced out. Here,

$$|\vec{R}|^2=\left(\sum_j p_j\vec{R}_j\right)\left(\sum_k p_k\vec{R}_k\right) \tag{26}$$

$$=\sum_j p_j^2|\vec{R}_j|^2+2\sum_{j<k} p_jp_k\vec{R}_j\cdot\vec{R}_k, \tag{27}$$

$$|\vec{r}|^2=\sum_j p_j^2|\vec{r}_j|^2+2\sum_{j<k} p_jp_k\vec{r}_j\cdot\vec{r}_k, \tag{28}$$

so by demanding $|\vec{R}|=|\vec{r}|$ and using $|\vec{R}_j|=|\vec{r}_j|$, we get

$$\sum_{j<k} p_jp_k\vec{R}_j\cdot\vec{R}_k=\sum_{j<k} p_jp_k\vec{r}_j\cdot\vec{r}_k. \tag{29}$$

By Eq. (25) none of the terms on the left-hand side (LHS) can be greater than the corresponding term on the right-hand side (RHS). The only way for this to be satisfied is that

$$\vec{R}_j\cdot\vec{R}_k=\vec{r}_j\cdot\vec{r}_k \tag{30}$$

for all pairs (j,k). By reversing the calculation leading to Eq. (25) we get that $|\vec{R}_j-\vec{R}_k|^2=|\vec{r}_j-\vec{r}_k|^2$ and

$$\|\rho_{AB}^j-\rho_{AB}^k\|_1=\|\rho_B^j-\rho_B^k\|_1. \tag{31}$$

The next step is to use Eq. (31) to find the structure of the support of ρ_{AB}. The difference $\rho_{AB}^j-\rho_{AB}^k$ must be on the same two-dimensional subspace that all the ρ_{AB}^j are confined to. Being the difference between two operators with trace one, it is also traceless, so in the spectral decomposition it can be written as

$$\rho_{AB}^j-\rho_{AB}^k=r|\psi_+\rangle\langle\psi_+|-r|\psi_-\rangle\langle\psi_-| \tag{32}$$

for some $r\geq 0$. The orthogonal vectors $|\psi_+\rangle$ and $|\psi_-\rangle$ define the two-dimensional support of ρ_{AB}^j and ρ_{AB}. From Eq. (31) and taking the trace norm of both sides of Eq. (32) it is clear that $\|\rho_B^j-\rho_B^k\|_1=2r$.

Let $\rho_B^+=\mathrm{tr}_A|\psi_+\rangle\langle\psi_+|$ and $\rho_B^-=\mathrm{tr}_A|\psi_-\rangle\langle\psi_-|$. Tracing out the A system in Eq. (32) and taking the trace norm gives $r\|\rho_B^+-\rho_B^-\|_1=\|\rho_B^j-\rho_B^k\|_1=2r$ or

$$\|\rho_B^+-\rho_B^-\|_1=2. \tag{33}$$

This is the maximal distance between two states in trace norm, and it means that ρ_B^+ and ρ_B^- have support on orthogonal subspaces. Since B is a qubit, ρ_B^+ and ρ_B^- must be orthogonal pure states which we denote $\rho_B^+=|0\rangle\langle 0|$, $\rho_B^-=|1\rangle\langle 1|$. This also means that $|\psi_+\rangle$ and $|\psi_-\rangle$ are product states,

$$|\psi_+\rangle=|\psi_0\rangle\otimes|0\rangle, \tag{34}$$

$$|\psi_-\rangle=|\psi_1\rangle\otimes|1\rangle, \tag{35}$$

where $|\psi_0\rangle$ and $|\psi_1\rangle$ are arbitrary.

Any state on the subspace spanned by $|\psi_+\rangle$ and $|\psi_-\rangle$ can be expressed as

$$\rho_{AB}=\sum_{m,n=0}^{1}\rho_{mn}|\psi_m\rangle\langle\psi_n|\otimes|m\rangle\langle n|, \tag{36}$$

with the reduced state being

$$\rho_B=\sum_{m,n=0}^{1}\rho_{mn}\langle\psi_n|\psi_m\rangle\otimes|m\rangle\langle n|. \tag{37}$$

Since ρ_{AB} is pure extendible, it satisfies Eq. (5) and for qubits this is equivalent to the condition that the purities of the global and reduced states are equal, $\mathrm{tr}(\rho_{AB}^2)=\mathrm{tr}(\rho_B^2)$. The purities are

$$\mathrm{tr}(\rho_{AB}^2)=\rho_{00}^2+\rho_{11}^2+|\rho_{01}|^2, \tag{38}$$

$$\mathrm{tr}(\rho_B^2)=\rho_{00}^2+\rho_{11}^2+|\rho_{01}|^2|\langle\psi_0|\psi_1\rangle|^2. \tag{39}$$

For the purities to be equal, either $\rho_{01}=0$ or $|\langle\psi_0|\psi_1\rangle|=1$. In the first case, the state would be

$$\rho_{AB}=\rho_{00}|\psi_0 0\rangle\langle\psi_0 0|+\rho_{11}|\psi_1 1\rangle\langle\psi_1 1|, \tag{40}$$

which is the sought separable form. In the other case $|\psi_0\rangle$ and $|\psi_1\rangle$ only differ by a phase, so all states in the subspace are product states of the form $|\psi_0\rangle\langle\psi_0|\otimes\rho_B$ which is the special case of Eq. (40) where $|\psi_0\rangle=|\psi_1\rangle$. ■

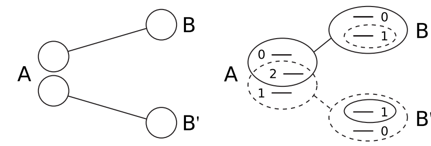

FIG. 1. Examples of tripartite states where ρ_{AB} satisfies spectrum condition (5) but does not have a symmetric extension. The 4×2 state from example 9 on the left and the 3×2 state from example 10 on the right.

V. COUNTEREXAMPLES FOR SYSTEMS WITH HIGHER DIMENSION

In Sec. IV, we have seen that spectrum condition (5) is not only necessary but also sufficient for the state to have a pure symmetric extension when the system considered is a pair of qubits. One may ask if the same might be true for any higher dimensional system. We show some counterexamples that exclude this possibility for any dimension greater than 2×2.

Example 9: (4×2). The simplest example is when Alice holds two qubits and Bob holds one. One of Alice's qubits is maximally mixed, while the other is maximally entangled with Bob's qubit,

$$\rho_{A_1A_2B}=\frac{I_{A_1}}{2}\otimes|\Phi^+\rangle\langle\Phi^+|_{A_2B}. \tag{41}$$

The global density matrix $\rho_{A_1A_2B}$ has nonzero eigenvalues $\{1/2,1/2\}$ and so has the local one ρ_B. The state therefore satisfies the spectrum condition but does not have a symmetric extension since by tracing out A_1, Alice can make a pure maximally entangled state. The purification of the state is illustrated in Fig. 1.

While the above example is conceptually simple, it does not exclude that the spectrum condition could be sufficient when Alice holds a qutrit. The following example is similar in spirit to the above and shows that for system of size 3×2 and higher, the spectrum condition cannot be sufficient.

Example 10: (3×2). Consider the (un-normalized) vectors of a tripartite system

$$|v_1\rangle=|001\rangle+|211\rangle, \tag{42}$$

$$|v_2\rangle = |110\rangle + |211\rangle, \tag{43}$$

where the registers are A, B, and B'. The vectors are illustrated in Fig. 1: the solid line corresponds to $|v_1\rangle$ and the dashed line corresponds to $|v_2\rangle$. The vector $|v_1\rangle$ is entangled between A and B, while $|v_2\rangle$ is entangled between A and B'. Interchanging 0 and 1 at A and swapping B and B' take $|v_1\rangle$ to $|v_2\rangle$ and vice versa. Adding the two vectors and normalizing give the state

$$|\psi\rangle = \frac{1}{\sqrt{6}}|001\rangle + \frac{1}{\sqrt{6}}|110\rangle + \sqrt{\frac{2}{3}}|211\rangle. \tag{44}$$

The reduced states are

$$\rho_{AB} = \tfrac{5}{6}|\psi_{1/5}\rangle\langle\psi_{1/5}| + \tfrac{1}{6}|11\rangle\langle 11|, \tag{45}$$

$$\rho_B = \tfrac{5}{6}|1\rangle\langle 1| + \tfrac{1}{6}|0\rangle\langle 0|, \tag{46}$$

where

$$|\psi_{1/5}\rangle = \frac{1}{\sqrt{5}}|00\rangle + \sqrt{\frac{4}{5}}|21\rangle. \tag{47}$$

The nonzero eigenvalues are the same for ρ_{AB} and ρ_B, so ρ_{AB} satisfies the spectrum condition. However, it does not have a symmetric extension. This is most easily seen by applying the filter $F=|0\rangle\langle 0|+|2\rangle\langle 2|$ to A. This succeeds with probability 5/6 and the state after the filter is the pure entangled state $|\psi_{1/5}\rangle$, which has no symmetric extension.

Both examples above are states that can be extended to states that are invariant under some $U_A \otimes P_{BB'}$, where U_A is a unitary on A but not under $I_A \otimes P_{BB'}$. For the 4×2 case, U_A was the unitary swapping A_1 and A_2, while in the 3×2 example it was $|0\rangle\langle 1|+|1\rangle\langle 0|+|2\rangle\langle 2|$. One can use the same arguments as in the proof of theorem 4 to show that any pure state that has a symmetry of the type $U_A \otimes P_{BB'}$ has a reduction to AB that satisfies condition (5).

The above examples show that condition (5) cannot be sufficient for pure extendibility for $M\times N$ systems where $M\geq 3$ and $N\geq 2$. This leaves open the question whether it is sufficient for $2\times N$ for any $N>2$. We therefore now give an example of a class of states with system dimension 2×3 that satisfies condition (5) but has no symmetric extension.

Example 11: (2×3). Consider states with spectral decomposition

$$\rho_{AB} = \sum_{j=0}^{2} \lambda_j |\psi_j\rangle\langle\psi_j|, \tag{48}$$

where the eigenvectors are $|\psi_0\rangle=|12\rangle$, $|\psi_1\rangle=|02\rangle$, and $|\psi_2\rangle=\sqrt{s}|00\rangle+\sqrt{1-s}|11\rangle$. For such a state to satisfy spectrum condition (5), the eigenvalues must be $\lambda_0=s/2$, $\lambda_1=(1-s)/2$, and $\lambda_2=1/2$. To ρ_{AB} we now apply a filter operation in the standard basis in the A system, $F=\sqrt{p}|0\rangle\langle 0|+|1\rangle\langle 1|$. This is a 1-LOCC operation (not trace-preserving) and cannot break a symmetric extension. After a successful filter, the global and local eigenvalues $\lambda_j(p)$ and $\lambda_j^B(p)$ are

$$\lambda_0(p) = \frac{s}{1+p}, \quad \lambda_0^B(p) = \frac{sp}{1+p},$$

$$\lambda_1(p) = \frac{(1-s)p}{1+p}, \quad \lambda_1^B(p) = \frac{1-s}{1+p},$$

$$\lambda_2(p) = \frac{1-s(1-p)}{1+p}, \quad \lambda_2^B(p) = \frac{1-(1-s)(1-p)}{1+p}.$$

Except when $s\in\{0,1/2,1\}$ or $p\in\{0,1\}$, the spectra of the local and global density matrices are different. Since a filtering like this will keep a pure symmetric extension if the original state had one, ρ_{AB} cannot have a pure symmetric extension. For $1/2<s<1$ and $0<p<1$, the state has no symmetric extension at all. This is because in this regime the coherent information $I(A\rangle B) := S(\rho_B)-S(\rho_{AB})$, where $S(\cdot)$ is the von Neumann entropy, is positive. This is a lower bound to the distillable entanglement with one-way communication from A to B [19]. By monogamy of entanglement, ρ_{AB} cannot have a symmetric extension.

VI. SYMMETRIC EXTENSION OF TWO-QUBIT STATES

In previous sections, we have characterized the *extremal* symmetric extendible two-qubit states as those states that are pure extendible—as characterized by theorem 7—but not of the (separable) form (17). We would also like to extend this to a characterization of *all* states with symmetric extension. In other words we want necessary and sufficient conditions for the ability to write a state as a convex combination of states that satisfy $\vec{\lambda}(\rho^i_{AB})=\vec{\lambda}(\rho^i_B)$. This is similar to the separability question, where the extremal states are pure product states, which are characterized by the more restrictive condition $\vec{\lambda}(\rho^i_{AB})=\vec{\lambda}(\rho^i_B)=(1,0,\ldots)$. Many years of entanglement theory have taught us that even though product states are easy to recognize, the separable states are not, except in special cases (two qubits is one of them). For one thing, even though the pure product states can be characterized through its local and global spectra, we need to know more about the structure to decide if a state is separable—even for two qubits [20]. Nevertheless, we conjecture that two-qubit symmetric extendible states can be characterized solely by the local and global eigenvalues. We present a conjectured necessary and sufficient condition which is supported by numerical evidence and we can prove in some special cases.

Conjecture 12. A two-qubit state ρ_{AB} with reduced state ρ_B has a symmetric extension if and only if

$$\mathrm{tr}(\rho_B^2) \geq \mathrm{tr}(\rho_{AB}^2) - 4\sqrt{\det(\rho_{AB})}. \tag{49}$$

Using techniques from previous sections, we prove the conjecture for states of rank 2. For Bell-diagonal states, necessary and sufficient conditions have been derived using techniques from semidefinite programming [15], and we show that our conjecture is equivalent to those conditions. Finally we show that the conjecture is also true for another special class of states.

A. Rank-2 states

When ρ_{AB} has rank 2, the determinant in Eq. (49) vanishes, and since the remaining inequality only compares the

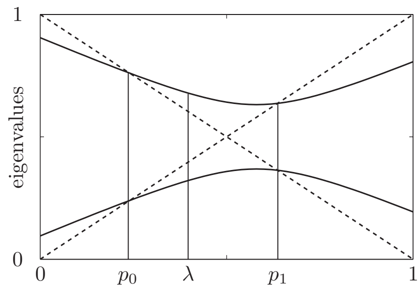

FIG. 2. Decomposition into pure-extendible states for two-qubit states of rank 2. The dashed lines are global eigenvalues as parametrized on the x axis. The solid lines are the local eigenvalues. The state with global eigenvalues $(1-\lambda,\lambda)$ is a convex combination of states with global eigenvalues $(1-p_0,p_0)$ and $(p_1,1-p_1)$, which have the same local eigenvalues and therefore a pure-symmetric extension.

purity of the states, we can as well use the maximum eigenvalues to compare it.

Theorem 13. A two-qubit state ρ_{AB} of rank 2 has a symmetric extension if and only if

$$\lambda_{\max}(\rho_{AB}) \leq \lambda_{\max}(\rho_B). \tag{50}$$

Proof. We first prove the "if" part. Assume that ρ_{AB} is a two-qubit state of rank 2 that satisfies Eq. (50). We can write it in the spectral decomposition

$$\rho_{AB} = (1-\lambda)|\psi_0\rangle\langle\psi_0| + \lambda|\psi_1\rangle\langle\psi_1|. \tag{51}$$

Consider the class of states with the same eigenvectors as above, parametrized by p, $\rho_{AB}^p=(1-p)|\psi_0\rangle\langle\psi_0|+p|\psi_1\rangle\langle\psi_1|$. Now, $\rho_{AB}=\rho_{AB}^\lambda$. For $p=0$ and $p=1$, the corresponding pure states satisfy $\lambda_{\max}(\rho_B^p)\leq\lambda_{\max}(\rho_{AB}^p)=1$. Since at $p=\lambda$, $\lambda_{\max}(\rho_B^p)\geq\lambda_{\max}(\rho_{AB}^p)$ by assumption and $\lambda_{\max}$ is a continuous function of the parameter p, there must exist parameters $p_0 \in[0,\lambda]$, $p_1\in[\lambda,1]$ such that $\lambda_{\max}(\rho_B^{p_0})=\lambda_{\max}(\rho_{AB}^{p_0})$ and $\lambda_{\max}(\rho_B^{p_1})=\lambda_{\max}(\rho_{AB}^{p_1})$ (see Fig. 2). From theorem 7, we know that $\rho_{AB}^{p_0}$ and $\rho_{AB}^{p_1}$ have pure symmetric extensions, $|\psi_{p_0}\rangle_{ABB'}$ and $|\psi_{p_1}\rangle_{ABB'}$. Since ρ_{AB}^λ is a convex combination $\rho_{AB}^\lambda=(1-q)\rho_{AB}^{p_0}+q\rho_{AB}^{p_1}$, where $q=(\lambda-p_0)/(p_1-p_0)$, a symmetric extension of ρ_{AB}^λ is $\rho_{ABB'}=(1-q)|\psi_{p_0}\rangle\langle\psi_{p_0}|_{ABB'} +q|\psi_{p_1}\rangle\langle\psi_{p_1}|_{ABB'}$.

Now, for the "only if" part, assume that ρ_{AB} is a bipartite state of rank 2 that has a symmetric extension to two copies of the qubit B (in this part we do not use that A is a qubit). Then by corollary 3 it can be written as a convex combination of pure-extendible states

$$\rho_{AB} = \sum_j p_j\rho_{AB}^j \tag{52}$$

and tracing out

$$\rho_B = \sum_j p_j\rho_B^j. \tag{53}$$

Like in the proof of proposition 8 we can use the fact that ρ_{AB} has support on a two-dimensional subspace to parametrize it using Pauli operators as in Eq. (20). Likewise we expand ρ_B as in Eq. (21), so Eqs. (52) and (53) become $\vec{R}=\sum_j p_j\vec{R}_j$ and $\vec{r}=\sum_j p_j\vec{r}_j$. We can proceed exactly as in the previous proof to arrive at Eq. (25) which says that $\vec{R}_j\cdot\vec{R}_k \leq\vec{r}_j\cdot\vec{r}_k$ for all j and k.

Since ρ_{AB}^j are pure-extendible states, they have the same eigenvalues as the corresponding ρ_B^j and therefore $|\vec{R}_j|=|\vec{r}_j|$. Now we can use this and Eq. (25) to compare $|\vec{R}|$ and $|\vec{r}|$,

$$|\vec{R}|^2 = \left(\sum_j p_j\vec{R}_j\right)\left(\sum_k p_k\vec{R}_k\right) \tag{54}$$

$$=\sum_j p_j^2|\vec{R}_j|^2 + 2\sum_{j<k} p_jp_k\vec{R}_j\cdot\vec{R}_k \tag{55}$$

$$\leq\sum_j p_j^2|\vec{r}_j|^2 + 2\sum_{j<k} p_jp_k\vec{r}_j\cdot\vec{r}_k \tag{56}$$

$$=|\vec{r}|^2. \tag{57}$$

From $|\vec{R}|\leq|\vec{r}|$ and the relations to eigenvalues (22) and (23) we can conclude that $\lambda_{\max}(\rho_{AB})\leq\lambda_{\max}(\rho_B)$ which completes the proof. ■

Remark 14. The assumption that system A is a qubit was only needed in the if part of the proof to conclude that states that satisfy the spectrum condition $\lambda_{\max}(\rho_B)=\lambda_{\max}(\rho_{AB})$ have a symmetric extension. The rest of the proof, in particular the only if part, is independent of this assumption. Therefore, no $N\times 2$ state of rank 2 that satisfies $\lambda_{\max}(\rho_B)<\lambda_{\max}(\rho_{AB})$ can have a symmetric extension.

B. Bell-diagonal states

Bell-diagonal states have eigenvectors $|\Phi^\pm\rangle=(|00\rangle+|11\rangle)/\sqrt{2}$ and $|\Phi^\pm\rangle=(|01\rangle+|10\rangle)/\sqrt{2}$ and are therefore defined by their eigenvalues p_I,p_X,p_Y,p_Z. Any two-qubit state with maximally mixed subsystems is Bell diagonal with the right choice of local basis [16]. For such states, necessary and sufficient conditions for symmetric extension have recently been found [15]. Parametrized by the following parameters,

$$\alpha_0 := p_I+p_X+p_Y+p_Z=1, \tag{58a}$$

$$\alpha_1 := p_I-p_X-p_Y+p_Z, \tag{58b}$$

$$\alpha_2 := \sqrt{2}(p_I-p_Z), \tag{58c}$$

$$\alpha_3 := \sqrt{2}(p_X-p_Y), \tag{58d}$$

a state admits a symmetric extension if and only if at least one of the following inequalities is satisfied:

$$4\alpha_1(\alpha_2^2-\alpha_3^2)-(\alpha_2^2-\alpha_3^2)^2-4\alpha_1^2(\alpha_2^2+\alpha_3^2)\geq 0, \tag{59a}$$

$$\alpha_2^2-\alpha_3^2-2\sqrt{2}\alpha_1|\alpha_2|\geq 0, \tag{59b}$$

$$\alpha_3^2-\alpha_2^2+2\sqrt{2}\alpha_1|\alpha_3|\geq 0. \tag{59c}$$

We now want to prove that these conditions are equivalent to the conjectured condition, Eq. (49) for Bell-diagonal states. Since these states have maximally mixed subsystems, Eq. (49) becomes

$$4\sqrt{\det(\rho_{AB})} \geq \mathrm{tr}(\rho_{AB}^2) - \tfrac{1}{2}, \tag{60}$$

where $\det(\rho_{AB})=p_I p_X p_Y p_Z$ and $\mathrm{tr}(\rho_{AB}^2)=p_I^2+p_X^2+p_Y^2+p_Z^2$. This is equivalent to at least one of the following inequalities holding

$$\mathrm{tr}(\rho_{AB}^2) \leq \tfrac{1}{2}, \tag{61a}$$

$$16 \det(\rho_{AB}) \geq \left[\mathrm{tr}(\rho_{AB}^2) - \tfrac{1}{2}\right]^2. \tag{61b}$$

For the two sets of inequalities to be equivalent, *each* of the inequalities (61a) and (61b) must imply *at least one* of Eqs. (59a)–(59c) and vice versa. By changing coordinates according to Eq. (58), it is straightforward to show that Eq. (61b) is equivalent to Eq. (59a). For the other inequalities, the relationship is more involved, but we prove that the sets of inequalities are equivalent in Appendix C. Therefore conjecture 12 holds for Bell-diagonal states.

C. *ZZ*-invariant states

Finally, we consider states of the form

$$\rho_{AB} = \begin{bmatrix} p_1 & 0 & 0 & x \\ 0 & p_2 & y & 0 \\ 0 & y & p_3 & 0 \\ x & 0 & 0 & p_4 \end{bmatrix} \tag{62}$$

in the product basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$. Without loss of generality, we can assume that $p_1 \geq p_2, p_3, p_4$ and that x and y are both real and non-negative since this can be accomplished by changing the local basis. This class includes the Bell-diagonal states as the special case where $p_1=p_4$ and $p_2=p_3$. In this section, we will show that condition (49) is necessary and sufficient in another special case of this class, namely, when $y=0$.

Let us first, however, simplify the problem for the whole class. The following lemma gives a necessary and sufficient condition for a state of form (62) to have a symmetric extension.

Lemma 15. A state of form (62) has symmetric extension if and only if there exist $s \in [0,p_2]$ and $t \in [0,\min(p_3,p_4)]$ such that

$$x \leq \sqrt{s}\sqrt{p_1 - t} + \sqrt{t}\sqrt{p_4 - s}, \tag{63a}$$

$$y \leq \sqrt{s}\sqrt{p_2 - t} + \sqrt{t}\sqrt{p_3 - s}. \tag{63b}$$

Proof. For the if part, we give an explicit symmetric extension of the state for the case when the inequalities are saturated. The extended state is then the rank-2 state $\rho_{ABB'} = p|\psi_1\rangle\langle\psi_1| + (1-p)|\psi_2\rangle\langle\psi_2|$ where

$$\sqrt{p}|\psi_1\rangle = \sqrt{p_1 - t}|000\rangle + \sqrt{p_2 - t}|011\rangle + \sqrt{s}|101\rangle + \sqrt{s}|110\rangle,$$

$$\sqrt{1-p}|\psi_2\rangle = \sqrt{t}|001\rangle + \sqrt{t}|010\rangle + \sqrt{p_3 - s}|100\rangle + \sqrt{p_4 - s}|111\rangle. \tag{64}$$

If a state has symmetric extension for a given x and y, then also states with smaller x or y have symmetric extension. This is because local unitaries can change the sign of either x or y. The qubit unitary $S := |0\rangle\langle 0| + i|1\rangle\langle 1|$ is often called the phase gate, and $S \otimes S$ will change the sign of x while $S \otimes S^{-1}$ does the same for y. The resulting states will also have a symmetric extension. Mixing the original state with one of these states will reduce either x or y of the original state, and convex combinations of extendible states also have a symmetric extension. Hence, we can have inequality instead of equality in Eqs. (63a) and (63b).

For the only if part, a generic symmetric operator on ABB' that reduces to form (62) when B' is traced out has the form

$$\begin{bmatrix} p_1 - t & \cdot & \cdot & \cdot & \cdot & k_1 & k_1 & \cdot \\ \cdot & t & \cdot & \cdot & l_1 & \cdot & \cdot & k_2 \\ \cdot & \cdot & t & \cdot & l_1 & \cdot & \cdot & k_2 \\ \cdot & \cdot & \cdot & p_2 - t & \cdot & l_2 & l_2 & \cdot \\ \cdot & l_1^* & l_1^* & \cdot & p_3 - s & \cdot & \cdot & \cdot \\ k_1^* & \cdot & \cdot & l_2^* & \cdot & s & \cdot & \cdot \\ k_1^* & \cdot & \cdot & l_2^* & \cdot & \cdot & s & \cdot \\ \cdot & k_2^* & k_2^* & \cdot & \cdot & \cdot & \cdot & p_4 - s \end{bmatrix}. \tag{65}$$

Here, $k_1+k_2=x$ and $l_1+l_2=y$. For this to be positive semidefinite, all subdeterminants must be positive. From positivity of the subdeterminants

$$\begin{vmatrix} p_1 - t & k_1 \\ k_1^* & s \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} t & k_2 \\ k_2^* & p_4 - s \end{vmatrix}$$

we get that $x=k_1+k_2 \leq |k_1|+|k_2| \leq \sqrt{s}\sqrt{p_1-t}+\sqrt{t}\sqrt{p_4-s}$. From the subdeterminants involving l_1 and l_2 we get $y=l_1+l_2 \leq |l_1|+|l_2| \leq \sqrt{t}\sqrt{p_3-s}+\sqrt{s}\sqrt{p_2-t}$. ■

Since $p_1 \geq p_2$ the possible values for t in Eqs. (63a) and (63b) are between 0 and p_2. The parameter s, however, is bounded from above by both p_3 and p_4. Before we go to the special case $y=0$ we treat the case $p_3 \geq p_4$ separately since knowing which of the two bounds applies will simplify the analysis. When $p_3 \geq p_4$, the state has a symmetric extension for any x and y since even the rank-2 state by taking the maximum $x=\sqrt{p_1 p_4}$ and $y=\sqrt{p_2 p_3}$ has symmetric extension by theorem 13. It is also easy to verify that in this case $\mathrm{tr}(\rho_B^2) \geq \mathrm{tr}(\rho_{AB}^2)$, so condition (49) is always satisfied.

In Appendix D, we show that when $y=0$, maximizing the bound for x in Eq. (63a) gives the condition

$$x \leq \begin{cases} \sqrt{p_1 p_4} & \text{for } p_1 p_3 + p_2 p_4 \geq p_1 p_4 \\ \sqrt{p_3}\sqrt{p_1 - p_2} + \sqrt{p_2}\sqrt{p_4 - p_3} & \text{otherwise}. \end{cases} \tag{66}$$

This is also what conjecture 12 reduces to in this case. Therefore, the conjecture holds for this class of states.

Any two-qubit state with three degenerate eigenvalues will be of this class. In this case, $|00\rangle$ and $|11\rangle$ can be taken as the Schmidt basis vectors of the nondegenerate eigenvector. We can then write the state as $(\lambda_1-\lambda)|\psi\rangle\langle\psi|+\lambda/4I$, where λ_1 is the nondegenerate eigenvalue, λ is the degenerate eigenvalue, and $|\psi\rangle$ is the nondegenerate eigenvector. Since I is diagonal and $|\psi\rangle\langle\psi|$ only has an off-diagonal entry in the x position, the state is of form (62) with $y=0$.

VII. APPLICATION TO (ANTI)DEGRADABLE CHANNELS

So far we have been interested in quantum states and whether they have a symmetric extension. We make the connection to *degradable* [21] and *antidegradable* [13] quantum channels which are related concepts in quantum channel theory. If a channel is degradable or antidegradable this greatly simplifies the evaluation of the quantum capacity of the channel.

A quantum channel can be represented by a unitary operator acting jointly on the system and the environment—where the environment starts out in a pure state—followed by tracing out the environment. Given a channel $\mathcal{N}:\mathcal{N}(\rho)=\mathrm{tr}_E[U(\rho\otimes|0\rangle\langle 0|_E)U^\dagger]$, the *complementary channel* is the channel to the environment where the system is traced out, $\mathcal{N}^C(\rho)=\mathrm{tr}_S[U(\rho\otimes|0\rangle\langle 0|_E)U^\dagger]$. The complementary channel is only defined up to a unitary on the output system, and the channel itself is a complementary channel of its complementary channel. A channel $\mathcal{N}$ is called degradable if there exists another channel $\mathcal{D}$, which will degrade the channel to the complementary channel when applied on the output, $\mathcal{N}^C=\mathcal{D}\circ\mathcal{N}$. Similarly, the channel is called antidegradable if the complementary channel is degradable, $\mathcal{N}=\mathcal{D}\circ\mathcal{N}^C$, for some channel $\mathcal{D}$.

Using the Choi-Jamiołkowski isomorphism [22,23], we can represent any channel by the bipartite quantum state resulting from the channel acting on one half of a maximally entangled state. We use the convention where Alice prepares a maximally entangled state and sends the second subsystem to Bob through the channel, a procedure that leaves the first subsystem maximally mixed [24],

$$\rho_{\mathcal{N}}=\frac{1}{d}\sum_{j,k=0}^{d-1}|i\rangle\langle j|\otimes\mathcal{N}(|i\rangle\langle j|). \tag{67}$$

Like in the rest of this paper, we always consider symmetric extensions to two copies of the second subsystem, which in the Choi-Jamiołkowski representation represents the output system.

Lemma 16. A channel $\mathcal{N}$ is antidegradable if and only if its Choi-Jamołkowski representation $\rho_{\mathcal{N}}$ has a symmetric extension.

Proof. Let the channel $\mathcal{N}$ be antidegradable, and let $\mathcal{D}$ be the channel that degrades the complementary channel, $\mathcal{N}=\mathcal{D}\circ\mathcal{N}^C$. Applying $\mathcal{N}$ on the second half of a maximally entangled state and applying $\mathcal{D}$ to the environment produce a tripartite state ρ_{ABE} where the reduced states satisfy $\rho_{AB}=\rho_{AE}=\rho_{\mathcal{N}}$, but it does not need to be invariant under P_{BE}. The state $(\rho_{ABE}+P_{BE}\rho_{ABE}P_{BE}^\dagger)/2$ has the same reduced states and is also invariant under exchange of B and E. It is therefore a symmetric extension of $\rho_{\mathcal{N}}=\rho_{AB}$.

Conversely, let the Choi-Jamiołkowski representation $\rho_{\mathcal{N}}$ have a symmetric extension $\rho_{ABB'}$. This satisfies $\rho_{AB}=\rho_{AB'}=\rho_{\mathcal{N}}$ and has a purification $|\psi\rangle_{ABB'R}$. The Choi-Jamiołkowski representation of the complementary channel is then $\rho_{AB'R}$ where $B'R$ is the output system. Clearly, a degrading channel is then $\mathcal{D}(\rho_{B'R})=\mathrm{tr}_R(\rho_{B'R})$. ■

This means that all necessary or sufficient conditions derived for symmetric extension are also necessary or sufficient conditions for the Choi-Jamiołkowski representation of an antidegradable channel. In particular, if conjecture 12 is true, it will also characterize the antidegradable qubit channels.

By interchanging the roles of the output and the environment, we can reduce to problem of deciding whether a channel is degradable to deciding whether the Choi-Jamiołkowski representation of the complementary channel has a symmetric extension. A channel $\mathcal{N}$ with d_A-dimensional input, d_B-dimensional output, and environment dimension of d_E is degradable if and only if $\rho_{\mathcal{N}^C}$ of dimension $d_A\times d_E$ and rank d_B has symmetric extension. Wolf and Pérez-García [13] found that when $d_E=2$, a qubit channel is either degradable, antidegradable, or both. This also follows from our theorem 13 about symmetric extension of rank-2 two-qubit states. For qubit channels with larger environment there are examples of channels that are neither—even close to the identity channel [25]. Using the following theorem, we can show that no qubit channels with $d_E\geq 2$ can be degradable.

Theorem 17. Any bipartite state ρ_{AB} of rank 2 with a symmetric extension has a reduced state that satisfies rank $(\rho_B)\leq 2$.

Proof. By corollary 3, ρ_{AB} can be decomposed into pure-extendible states

$$\rho_{AB}=\sum_j p_j\rho_{AB}^j, \tag{68}$$

where the ρ_{AB}^j all satisfy spectrum condition (5). Since ρ_{AB} is of rank 2, rank $(\rho_{AB}^j)\leq 2$ for all j.

If $\max_j$ rank$(\rho_{AB}^j)=1$, all the pure-extendible states are pure product states $\rho_{AB}^j=|\psi_j\otimes\phi_j\rangle\langle\psi_j\otimes\phi_j|$ by condition (5). Because the rank of ρ_{AB} is 2, there can only be two independent product vectors, say $|\phi_1\otimes\psi_1\rangle$ and $|\phi_2\otimes\psi_2\rangle$, so the support of ρ_B is spanned by ψ_1 and ψ_2 and is therefore at most two dimensional.

If there is at least one j such that rank $(\rho_{AB}^j)=2$, this defines a two-dimensional subspace where all other ρ_{AB}^j must have their support. Let ρ_{AB}^1 be one of the ρ_{AB}^j with rank 2. Let the spectral decomposition for it and its reduction to B be $\rho_{AB}^1=\gamma|\phi_0\rangle\langle\phi_0|+(1-\gamma)|\phi_1\rangle\langle\phi_1|$ and $\rho_B^1=\gamma|0\rangle\langle 0|+(1-\gamma)|1\rangle\langle 1|$, respectively, in accordance with condition (5). The eigenvectors of ρ_{AB}^1 can be decomposed as $|\phi_k\rangle=|\tilde{\psi}_{k0}\rangle_A|0\rangle_B+|\tilde{\psi}_{k1}\rangle_A|1\rangle_B$, where maximum one of the four un-normalized $|\tilde{\psi}_{kl}\rangle_A$ can be the zero vector. Since all the other ρ_{AB}^j have to have support within span $\{|\phi_1\rangle,|\phi_2\rangle\}$, they can only ever have reduced states ρ_B^j that are supported on span $\{|0\rangle,|1\rangle\}$. Therefore, also ρ_B is supported on span $\{|0\rangle,|1\rangle\}$ and has rank $(\rho_B)\leq 2$. ■

This reduces the $N\times M$ symmetric extension problem for states of rank 2 to $N\times 2$. From remark 14, we already have a necessary condition for this case, namely, that $\lambda_{\max}(\rho_B)\geq\lambda_{\max}(\rho_{AB})$. This also generalizes theorem 13 to give necessary and sufficient conditions for symmetric extension of a

$2\times N$ state of rank 2. Such a state has symmetric extension if and only if $\lambda_{\max}(\rho_B)\geq\lambda_{\max}(\rho_{AB})$ and ρ_B is of rank 2.

From the connection between symmetric extension and antidegradable channels in lemma 16, the following corollary automatically follows.

Corollary 18. Any antidegradable channel $\mathcal{N}$ with qubit environment has output of rank 2. If $\rho_{\mathcal{N}}$ is the Choi-Jamiołkowski state representing the channel, $\lambda_{\max}(\rho_{\mathcal{N}})\leq\lambda_{\max}(\mathrm{tr}_A[\rho_{\mathcal{N}}])$.

Exchanging the output and the environment changes antidegradability into degradability.

Corollary 19. Any degradable channel with qubit output has $d_E\leq 2$. If $\rho_{\mathcal{N}}$ is the Choi-Jamiołkowski state representing the channel, $\lambda_{\max}(\mathrm{tr}_A[\rho_{\mathcal{N}}])\leq\lambda_{\max}(\rho_{\mathcal{N}})$.

This result has recently been independently obtained by Cubitt *et al.* [26] by other methods. One could imagine that theorem 17 would generalize to higher rank so that the rank of the ρ_B system always would be bounded by the rank of ρ_{AB} for symmetric extendible states. This would mean that the dimension of the environment always would be bounded by the output rank for degradable channels. However, Cubitt *et al.* [26] proved that this only holds for channels with qubit and qutrit outputs. If the rank of a symmetric extendible state is R, the technique from the proof of theorem 17 can fail only if $1<\max_j \mathrm{rank}(\rho_{AB}^j)<R$. This gives the following corollary.

Corollary 20. If ρ_{AB} has a (1,2)-symmetric extension and $\mathrm{rank}(\rho_B)>\mathrm{rank}(\rho_{AB})$, then for any decomposition into pure-extendible states

$$\rho_{AB}=\sum_j p_j\rho_{AB}^j,$$

$\mathrm{rank}(\rho_{AB}^j)<\mathrm{rank}(\rho_{AB})$ for all j.

Proof. Assume that $\max_j \mathrm{rank}(\rho_{AB}^j)=\mathrm{rank}(\rho_{AB}^1)=\mathrm{rank}(\rho_{AB})=:R$. Let the spectral decomposition of ρ_{AB}^1 and its reduced state be $\rho_{AB}^1=\Sigma_{k=1}^R\gamma_k|\phi_k\rangle\langle\phi_k|$ and $\rho_B^1=\Sigma_{k=1}^R\gamma_k|k\rangle\langle k|$. The eigenvectors of ρ_{AB}^1 can then be written as $|\phi_k\rangle=\Sigma_{m=1}^R\langle\tilde{\psi}_{km}||m\rangle$. Since ρ_{AB}^1 has the full rank of ρ_{AB}, the support of ρ_{AB} must be the space spanned by the eigenvectors of ρ_{AB}^1. This means that ρ_B has support on span $\{\{|m\rangle\}_{m=1}^R\}$ and therefore has rank R. Therefore, if rank $(\rho_B)>R$ we cannot have $\max_j \mathrm{rank}(\rho_{AB}^j)=\mathrm{rank}(\rho_{AB})$. ■

VIII. DISCUSSION AND CONCLUSIONS

In this work, we have characterized states with symmetric extension by decomposing them into states with a pure symmetric extension. For two qubits we have fully characterized these pure-extendible states and quite remarkably this characterization only depends on the global and one of the local spectra of the density matrix. Even given this result, it is rather surprising that knowledge of this information also seems to be sufficient for deciding whether or not a generic two-qubit state has a symmetric extension. Although we cannot prove this in general, the special cases for which we prove it and extensive numerical testing suggest that our conjecture holds for all two-qubit states. Actually, proving that inequality (49) describes a convex set will be sufficient for proving that it is a necessary condition for symmetric extension since we have proven that the extremal extendible states are all contained in this set. One way to prove the sufficiency of the condition is to find a way to decompose any state that satisfies it either into pure-extendible states or into extendible states of any of the classes for which we have proven that the conjecture holds.

When either of the subsystems is larger than a qubit, symmetric extendibility does not only depend on local and global eigenvalues. In any higher dimension, there are states without symmetric extension which have the same spectra as states with pure symmetric extension. It would nevertheless be interesting to know if the convex hull of the states that satisfies spectrum condition (5) can be characterized in a way similar to inequality (49). Such a condition would provide a useful necessary condition for a state to have a symmetric extension.

The isomorphism between quantum channels and bipartite quantum states allows us to use our results for quantum states to make some interesting statements about quantum channels. States with symmetric extension correspond to antidegradable channels, and by interchanging the output and the environment we can also make statements about degradable channels. Our corollary 19 says that if the output of a quantum channel is a qubit, it can only be degradable if the environment also is a qubit, a result that follows from our conditions on symmetric extendible states of rank 2. When the dimension of the channel output is higher, the environment dimension of degradable channels is not always bounded by this. Corollary 20 gives a condition on the structure of degradable channels with higher environment dimension than output dimension.

ACKNOWLEDGMENTS

The authors would like to thank Debbie Leung, Matthias Christandl, Andrew Doherty, Joseph Renes, and Barbara Kraus and especially Marco Piani and Tobias Moroder for stimulating discussions. Marco Piani has also provided the last part of the proof for proposition 8 and Tobias Moroder has provided some of the MATLAB scripts based on YALMIP [27] and SDPT3 [28] that were used for numerical testing. This work was funded by the Research Council of Norway (Project No. 166842/V30), by the European Union through the IST Integrated Project SECOQC and the IST-FET Integrated Project QAP, and by the NSERC Discovery Grant.

APPENDIX A: BOSONIC AND FERMIONIC EXTENSIONS

In this paper we have used the term symmetric extension for extensions that are invariant under exchange of two systems, without considering if its support is on the symmetric or antisymmetric subspace or both. We call an extension that resides only on the symmetric subspace of $\mathcal{H}_B\otimes\mathcal{H}_{B'}$ a *bosonic extension*, while one that resides on the antisymmetric subspace is a *fermionic extension*. Generic symmetric extensions are mixtures of bosonic and fermionic extensions. Bosonic (+) and fermionic (−) extensions satisfy $\pi_\pm\rho_{ABB'}\pi_\pm^\dagger$ for $\pi_\pm:=\frac{1}{2}(I\pm P_{BB'})$, the projector onto the symmetric/

antisymmetric subspace and $P_{BB'}\rho_{ABB'}=\pm\rho_{ABB'}$. Here we show that when the subsystem to be extended is a qubit, the states with symmetric and bosonic extension coincide and that this is not true in general.

Proposition 21. If a quantum state ρ_{AB} of dimension $N\times 2$ has a symmetric extension to $\rho_{ABB'}$ it also has a bosonic extension $\sigma_{ABB'}$, i.e., that satisfies also

$$\sigma_{ABB'}=\tfrac{1}{2}(I+P_{BB'})\sigma_{ABB'}\tfrac{1}{2}(I+P_{BB'}). \quad (A1)$$

Proof. Decompose the extended state $\rho_{ABB'}$ with the spectral decomposition as in lemma 2,

$$\rho_{ABB'}=\sum_j \lambda_j^+|\phi_j^+\rangle\langle\phi_j^+|+\sum_k \lambda_k^-|\phi_k^-\rangle\langle\phi_k^-|, \quad (A2)$$

where $|\phi_j^+\rangle=P_{BB'}|\phi_j^+\rangle$ and $|\phi_k^-\rangle=-P_{BB'}|\phi_k^-\rangle$ are symmetric and antisymmetric, respectively. The vectors are of the form $|\phi_j^\pm\rangle=\Sigma_k\alpha_{jk}|\psi_{jk}\rangle_A|\psi_k^\pm\rangle_{BB'}$, where $|\psi_k^\pm\rangle_{BB'}$ are in the symmetric and antisymmetric subspaces of $\mathcal{H}_B\otimes\mathcal{H}_{B'}$. When B and B' are qubits, the antisymmetric space is one-dimensional and is spanned by the vector $|\Psi^-\rangle=(1/\sqrt{2})(|01\rangle-|10\rangle)$. The antisymmetric vectors are therefore of the product form $|\phi_k^-\rangle=|\psi_k\rangle_A|\Psi^-\rangle_{BB'}$. Replacing them with symmetric vectors of the form $|\xi_k^+\rangle=|\psi_k\rangle_A|\Psi^+\rangle_{BB'}$ where $|\Psi^+\rangle=(1/\sqrt{2})(|01\rangle+|10\rangle)$ yields a state

$$\sigma_{ABB'}=\sum_j \lambda_j^+|\phi_j^+\rangle\langle\phi_j^+|+\sum_k \lambda_k^-|\xi_k^+\rangle\langle\xi_k^+|, \quad (A3)$$

which has support on the symmetric subspace. Note that λ_j^+ and λ_k^- are no longer eigenvalues of this state. But since the reduced states of $|\xi_k^+\rangle$ are the same as for $|\phi_k^-\rangle$, we have that $\rho_{AB}:=\mathrm{tr}_{B'}[\rho_{ABB'}]=\mathrm{tr}_{B'}[\sigma_{ABB'}]$, so $\sigma_{ABB'}$ is a valid bosonic extension of ρ_{AB}. ■

To show that this is an effect of the low dimension of the B system, we give an example of a state of two qutrits that has a fermionic and not a bosonic extension.

Example 22. Consider a tripartite pure state on ABB' of the form

$$|\psi\rangle=\alpha(|012\rangle-|021\rangle)+\beta(|120\rangle-|102\rangle)+\gamma(|201\rangle-|210\rangle), \quad (A4)$$

where $\alpha,\beta,\gamma\neq 0$. This is a fermionic extension of the reduced state $\rho_{AB}=\mathrm{tr}_{B'}[|\psi\rangle\langle\psi|]$. If ρ_{AB} had a bosonic extension, a trace preserving and completely positive (TPCP) map on a purifying system (here B') would be able to convert any purification of ρ_{AB} into this bosonic extension. If the TPCP map is given by its Kraus operators K_j which satisfy $\Sigma_j K_j^\dagger K_j=I_{B'}$, the output state when applied to $|\psi\rangle$ would be

$$\sigma_{ABB'}=\sum_j (I_A\otimes I_B\otimes K_j)|\psi\rangle\langle\psi|(I_A\otimes I_B\otimes K_j)^\dagger. \quad (A5)$$

If $\sigma_{ABB'}$ is a bosonic extension, all the terms in this sum must be on the symmetric subspace. Consider one of the Kraus operators, K. Applying it to $|\psi\rangle$ gives

$$(I_A\otimes I_B\otimes K)|\psi\rangle=\alpha|0\rangle|\psi_0\rangle+\beta|1\rangle|\psi_1\rangle+\gamma|2\rangle|\psi_2\rangle, \quad (A6)$$

where $|\psi_0\rangle=|1\rangle\otimes K|2\rangle-|2\rangle\otimes K|1\rangle$, $|\psi_1\rangle=|2\rangle\otimes K|0\rangle-|0\rangle\otimes K|2\rangle$, and $|\psi_2\rangle=|0\rangle\otimes K|1\rangle-|1\rangle\otimes K|0\rangle$. Each of the $|\psi_j\rangle$ needs to be on the symmetric subspace of $\mathcal{H}_B\otimes\mathcal{H}_{B'}$. Expressing K as $\Sigma_{jk}k_{jk}|j\rangle\langle k|$ and imposing $P_{BB'}|\psi_1\rangle=|\psi_1\rangle$ give us that $k_{01}=k_{02}=0$ and $k_{22}=-k_{11}$. Doing the same with the other vectors we get that $k_{jk}=0$ for any $j\neq k$, $k_{00}=-k_{22}$, and $k_{11}=-k_{00}$. The only possible solution to this is that K vanishes, so no nonzero K applied on B' can give a vector which is on the symmetric subspace. Hence, the state ρ_{AB} cannot have a bosonic extension.

This means that there are states ρ_{AB} with a symmetric extension that cannot be extended to a pure state on four systems $|\psi\rangle_{ABB'R}$ in such a way that $|\psi\rangle_{ABB'R}=\pm P_{BB'}|\psi\rangle_{ABB'R}$. This condition means that the extension is bosonic (+) or fermionic (−), but some states with symmetric extension admit neither. One example is if ρ_{AB} does not admit a fermionic extension and σ_{AB} does not admit a bosonic extension. Then the state $(|0\rangle\langle 0|_{A'}\otimes\rho_{AB}+|1\rangle\langle 1|_{A'}\otimes\sigma_{AB})/2$ cannot admit bosonic nor fermionic extensions.

APPENDIX B: CALCULATIONS LEADING TO EQS. (13) and (14)

In this appendix, we show that if on a generic three-qubit state $a|000\rangle+b|001\rangle+c|010\rangle+d|011\rangle+e|100\rangle+f|101\rangle+g|110\rangle+h|111\rangle$ we impose that its reductions ρ_B and $\rho_{B'}$ are equal, diagonal, and not maximally mixed, then $|b|=|c|$ and $|f|=|g|$ and $|c||g|(\exp[i(\phi_b-\phi_c)]-\exp[i(\phi_f-\phi_g)])=0$.

The two reduced density matrices of this generic state are in the computational basis

$$\rho_B=\begin{bmatrix}|a|^2+|b|^2+|e|^2+|f|^2 & ac^*+bd^*+eg^*+fh^* \\ a^*c+b^*d+e^*g+f^*h & |c|^2+|d|^2+|g|^2+|h|^2\end{bmatrix}, \quad (B1)$$

$$\rho_{B'}=\begin{bmatrix}|a|^2+|c|^2+|e|^2+|g|^2 & ab^*+cd^*+ef^*+gh^* \\ a^*b+c^*d+e^*f+g^*h & |b|^2+|d|^2+|f|^2+|h|^2\end{bmatrix}. \quad (B2)$$

The equations we get are

$$|b|^2+|f|^2=|c|^2+|g|^2, \quad (B3a)$$

$$ac^*+bd^*+eg^*+fh^*=0, \quad (B3b)$$

$$ab^*+cd^*+ef^*+gh^*=0, \quad (B3c)$$

where the first is from the diagonal entries of ρ_B being equal to those of $\rho_{B'}$ and the others from the off-diagonal elements being 0.

Assume that $|b|\neq|c|$. Then by Eq. (B3a) also $|f|\neq|g|$. From Eqs. (B3b) and (B3c) one can then isolate e and h^*,

$$e=\frac{a(b^*f-c^*g)+d^*(cf-bg)}{|g|^2-|f|^2}, \quad (B4a)$$

$$h^* = \frac{a(c^* f^* - b^* g^*) + d^*(bf^* - cg^*)}{|g|^2 - |f|^2}. \tag{B4b}$$

From this one can compute $|e|^2 - |h|^2$ and by using Eq. (B3a) this simplifies to

$$|e|^2 - |h|^2 = |d|^2 - |a|^2. \tag{B5}$$

Taken together with Eq. (B3a), this is exactly the condition that the two diagonal elements in ρ_B and $\rho_{B'}$ are equal, so they are completely mixed. If the subsystems are not completely mixed, we must therefore have $|b| = |c|$ and $|f| = |g|$, which is Eq. (13).

Now we want to find the relations between the complex phases of b, c, f, and g. Denote $b = |b| \exp[i\phi_b]$, $c = |c| \exp[i\phi_c]$, $f = |f| \exp[i\phi_f]$, and $g = |g| \exp[i\phi_g]$. Multiplying Eq. (B3b) by g, Eq. (B3c) by f, taking the difference, and using $|f| = |g|$, we obtain

$$a(c^* g - b^* f) + d^*(bg - cf) = 0. \tag{B6}$$

Since $|c||g| = |c||f| = |b||f| = |b||g|$, this becomes

$$\exp[i\phi_g]|c||g|(a \exp[-i\phi_b] + d^* \exp[i\phi_c]) \times (\exp[i(\phi_b - \phi_c)] - \exp[i(\phi_f - \phi_g)]) = 0. \tag{B7}$$

Then at least one of the following two equations must hold. Either

$$|c||g|(\exp[i(\phi_b - \phi_c)] - \exp[i(\phi_f - \phi_g)]) = 0, \tag{B8}$$

which is Eq. (14) that we want to show or

$$d^* \exp[i\phi_c] = -a \exp[-i\phi_b]. \tag{B9}$$

In the case that Eq. (B8) does not hold, Eq. (B9) must hold, and we will now see that this case implies that subsystem B is completely mixed.

If we insert Eq. (B9) into Eq. (B3c) and use $|b| = |c|$ and $|f| = |g|$, we obtain

$$h^* g = -ef^*. \tag{B10}$$

Since Eq. (B8) does not hold, $|f| = |g| \neq 0$ and therefore Eq. (B10) implies $|e| = |h|$. Condition (B9) already means that $|a| = |d|$, so again we have that Eq. (B5) holds so the diagonal terms in ρ_B are equal and we are in the maximally mixed case.

Hence, if Eq. (B8) does not hold, Eq. (B9) cannot hold either since ρ_B and $\rho_{B'}$ are not maximally mixed and therefore Eq. (B8) which is the same as Eq. (14) must hold.

APPENDIX C: INEQUALITY RELATIONS FOR BELL-DIAGONAL STATES

In this appendix, we show that each of inequalities (61a) and (61b) implies at least one of Eqs. (59a)–(59c) and vice versa. More precisely, Eqs. (59a) and (61b) are equivalent, either of Eqs. (59b) and (59c) implies Eq. (61a), while Eq. (61a) only implies that at least one of Eqs. (59a)–(59c) is satisfied.

We first change variables in Eqs. (61a) and (61b) so that they use the same parameters as Eqs. (59a)–(59c). This gives the two inequalities

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 \leq 1, \tag{C1a}$$

$$4\alpha_1(\alpha_2^2 - \alpha_3^2) - (\alpha_2^2 - \alpha_3^2)^2 - 4\alpha_1^2(\alpha_2^2 + \alpha_3^2) \geq 0. \tag{C1b}$$

Inequality (C1b) which comes from Eq. (61b) is the same as Eq. (59a) so they are all equivalent.

Next, we prove that either of Eqs. (59b) and (59c) implies Eq. (C1a) and therefore also Eq. (61a). Each of Eqs. (59b) and (59c) can be split into two inequalities for the cases when the variable inside the absolute value is negative or non-negative. For each of the four inequalities, an orthogonal change of variables allows us to express them on a standard form. The transformation for Eq. (59b) is $\alpha_1 = \sqrt{2/3}x - \sqrt{1/3}y$, $\alpha_2 = \pm\sqrt{1/3}x \pm \sqrt{2/3}y$, and $\alpha_3 = z$ for the cases $\pm\alpha_2 \geq 0$. The transformations for Eq. (59c) are obtained by interchanging α_2 with α_3 and α_1 with $-\alpha_1$. All four inequalities then become simply $x^2 + z^2 \leq 2y^2$. Purity condition (C1a) becomes $x^2 + y^2 + z^2 \leq 1$ for all the transformations. By noting that for each transformation one of the positivity conditions for the eigenvalues translates into $y \leq 1/\sqrt{3}$, we get $x^2 + y^2 + z^2 \leq 3y^2 \leq 1$.

The last implication we need to show is that any state that satisfies Eq. (61a), $\mathrm{tr}(\rho_{AB}^2) \leq 1/2$, or equivalently Eq. (C1a) also satisfies at least one of Eqs. (59a)–(59c). For this we use the proven fact that these inequalities are necessary and sufficient conditions for the state to have a symmetric extension, and therefore the set must be convex. Any Bell-diagonal state that satisfies inequality (61a) can be written as a convex combination of states that saturates it. One way to achieve this is to choose the four states $\rho_I := (1 - q_I)|\Phi^+\rangle\langle\Phi^+| + q_I \rho_{AB}$, $\rho_X := (1 - q_X)|\Psi^+\rangle\langle\Psi^+| + q_X \rho_{AB}$, etc., with the choice of q_j that gives $\mathrm{tr}[\rho_j^2] = 1/2$, and to write ρ_{AB} as a convex combination of these. Since the determinant of a state always is non-negative, all these extremal states satisfy Eq. (61b) and therefore also Eq. (59a), so they must have a symmetric extension. Convex combinations of states with symmetric extension also have symmetric extension, so any state with $\mathrm{tr}(\rho_{AB}^2) \leq 1/2$ has symmetric extension and therefore satisfies one of Eqs. (59a)–(59c).

APPENDIX D: EQUIVALENCE FOR ZZ-INVARIANT STATES

In this appendix, we show that for states of class (62) with $y = 0$ and $p_4 \geq p_3$, the necessary and sufficient conditions for symmetric extension from lemma 15 simplify to Eq. (66). Next, we show that conjecture 12 also reduces to Eq. (66) for this class of states.

Since $y = 0$, Eq. (63b) is satisfied for any $s \in [0, p_3]$ and $t \in [0, p_2]$. Our only objective is therefore to maximize the right-hand side of Eq. (63a), $f(s,t) := \sqrt{s}\sqrt{p_1 - t} + \sqrt{t}\sqrt{p_4 - s}$, on this domain. Without the constraints on s and t, this reaches its maximum value of $\sqrt{p_1 p_4}$ for any value of (s,t) that satisfies $p_1 s + p_4 t = p_1 p_4$. Since $s \leq p_3$ and $t \leq p_2$ this maximum value may or may not be obtainable. The maximum value of $p_1 s + p_4 t$ is $p_1 p_3 + p_2 p_4$, so if $p_1 p_3 + p_2 p_4 \geq p_1 p_4$, then $x = \sqrt{p_1 p_4}$ can be obtained by choosing $s = p_3$ and $t = (p_1 p_4 - p_1 p_3)/p_4 \leq p_2$. When $p_1 p_3 + p_2 p_4 < p_1 p_4$, however, we will have $f(s,t) < \sqrt{p_1 p_4}$ for all possible (s,t). In this case the optimal choice of (s,t) is (p_3,p_2) since in the region where $p_1p_3+p_2p_4<p_1p_4$ the $f(s,t)$ increases both when s and t increases. The maximum value for x is then $\sqrt{p_3}\sqrt{p_1-p_2}+\sqrt{p_2}\sqrt{p_4-p_3}$. Summing up, a state of form (62) with $y=0$ has a symmetric extension if and only if

$$x \leq \begin{cases} \sqrt{p_1p_4} & \text{for } p_1p_3+p_2p_4 \geq p_1p_4 \\ \sqrt{p_3}\sqrt{p_1-p_2}+\sqrt{p_2}\sqrt{p_4-p_3} & \text{otherwise,} \end{cases} \tag{D1}$$

which is the same as Eq. (66).

The remaining part is to show that condition (49) from conjecture 12 is equivalent to this. The condition is equivalent to at least one of the following two inequalities holding

$$\mathrm{tr}(\rho_{AB}^2)-\mathrm{tr}(\rho_B^2) \leq 0, \tag{D2a}$$

$$4\sqrt{\det \rho_{AB}} \geq |\mathrm{tr}(\rho_{AB}^2)-\mathrm{tr}(\rho_B^2)|. \tag{D2b}$$

Since $y=0$, we get $\det(\rho_{AB})=p_2p_3(p_1p_4-x^2)$ and $\mathrm{tr}(\rho_{AB}^2)-\mathrm{tr}(\rho_B^2)=2(x^2-p_1p_3-p_2p_4)$. Inserting this into Eqs. (D2a) and (D2b) and solving for x give

$$x \leq \sqrt{p_1p_3+p_2p_4}, \tag{D3a}$$

$$x \leq \sqrt{p_3}\sqrt{p_1-p_2}+\sqrt{p_2}\sqrt{p_4-p_3}. \tag{D3b}$$

Only one of these inequalities has to be satisfied for a state with symmetric extension, so the upper bound on x is the maximum of the two. By comparing the two bounds, we find in which region each of the two is valid and get

$$x \leq \begin{cases} \sqrt{p_1p_3+p_2p_4} & \text{for } p_1p_3+p_2p_4 \geq p_1p_4 \\ \sqrt{p_3}\sqrt{p_1-p_2}+\sqrt{p_2}\sqrt{p_4-p_3} & \text{otherwise.} \end{cases} \tag{D4}$$

The only region where $\sqrt{p_1p_3+p_2p_4}$ is the valid upper bound is when it is greater than $\sqrt{p_1p_4}$. Since x never can exceed $\sqrt{p_1p_4}$ for any state, this is the same as Eq. (D1).

[1] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **80**, 5239 (1998).
[2] M. Horodecki and P. Horodecki, Phys. Rev. A **59**, 4206 (1999).
[3] M. Curty, M. Lewenstein, and N. Lütkenhaus, Phys. Rev. Lett. **92**, 217903 (2004).
[4] V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A **61**, 052306 (2000).
[5] M. Koashi and A. Winter, Phys. Rev. A **69**, 022309 (2004).
[6] B. Toner, Proc. R. Soc. London, Ser. A **465**, 59 (2009).
[7] A. C. Doherty, P. A. Parrilo, and F. M. Spedalieri, Phys. Rev. Lett. **88**, 187904 (2002).
[8] S. Bravyi, Quantum Inf. Comput. **4**, 012 (2004).
[9] Y.-J. Han, Y.-S. Zhang, and G.-C. Guo, Phys. Rev. A **72**, 054302 (2005).
[10] P. Butterley, A. Sudbery, and J. Szulc, Found. Phys. **36**, 83 (2006).
[11] Y.-K. Liu, in *Proceedings of the RANDOM 2006*, Lecture Notes in Computer Science Vol. 4110 (Springer, Berlin, 2006), pp. 438–449.
[12] W. Hall, Phys. Rev. A **75**, 032102 (2007).
[13] M. M. Wolf and D. Pérez-García, Phys. Rev. A **75**, 012303 (2007).
[14] M. L. Nowakowski and P. Horodecki, J. Phys. A: Math. Theor. **42**, 135306 (2009).
[15] G. O. Myhr, J. M. Renes, A. C. Doherty, and N. Lütkenhaus, Phys. Rev. A **79**, 042329 (2009).
[16] R. Horodecki and M. Horodecki, Phys. Rev. A **54**, 1838 (1996)
[17] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A **54**, 3824 (1996).
[18] M.-B. Ruskai, Rev. Math. Phys. **6**, 1147 (1994).
[19] I. Devetak and A. Winter, Proc. R. Soc. London, Ser. A **461**, 207 (2005).
[20] M. A. Nielsen and J. Kempe, Phys. Rev. Lett. **86**, 5184 (2001).
[21] I. Devetak and P. W. Shor, Commun. Math. Phys. **256**, 287 (2005).
[22] M.-D. Choi, Linear Algebr. Appl. **10**, 285 (1975).
[23] A. Jamiołkowski, Rep. Math. Phys. **3**, 275 (1972).
[24] A more common convention is to let the channel act on the first subsystem of the operator $M=\Sigma_{j,k=0}^{d-1}|jj\rangle\langle kk|$ which is a maximally entangled state, normalized such that $\mathrm{tr}_A(M)=I_B$.
[25] G. Smith and J. A. Smolin, Phys. Rev. Lett. **98**, 030501 (2007).
[26] T. S. Cubitt, M. B. Ruskai, and G. Smith, J. Math. Phys. **49**, 102104 (2008).
[27] J. Löfberg, in Proceedings of the CACSD Conference (IEEE, Taipei, Taiwan, 2004, pp. 284–289 (http://control.ee.ethz.ch/~joloef/yalmip.php).
[28] K. C. Toh, R. H. Tutuncu, and M. J. Todd, Optim. Methods Software **11**, 545 (1999).