Arrival times, complex potentials, and decoherent histories

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We carry out a decoherent-histories analysis of the arrival-time problem, taking advantage of a recently demonstrated connection between time-ordered strings of projection operators and evolution in the presence of a complex potential of step-function form. We concentrate on the limit of a weak potential, in which the resulting arrival-time distribution function is closely related to the quantum-mechanical current. We first consider the analogous classical arrival-time problem involving an absorbing potential, and this sheds some light on certain aspects of the quantum case. We use the path-decomposition expansion to give a derivation of the standard arrival-time distribution defined using a complex potential. This derivation is then used in the decoherent-histories analysis to obtain very simple and plausible expressions for the class operators (describing the amplitudes for crossing the origin during intervals of time). We show that decoherence of histories is obtained for a wide class of initial states (such as simple wave packets and superpositions of wave packets). We find that the decoherent-histories approach gives results with a sensible classical limit that are fully compatible with standard results on the arrival-time problem. We also find some interesting connections between backflow and decoherence.

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I. INTRODUCTION

A. Arrival-time problem

The arrival-time problem has attracted some considerable interest in recent years [1]. In the one-dimensional statement of this problem, one considers an initial wave function $|\psi\rangle$ concentrated in the region $x>0$ and consisting entirely of negative momenta. The question is then to find the probability $\Pi(\tau)d\tau$ that the particle crosses $x=0$ between time τ and $\tau+d\tau$. (See Fig. 1.)

Two particular candidate expressions for the arrival-time distribution $\Pi(\tau)$ are central to the discussion. First, there is the current density

$$J(\tau) = -\frac{1}{2m}\langle\psi_f(\tau)|[\hat{p}\,\delta(\hat{x}) + \delta(\hat{x})\hat{p}]|\psi_f(\tau)\rangle, \qquad (1.1)$$

where $|\psi_f(\tau)\rangle$ is the freely evolved state, which arises from elementary considerations of the Schrödinger equation. (For convenience we work in units in which $\hbar=1$.) This is sensible classically but can be negative in the quantum case for certain states consisting of superpositions of different momenta [2–4]. Second, a simple operator reordering of $J(t)$ gives the "ideal" arrival-time distribution of Kijowski [5],

$$\Pi_K(\tau) = \frac{1}{m}\langle\psi_f(\tau)||\hat{p}|^{1/2}\delta(\hat{x})|\hat{p}|^{1/2}|\psi_f(\tau)\rangle, \qquad (1.2)$$

which is clearly positive. Both of these distributions are measurable [6,7].

B. Complex potentials

An interesting question is the extent to which such expressions emerge from more elaborate measurement or axiomatic schemes. There are many such schemes. Here, we will focus on expressions for the arrival-time distribution arising from the inclusion of a complex potential

$$V(x) = -iV_0\theta(-x) \qquad (1.3)$$

in the Schrödinger equation. With such a potential, the state at time τ is

$$|\psi(\tau)\rangle = \exp[-iH_0\tau - V_0\theta(-x)\tau]|\psi\rangle, \qquad (1.4)$$

where H_0 is the free Hamiltonian. The idea here is that the part of the wave packet that reaches the origin during the time interval $[0,\tau]$ should be absorbed, so that

$$N(\tau) = \langle\psi(\tau)|\psi(\tau)\rangle \qquad (1.5)$$

is the probability of not crossing $x=0$ during the time interval $[0,\tau]$. The probability of crossing between τ and $\tau+d\tau$ is then

$$\Pi(\tau) = -\frac{dN}{d\tau}. \qquad (1.6)$$

Complex potentials such as Eq. (1.3) were originally considered by Allcock [8] in his seminal work on arrival time and have subsequently appeared in detector models [6,9]. (See also Refs. [7,10] for further work with complex potentials.) A recent interesting result of Echanobe *et al.* [11] is that under certain conditions, evolution according to Eq. (1.4) is essentially the same as pulsed measurements, in which the wave function is projected onto $x>0$ at discrete time intervals.

For large V_0, the wave function defined by evolution equation (1.4) undergoes significant reflection, with total reflection in the "Zeno limit," $V_0\to\infty$ [12]. Here, we are interested in the opposite case of small V_0, where there is small reflection and Eq. (1.6) can give reasonable expressions for the arrival-time distribution. A number of different authors [6–8] indicate that the resulting distribution is of the form

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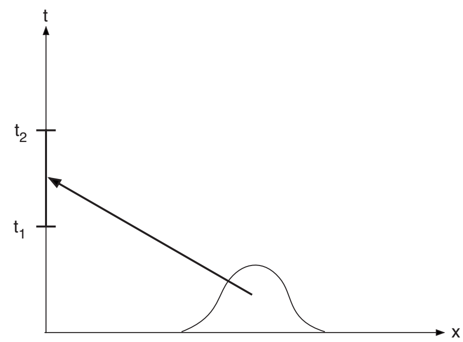

FIG. 1. The quantum arrival-time problem. We prepare an initial state localized entirely in $x>0$ and consisting entirely of negative momenta. What is the probability that the particle crosses the origin during the time interval $[t_1,t_2]$?

$$\Pi(\tau) = \int_{-\infty}^{\infty} dt R(V_0, \tau - t) J(t), \tag{1.7}$$

where $J(t)$ is the current, equation (1.1) and

$$R(V_0,t) = 2V_0 \theta(t) \exp(-2V_0 t). \tag{1.8}$$

It is therefore closely related to the current $J(t)$ but also includes the influence of the complex potential via the "apparatus resolution function" $R(V_0,t)$.

The first aim of this paper is to look in some detail at the calculation and properties of arrival-time distribution (1.7) defined using a complex potential. In particular, we will use path-integral methods, which in some ways are more concise and transparent than previous derivations [and also suggest generalizations to complex potentials more general than Eq. (1.3)]. We will also explore some of the properties of the result [Eq. (1.7)].

The second aim of this paper is to carry out a decoherent-histories analysis of the arrival-time problem. This turns out to be closely related to the complex potential definition of arrival time and was in fact the original motivation for exploring complex potentials. In brief, our aim is to see if standard results for $\Pi(\tau)$, such as Eq. (1.1) and (1.7), have a natural place in the decoherent-histories approach.

C. Decoherent-histories approach generally

In the decoherent-histories approach to quantum theory [13–19], probabilities are assigned to histories via the formula

$$p(\alpha_1, \alpha_2, \ldots) = \mathrm{Tr}(C_{\underline{\alpha}} \rho C_{\underline{\alpha}}^{\dagger}), \tag{1.9}$$

where the class operator $C_{\underline{\alpha}}$ denotes a time-ordered string of projectors P_α interspersed with unitary evolution,

$$C_{\underline{\alpha}} = P_{\alpha_n} e^{-iH(t_n - t_{n-1})} P_{\alpha_{n-1}} \cdots e^{-iH(t_2 - t_1)} P_{\alpha_1}, \tag{1.10}$$

and $\underline{\alpha}$ denotes the string $\alpha_1, \alpha_2, \ldots, \alpha_n$. The class operator satisfies the condition

$$\sum_{\underline{\alpha}} C_{\underline{\alpha}} = e^{-iH\tau}, \tag{1.11}$$

where τ is the total time interval, $\tau = t_n - t_1$. Interference between pairs of histories is measured by the decoherence functional

$$D(\underline{\alpha}, \underline{\alpha}') = \mathrm{Tr}(C_{\underline{\alpha}} \rho C_{\underline{\alpha}'}^{\dagger}). \tag{1.12}$$

It satisfies the relations

$$D(\underline{\alpha}, \underline{\alpha}') = D^*(\underline{\alpha}', \underline{\alpha}), \tag{1.13}$$

$$\sum_{\underline{\alpha}, \underline{\alpha}'} D(\underline{\alpha}, \underline{\alpha}') = 1, \tag{1.14}$$

$$\sum_{\underline{\alpha}} D(\underline{\alpha}, \underline{\alpha}) = \sum_{\underline{\alpha}} p(\underline{\alpha}) = 1. \tag{1.15}$$

Of particular interest are sets of histories which satisfy the condition of decoherence, which is

$$D(\underline{\alpha}, \underline{\alpha}') = 0 \quad \text{if} \quad \underline{\alpha} \neq \underline{\alpha}'. \tag{1.16}$$

Decoherence implies the weaker condition of consistency, which is that $\mathrm{Re}\, D(\underline{\alpha}, \underline{\alpha}') = 0$ for $\underline{\alpha} \neq \underline{\alpha}'$. This is equivalent to the requirement that the above probabilities satisfy the probability sum rules. In most situations of physical interest, both the real and imaginary parts of $D(\underline{\alpha}, \underline{\alpha}')$ vanish for $\underline{\alpha} \neq \underline{\alpha}'$, a condition we shall simply call decoherence, and is related to the existence of records [14,19]. Decoherence is only approximate in general, which raises the question of how to measure approximate decoherence. The decoherence functional satisfies the inequality [18]

$$|D(\underline{\alpha}, \underline{\alpha}')|^2 \le p(\underline{\alpha}) p(\underline{\alpha}'). \tag{1.17}$$

This suggests that a sensible measure of approximate decoherence is

$$|D(\underline{\alpha}, \underline{\alpha}')|^2 \ll p(\underline{\alpha}) p(\underline{\alpha}'). \tag{1.18}$$

The approach also permits other types of class operators which are not simply strings of projections, but sums of such strings. These are often called inhomogeneous histories [20] and are relevant to questions involving time in a nontrivial way. For example, for a given class operator $C_{\underline{\alpha}}$, the object $1 - C_{\underline{\alpha}}$ is also a class operator but it not equal to a simple string of projections. Unlike homogenous histories, inhomogeneous histories do not satisfy condition (1.15) in general, except when there is decoherence.

A quantity closely related to the probabilities is the quasiprobability

$$q(\underline{\alpha}) = \mathrm{Tr}(C_{\underline{\alpha}} \rho e^{iH\tau}). \tag{1.19}$$

Using Eq. (1.11), this satisfies

$$q(\underline{\alpha}) = \sum_{\underline{\alpha}'} \mathrm{Tr}(C_{\underline{\alpha}} \rho C_{\underline{\alpha}'}^{\dagger}) = p(\underline{\alpha}) + \sum_{\underline{\alpha}' \neq \underline{\alpha}} D(\underline{\alpha}, \underline{\alpha}') \tag{1.20}$$

(where $\underline{\alpha}$ is fixed in the sum). This means that when there is decoherence, the probabilities for histories $p(\underline{\alpha})$ are equal to the simpler expression $q(\underline{\alpha})$. (The converse is generally not true except for in very simple circumstances.) Note that $q(\underline{\alpha})$ is not positive (or even real) in general, but it is positive and real when there is decoherence. These facts turn out to be relevant to our analysis of the arrival-time problem.

D. Decoherent-histories approach to the arrival-time problem

We now consider the decoherent-histories approach applied to the arrival-time problem [21–26]. We ultimately seek a decoherent-histories account of the arrival-time probability $\Pi(t)dt$, the probability for the particle to arrive in an infinitesimal interval $[t,t+dt]$. However, for simplicity, we first consider the simpler problem of computing the probability of arriving during a finite (possibly large) interval $[0,\tau]$. We consider an incoming state entirely localized in $x>0$, and we partition the system's histories into two classes: histories that either cross or do not cross $x=0$ during the time interval $[0,\tau]$. We seek class operators C_c and C_{nc} corresponding to these two classes of histories.

Some earlier papers on the decoherent-histories approach adopted the following definition of the class operators [21–25]. (This definition is problematic, as we shall see, but sets the stage for the corrected version we shall use here.) Let $P=\theta(\hat{x})$ denote the projection onto the positive x axis. To define the class operator for histories which do not cross $x=0$, we split the time interval into N parts of size ϵ, and the class operator is defined by

$$C_{nc}=\lim_{\epsilon\to 0} Pe^{-iH\epsilon}P\cdots e^{-iH\epsilon}P, \tag{1.21}$$

where there are $N+1$ projections and N unitary evolution operators, and $N\to\infty$ in such a way that $\tau=N\epsilon$ is constant. The limit actually yields the so-called restricted propagator

$$C_{nc}=g_r(\tau,0). \tag{1.22}$$

This object is also given by the path-integral expression

$$\langle x_1|g_r(\tau,0)|x_0\rangle=\int_r \mathcal{D}x\,\exp(iS), \tag{1.23}$$

where the integral is over all paths from $x(0)=x_0$ to $x(\tau)=x_1$ that always remain in $x(t)>0$. The class operator for crossing the surface is then defined by

$$C_c=e^{-iH\tau}-C_{nc}. \tag{1.24}$$

However, as indicated, there is a problem with this definition. Class operator (1.21) suffers from the quantum Zeno effect [12]—projecting continually in time onto the region $x>0$ prevents the system from leaving it and the probability for not crossing $x=0$ is unity,

$$p_{nc}=\mathrm{Tr}(C_{nc}\rho C_{nc}^{\dagger})=1, \tag{1.25}$$

for any initial state. This is easily seen from the observation that the restricted propagator defined by the limit of Eq. (1.21) may actually be written as

$$g_r(\tau,0)=P\,\exp(-iPHP\tau) \tag{1.26}$$

and so is unitary in the Hilbert space of states with support only in $x>0$ [27,28]. Differently put, an incoming wave packet evolving according to the restricted propagator undergoes total reflection, and so never crosses $x=0$. These results are clearly unphysical and have no sensible classical limit.

The problem here is that the system is monitored too closely to allow the wave packet to actually cross $x=0$. The resolution is therefore to relax the monitoring in such a way that something interesting can happen. There are two obvious ways in which this may be achieved.

The first option is to simply decline to take the limit $\epsilon\to 0$ in Eq. (1.21), so we define the class operator for not crossing to be

$$C_{nc}^{\epsilon}=Pe^{-iH\epsilon}P\cdots e^{-iH\epsilon}P, \tag{1.27}$$

where again $\tau=N\epsilon$, but N is finite and $\epsilon>0$. Clearly if ϵ is large enough the system will be monitored sufficiently infrequently to let the wave packet cross $x=0$ without too much reflection. Studies of the quantum Zeno effect suggest that the appropriate lower limit on ϵ is

$$\epsilon>\frac{1}{\Delta H_0}, \tag{1.28}$$

where H_0 is the free Hamiltonian [12].

The second option is to retain the limit in Eq. (1.21), but "soften" the projections to positive-operator-valued measures (POVMs), that is, to replace $P=\theta(\hat{x})$ with a function which is approximately 1 for large positive x, approximately 0 for large negative x, and with a smooth transition in between.

What we will do in this paper is adopt a definition of the class operator which involves elements of both of these options. In particular, we exploit the useful observation of Echanobe *et al.* [11], which is that the string of projections interspersed with unitary evolution equation (1.27) is approximately equivalent to evolution in the presence of complex potential (1.3),

$$Pe^{-iH\epsilon}P\cdots e^{-iH\epsilon}P\approx\exp[-iH_0\tau-V_0\theta(-\hat{x})\tau]. \tag{1.29}$$

This connection follows from first noting that

$$P=\theta(\hat{x})\approx\exp[-V_0\theta(-\hat{x})\epsilon] \tag{1.30}$$

as long as

$$V_0\epsilon\gg 1. \tag{1.31}$$

We now note that the approximate equivalence

$$\exp(-iH_0\epsilon)\exp[-V_0\theta(-\hat{x})\epsilon]\approx\exp[-iH_0\epsilon-V_0\theta(-\hat{x})\epsilon] \tag{1.32}$$

will hold as long as

$$V_0\epsilon^2|\langle[H_0,\theta(-\hat{x})]\rangle|\ll 1. \tag{1.33}$$

Echanobe *et al.* [11] put an upper bound on the left-hand side using the Schrödinger-Robertson inequality, and the two inequalities together read

$$1\ll V_0^2\epsilon^2\ll\frac{V_0}{\Delta H_0}, \tag{1.34}$$

which can be satisfied as long as

$$V_0 \gg \Delta H_0. \tag{1.35}$$

(Note that Zeno limit restriction (1.28) is not in fact relevant here since the quantum Zeno effect goes away when exact projectors are replaced by quasiprojectors, as in Eq. (1.30) [26].) This demonstration of approximate equivalence (1.29) is somewhat heuristic, but appears to be backed up by numerical evidence and more detailed analytic results [11,29], so we will accept it in what follows.

We stress the key point in this subsection: decoherent histories are usually concerned with class operators defined by strings of projection operators as in Eq. (1.27). However, the above arguments indicate that these strings of projection operators are approximately equivalent to evolution in the presence of a complex potential [Eq. (1.29)]. This is a very useful result for the decoherent-histories approach generally, since the problem of evolution in the presence of the complex potential is straightforward to solve, but the evolution described by Eq. (1.27) could be difficult to solve analytically. It means that in our decoherent-histories analysis of the arrival-time problem, instead of having to work with class operators (1.27), we may instead define the class operator for not crossing to be

$$C_{nc} = \exp[-iH_0\tau - V_0\theta(-\hat{x})\tau], \tag{1.36}$$

with the crossing class operator defined by Eq. (1.24).

These definitions will be extended to class operators C_c^k for crossing during any one of a set of small intervals $[t_k, t_{k+1}]$ of size Δ. These are the class operators we need to give a decoherent-histories account of the origin of $\Pi(\tau)$.

Our main result is that for intervals of size $\Delta \gg 1/V_0$, these class operators are given approximately by

$$\begin{aligned} C_c^k &= e^{-iH_0\tau}\int_{t_k}^{t_{k+1}} dt \frac{(-1)}{2m}[\hat{p}\,\delta(\hat{x}_t) + \delta(\hat{x}_t)\hat{p}] \\ &= e^{-iH_0\tau}[\theta(\hat{x}(t_k)) - \theta(\hat{x}(t_{k+1}))]. \end{aligned} \tag{1.37}$$

Significantly, the dependence on the complex potential has dropped out entirely. We will show that there is decoherence for an interesting class of states, and for such states the probabilities are then given by Eq. (1.19), which has the form

$$q(t_k, t_{k+1}) = \int_{t_k}^{t_{k+1}} dt J(t). \tag{1.38}$$

This, we will show, coincides with the expected result (1.7), when $\Pi(t)$ is integrated over a range of time much greater than $1/V_0$. Hence there is complete agreement with standard results on the arrival-time distribution at sufficiently coarse-grained time scales. Furthermore, our results shed some light on the problem of backflow—we find that the situations when Eq. (1.38) is negative are those in which there is no decoherence, in which case probabilities cannot be assigned.

In another paper, we compute the crossing class operators in a simpler but more heuristic way, by exploring a semiclassical approximation to Eq. (1.27), keeping ϵ finite [30]. The results are essentially the same.

E. This paper

The overall aim of this paper is to give a decoherent-histories account of the arrival-time question which has a sensible classical limit. As stated, we seek a decoherent-histories analysis account of standard results, such as Eq. (1.7).

In Sec. II, we consider the classical analog of the arrival time defined by a complex potential. This sheds light on the form of result (1.7) and in particular, the origin of resolution function (1.8). In Sec. III, we carry out a calculation of standard arrival-time distribution function (1.6), and very readily obtain the expected result of form (1.7). In Secs. IV and V, we use the results of Secs. II and III to carry out the decoherent-histories analysis, as outlined in Sec. I D. We summarize and conclude in Sec. VI.

The main thrust of our story is described in Secs. II–VI. However, we have also included in Appendices A, B, and C some materials which are either tangential to our main story or essential but lengthy background material. In Appendix A we describe some properties of the current and in particular how it may become positive when integrated over a range of time. In Appendix B, we review the path-decomposition expansion (PDX), which is a useful path-integral technique for factoring the propagator across the surface $x=0$, and so is very useful for systems with a step-function potential. In Appendix C, we use the PDX to derive the scattering wave functions corresponding to evolution with a complex step potential. These are of course known, but the PDX gives a useful and concise derivation of them. These results are used in the decoherent-histories analysis and are also valuable for checking a certain semiclassical approximation we use in Sec. III.

II. CLASSICAL CASE

Before looking at the quantum arrival-time problem, it is enlightening to look at the corresponding classical arrival-time problem defined using an absorbing potential. This gives some understanding of the expected form of the result in the quantum case [Eq. (1.7)].

We consider a classical phase-space distribution $w_t(p,q)$, with initial value of $w_0(p,q)$ concentrated entirely in $q>0$ with only negative momenta. The appropriate evolution equation corresponding to quantum case (1.4) is

$$\frac{\partial w}{\partial t} = -\frac{p}{m}\frac{\partial w}{\partial q} - 2V(q)w, \tag{2.1}$$

where $V(q)=V_0\theta(-q)$. This form may be deduced, for example, by computing the evolution equation of the Wigner function corresponding to Eq. (1.4) and dropping the higher-order terms (involving powers of $\hbar$). Equation (2.1) is readily solved and has solution

$$w_\tau(p,q) = \exp\left[-2\int_0^\tau ds V(q - ps/m)\right] w_0(p, q - p\tau/m). \tag{2.2}$$

Following the corresponding steps in the quantum case, the survival probability is

$$N(\tau) = \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq w_\tau(p,q) \tag{2.3}$$

and the arrival-time distribution is

$$\Pi(\tau) = -\frac{dN}{d\tau} = 2V_0 \int_{-\infty}^{\infty} dp \int_{-\infty}^{0} dq w_\tau(p,q), \tag{2.4}$$

where we have made use of Eq. (2.1). This expression is conveniently rewritten by noting that, again using Eq. (2.1), $\Pi(\tau)$ obeys the equation

$$\frac{d\Pi}{d\tau} + 2V_0\Pi = -2V_0 \int_{-\infty}^{\infty} dp \frac{p}{m} w_\tau(p,0). \tag{2.5}$$

This may be solved to yield

$$\Pi(\tau) = -2V_0 \int_0^{\tau} dt e^{-2V_0(\tau-t)} \int_{-\infty}^{\infty} dp \frac{p}{m} w_t(p,0). \tag{2.6}$$

From Eq. (2.2), we see that

$$w_t(p,0) = \exp\left[-2V_0 \int_0^t ds\, \theta(ps/m)\right] w_0(p,-pt/m) \tag{2.7}$$

but since the momenta are all negative the exponential factor makes no contribution. We thus obtain

$$\Pi(\tau) = 2V_0 \int_0^{\tau} dt e^{-2V_0(\tau-t)} J(t), \tag{2.8}$$

where

$$J(t) = -\int_{-\infty}^{\infty} dp \frac{p}{m} w_0\left(p, -\frac{pt}{m}\right). \tag{2.9}$$

The current $J(t)$ is the usual classical arrival-time distribution that would have been obtained in the absence of the absorbing potential.

Result (2.8) is very close in form to the expected quantum result [Eq. (1.7)], differing only in the range of integration. The lower limit of $-\infty$ in Eq. (1.7) arises as a result of the approximations used in deriving it. This difference is not significant since we expect the current to be approximately zero anyway for $t \le 0$.

The classical result (2.8) helps to understand the role of the resolution function $R(t)$ in the quantum case [Eq. (1.7)]—it in essence describes the classical influence of the absorbing potential used to model the detector. More precisely, $R(t)$ is actually related to a sort of coarse graining in time, and this we now demonstrate. Equation (2.8) gives the probability $\Pi(\tau)d\tau$ for arriving during the infinitesimal time interval $[\tau, \tau+d\tau]$. Suppose we consider the probability for arriving during a finite time interval $[\tau_1, \tau_2]$. This is given by

$$p(\tau_2,\tau_1) = \int_{\tau_1}^{\tau_2} d\tau \Pi(\tau) = \int_{\tau_1}^{\tau_2} d\tau \int_0^{\tau} dt 2V_0 e^{-2V_0(\tau-t)} J(t). \tag{2.10}$$

Rearranging the order of integration and integrating over τ yields

$$p(\tau_2,\tau_1) = \int_0^{\tau_1} dt \int_{\tau_1}^{\tau_2} d\tau 2V_0 e^{-2V_0(\tau-t)} J(t) + \int_{\tau_1}^{\tau_2} dt \int_t^{\tau_2} d\tau 2V_0 e^{-2V_0(\tau-t)} J(t) \tag{2.11}$$

$$= \int_0^{\tau_1} dt (e^{-2V_0(\tau_1-t)} - e^{-2V_0(\tau_2-t)}) J(t) + \int_{\tau_1}^{\tau_2} dt (1 - e^{-2V_0(\tau_2-t)}) J(t). \tag{2.12}$$

We will see in what follows that $1/V_0$ plays a role as a fundamental short time scale in the problem. So now suppose we assume that τ_1, τ_2, and $(\tau_2-\tau_1)$ are all much greater than $1/V_0$. It follows that all the exponential terms may be dropped in Eq. (2.12) and we obtain the very simple result

$$p(\tau_2,\tau_1) \approx \int_{\tau_1}^{\tau_2} dt J(t). \tag{2.13}$$

That is, all dependence on the resolution function $R(t)$ and the complex potential parameter V_0 completely drops out when we look at probabilities defined on time scales much greater than $1/V_0$. This result is very relevant to the decoherent-histories analysis considered later where it is natural to look at the arrival time during a finite time interval.

III. CALCULATION OF THE ARRIVAL-TIME DISTRIBUTION

In this section we present an alternative calculation of the standard approach to the arrival-time distribution using complex potentials (1.3)–(1.6). This has been calculated previously using the stationary scattering states (derived in Appendix C), but here we give a derivation using the path-decomposition expansion (see Appendix B). This alternative derivation may be of some value for possible generalizations (to more complicated potentials, for example), but its main value in this paper is that we use the same calculational steps, twice, in Secs. IV and V on the decoherent-histories analysis.

With general complex potential (1.3), arrival-time distribution (1.6) is given by

$$\Pi(\tau) = 2V_0 \langle \psi_\tau | \theta(-\hat{x}) | \psi_\tau \rangle, \tag{3.1}$$

where

$$|\psi_\tau\rangle = \exp(-iH\tau)|\psi\rangle = \exp[-iH_0\tau - V_0\theta(-\hat{x})\tau]|\psi\rangle \tag{3.2}$$

[so we use $H=H_0-iV_0\theta(-\hat{x})$ to denote the total non-Hermitian Hamiltonian]. We are interested in calculating this expression for the case of weak potential,

$$V_0 \ll E_0, \tag{3.3}$$

where E_0 is the energy scale of the initial state. (The very different limit of $V_0\to\infty$, the Zeno limit, has been explored elsewhere [31].)

We use the first crossing PDX [Eq. (B4)], which is conveniently rewritten as the operator expression

$$\langle x|\exp(-iH\tau)|\psi\rangle = -\frac{1}{m}\int_0^\tau dt\langle x|\exp[-iH(\tau-t)]\delta(\hat{x})\hat{p} \times\exp(-iH_0 t)|\psi\rangle. \tag{3.4}$$

Now note that the operator $\delta(\hat{x})=|0\rangle\langle 0|$ has the simple property that for any operator A

$$\delta(\hat{x})A\,\delta(\hat{x}) = \delta(\hat{x})\langle 0|A|0\rangle \tag{3.5}$$

(where recall that $|0\rangle$ denotes the position eigenstate $|x\rangle$ at $x=0$). Inserting Eq. (3.4) into Eq. (3.1), together with property (3.5) and the change of variables $s=\tau-t$, $s'=\tau-t'$, yields

$$\Pi(\tau) = \frac{2V_0}{m^2}\int_0^\tau ds'\int_0^\tau ds\int_{-\infty}^0 dx\langle 0|\exp(iH^\dagger s')|x\rangle\langle x| \times\exp(-iHs)|0\rangle\langle\psi|\exp[iH_0(\tau-s')]\hat{p}\,\delta(\hat{x})\hat{p} \times\exp[-iH_0(\tau-s)]|\psi\rangle. \tag{3.6}$$

We aim to show that this coincides with Eq. (1.7) with current equation (1.1), and the main challenge is to show how the $\hat{p}\,\delta(\hat{x})\hat{p}$ combination turns into the combination $\hat{p}\,\delta(\hat{x}) + \delta(\hat{x})\hat{p}$ in current equation (1.1).

Consider first the x integral. Our assumption that we are in the regime $V_0\ll E_0$ allows us to use semiclassical approximation (C10), which reads

$$\langle x|\exp(-iHs)|0\rangle \approx \left(\frac{m}{2\pi is}\right)^{1/2}\exp\left(i\frac{mx^2}{2s} - V_0 s\right). \tag{3.7}$$

The x integral may now be carried out, with the result

$$\Pi(\tau) = \frac{V_0}{m^2}\int_0^\tau ds'\int_0^\tau ds\left(\frac{m}{2\pi i}\right)^{1/2}\frac{e^{-V_0(s+s')}}{(s-s')^{1/2}} \times\langle\psi_\tau|\exp(-iH_0 s')\hat{p}\,\delta(\hat{x})\hat{p}\,\exp(iH_0 s)|\psi_\tau\rangle, \tag{3.8}$$

where $|\psi_\tau\rangle$ denotes the free evolution of the initial state.

We now carry out one of the time integrals. Note that

$$\int_0^\tau ds'\int_0^\tau ds = \int_0^\tau ds'\int_{s'}^\tau ds + \int_0^\tau ds\int_s^\tau ds'. \tag{3.9}$$

In the first integral, we set $u=s'$ and $v=s-s'$, and in the second integral we set $u=s$ and $v=s'-s$. We thus obtain

$$\Pi(\tau) = \frac{V_0}{m^2}\left(\frac{m}{2\pi}\right)^{1/2}\int_0^\tau du e^{-2V_0 u}\int_0^{\tau-u} dv\frac{e^{-V_0 v}}{v^{1/2}} \times\left[\frac{1}{i^{1/2}}\langle\psi_\tau|\exp(-iH_0 u)\hat{p}\,\delta(\hat{x})\hat{p}\,\exp[iH_0(u+v)]|\psi_\tau\rangle + \frac{1}{(-i)^{1/2}}\langle\psi_\tau|\exp[-iH_0(u+v)]\hat{p}\,\delta(\hat{x})\hat{p} \times\exp(iH_0 u)|\psi_\tau\rangle\right]. \tag{3.10}$$

The factors of $1/(\pm i)^{1/2}$ in front of each term come from a careful consideration of the square root in the free propagator prefactor [and must have this form because $\Pi(\tau)$ is real].

We will assume that

$$V_0\tau \gg 1, \tag{3.11}$$

which means that the integrals are concentrated around $u=v=0$. This means that we may take the upper limit of the v integral to be ∞, and it may be carried out, to yield

$$\Pi(\tau) = 2V_0\int_0^\tau du e^{-2V_0 u}\frac{1}{2m}\langle\psi_{\tau-u}|\hat{p}\,\delta(\hat{x})\Sigma(\hat{p}) + \Sigma^\dagger(\hat{p})\delta(\hat{x})\hat{p}|\psi_{\tau-u}\rangle, \tag{3.12}$$

where the operator $\Sigma(\hat{p})$ is given by

$$\Sigma(\hat{p}) = \frac{\hat{p}}{[2m(H_0+iV_0)]^{1/2}}. \tag{3.13}$$

Since $V_0\ll E_0$, we have

$$\Sigma(\hat{p}) \approx \frac{\hat{p}}{|\hat{p}|}, \tag{3.14}$$

so $\Sigma(\hat{p})$ is simply the sign function of the momentum, which is -1 in this case, since the initial state consists entirely of negative momenta. Finally, writing $u=\tau-t$, we obtain

$$\Pi(\tau) = 2V_0\int_0^\tau dt e^{-2V_0(\tau-t)}\frac{(-1)}{2m}\langle\psi_t|[\hat{p}\,\delta(\hat{x}) + \delta(\hat{x})\hat{p}]|\psi_t\rangle = 2V_0\int_0^\tau dt e^{-2V_0(\tau-t)}J(t). \tag{3.15}$$

We therefore have precise confirmation of classical result (2.8), and also agreement with the expected quantum result, Eq. (1.7), modulo the issues already discussed concerning the range of integration of t. This result is valid under conditions (3.3) and (3.11), which may be combined to read

$$\tau \gg \frac{1}{V_0} \gg \frac{1}{E_0}. \tag{3.16}$$

Some comments are in order concerning the positivity of the result for $\Pi(\tau)$. Expression (3.12) is positive because it was derived from the manifestly positive expression (3.6) [and note that conditions (3.16) do not affect the positivity].

However, to obtain the final result (3.15) we took the limit $V_0\to 0$ in the current part only of Eq. (3.12), leaving behind the V_0-dependent term in the exponential part, and the resulting expression is not guaranteed to be positive. In Eq. (3.15), $J(t)$ is not always positive due to the backflow effect [2,3] and integration over time does not necessarily remedy the situation. (See Appendix A for a more thorough discussion of this point.) The lack of positivity for a $\Pi(\tau)$ obtained in this way is not surprising since taking the limit $V_0 \to 0$ in one part of expression (3.12) only but not the other will not necessarily preserve its property of positivity. The violation of positivity is generally small, however, so Eq. (3.15) may still be a good approximation to the manifestly positive expression (3.12).

It should also be added that it would be misleading to explore the first-order corrections in V_0 in going from Eq. (3.12) to Eq. (3.15), since comparable correction terms have already been dropped in using semiclassical approximation (C10).

IV. DECOHERENT-HISTORIES ANALYSIS FOR A SINGLE LARGE TIME INTERVAL

We now consider the decoherent-histories analysis of this system. We consider an incoming wave packet approaching the origin from $x>0$ and ask for the probability of crossing during a given time interval. We do this in two parts: first in this section, using a large time interval $[0,\tau]$, and second in Sec. V, using a set of intervals of arbitrary size.

A. Class operators and probabilities

We consider first the following simple question. What is the probability of crossing or not crossing during the time interval $[0,\tau]$? The class operators for not crossing and crossing are

$$C_{nc} = \exp[-iH_0\tau - V(x)\tau], \tag{4.1}$$

$$C_c = \exp(-iH_0\tau) - \exp[-iH_0\tau - V(x)\tau], \tag{4.2}$$

and they satisfy

$$C_{nc} + C_c = e^{-iH_0\tau}. \tag{4.3}$$

We are interested in the probabilities for not crossing and crossing,

$$p_{nc}(\tau) = \mathrm{Tr}(C_{nc}\rho C_{nc}^{\dagger}), \tag{4.4}$$

$$p_c(\tau) = \mathrm{Tr}(C_c\rho C_c^{\dagger}), \tag{4.5}$$

and the off-diagonal term of the decoherence functional,

$$D_{c,nc} = \mathrm{Tr}(C_{nc}\rho C_c^{\dagger}) = \mathrm{Tr}(C_{nc}\rho e^{iH_0\tau}) - p_{nc}. \tag{4.6}$$

These quantities obey the relation

$$p_{nc} + p_c + D_{c,nc} + D_{c,nc}^{*} = 1. \tag{4.7}$$

We look for situations where there is decoherence,

$$D_{c,nc} = 0 \tag{4.8}$$

(which is usually only approximate), in which case the probabilities then sum to 1,

$$p_c(\tau) + p_{nc}(\tau) = 1. \tag{4.9}$$

It is useful to relate some of these expressions to the standard expressions for arrival time $\Pi(t)$ defined in Eqs. (1.5) and (1.6) [or Eq. (3.1)]. To do this, note that p_{nc} is in fact the same as the survival probability, $N(\tau)$ defined in Eq. (1.5), and that p_{nc} obeys the trivial identity

$$p_{nc}(\tau) = 1 + \int_0^{\tau} dt \frac{dp_{nc}}{dt} \tag{4.10}$$

since $p_{nc}(0)=1$. It follows that

$$p_{nc}(\tau) = 1 - \int_0^{\tau} dt \Pi(t). \tag{4.11}$$

When there is decoherence, Eq. (4.9) holds and we may deduce that

$$p_c(\tau) = \int_0^{\tau} dt \Pi(t). \tag{4.12}$$

Hence the decoherent-histories analysis is compatible with the standard result, but only when there is decoherence.

B. Calculation of the decoherence functional

We now give two methods for checking for the decoherence of histories. The first involves expressing the probabilities and decoherence functional in terms of the transmitted and reflected waves defined in Eq. (B12), which implies that

$$C_{nc}|\psi\rangle = \theta(-\hat{x})|\psi_{tr}\rangle + \theta(\hat{x})(|\psi_{ref}\rangle + |\psi_f\rangle), \tag{4.13}$$

$$C_c|\psi\rangle = \theta(-\hat{x})(|\psi_f\rangle - |\psi_{tr}\rangle) - \theta(\hat{x})|\psi_{ref}\rangle. \tag{4.14}$$

The probabilities and decoherence functional are therefore given by

$$p_{nc} = \langle\psi_{tr}|\psi_{tr}\rangle + \langle\psi_{ref}|\psi_{ref}\rangle + \langle\psi_{ref}|\psi_f\rangle + \langle\psi_f|\psi_{ref}\rangle + \langle\psi_f|\theta(\hat{x})|\psi_f\rangle, \tag{4.15}$$

$$p_c = \langle\psi_{tr}|\psi_{tr}\rangle + \langle\psi_{ref}|\psi_{ref}\rangle - \langle\psi_{tr}|\psi_f\rangle - \langle\psi_f|\psi_{tr}\rangle + \langle\psi_f|\theta(-\hat{x})|\psi_f\rangle, \tag{4.16}$$

$$D_{c,nc} = \langle\psi_{tr}|\psi_f\rangle - \langle\psi_{tr}|\psi_{tr}\rangle - \langle\psi_{ref}|\psi_{ref}\rangle - \langle\psi_f|\psi_{ref}\rangle. \tag{4.17}$$

[Here for notational convenience we assume that the definition of $|\psi_{tr}\rangle$ includes $\theta(-\hat{x})$ and that of $|\psi_{ref}\rangle$ includes $\theta(\hat{x})$, but the definition of the freely evolving part $|\psi_f\rangle$ does not include a θ function.]

The magnitude of the off-diagonal term in the decoherence functional may be estimated from the explicit solution for the scattering states, Eqs. (C6) and (C8). If there is substantial reflection, it is easily seen that the decoherence functional will not be small. So the interesting regime is the one explored in Secs. II and III, namely, $V_0 \ll E$ (where E is a typical energy scale). In this regime the reflected wave functions are on the order of V_0/E. Furthermore, one can see from Eq. (C6) that the difference between $|\psi_\tau\rangle$ and $|\psi_{tr}\rangle$ is on the order of V_0/E. Therefore, the off-diagonal terms and the probability for not crossing are on the order of V_0/E, and the probability for crossing is on the order of 1, up to corrections on the order of V_0/E. Hence there is decoherence of histories in the regime $V_0 \ll E$.

There is a second method of demonstrating decoherence which gives a different picture and will be useful later. Following the general pattern described in Eqs. (1.19) and (1.20), consider the quantity

$$q_{nc}(\tau) = \mathrm{Tr}(C_{nc}\rho e^{iH_0\tau}). \tag{4.18}$$

From Eq. (4.6), we see that the decoherence functional may be written as

$$D_{c,nc} = q_{nc}(\tau) - p_{nc}(\tau). \tag{4.19}$$

This means that $q_{nc}=p_{nc}$ when there is decoherence. Or to put it the other way around, decoherence of histories may be checked by comparing q_{nc} with p_{nc} and this is what we now do. Recall that p_{nc} is given by Eqs. (4.10) and (4.11) (which hold in the absence of decoherence). We may write q_{nc} in a similar form:

$$q_{nc}(\tau) = 1 + \int_0^\tau dt \frac{dq_{nc}}{dt}. \tag{4.20}$$

The integrand is similar to $\Pi(t)$ defined in Eq. (3.1), so we define

$$\tilde{\Pi}(t) \equiv - \frac{dq_{nc}}{dt}. \tag{4.21}$$

We now have that

$$q_{nc}(\tau) = 1 - \int_0^\tau dt \tilde{\Pi}(t). \tag{4.22}$$

It then follows that the decoherence functional is

$$D_{c,nc} = \int_0^\tau dt[\Pi(t) - \tilde{\Pi}(t)]. \tag{4.23}$$

To compute the decoherence functional we need to calculate $\tilde{\Pi}(t)$, which is given by

$$\tilde{\Pi}(t) = V_0 \langle \psi | \exp(iH_0 t)\theta(-\hat{x})\exp[-iH_0 t - V_0\theta(-\hat{x})t] | \psi \rangle. \tag{4.24}$$

This is almost the same as $\Pi(t)$ except that the exponential on the left involves only H_0 and not the complex potential (and also an overall factor of 2). We therefore follow the calculation of $\Pi(t)$ in Sec. V with small modifications. With little care, one may see that the final result is the same as that for $\Pi(t)$ [Eq. (3.15)] except that $2V_0$ is replaced with V_0, that is,

$$\tilde{\Pi}(t) = V_0 \int_0^t ds e^{-V_0(\tau-s)} J(s). \tag{4.25}$$

This result holds for time scales greater than $1/V_0$ and under semiclassical approximation (C10) (which requires $E \gg V_0$ and so is equivalent to the requirement of negligible reflection encountered above). Finally, a calculation similar to that of Eqs. (2.10)–(2.13) implies that

$$\int_0^\tau dt \Pi(t) \approx \int_0^\tau dt J(t) \tag{4.26}$$

as long as $V_0\tau \gg 1$. Since this result is independent of V_0, $\tilde{\Pi}(t)$ will satisfy the same relation. We thus deduce that

$$D_{c,nc} \approx 0. \tag{4.27}$$

Hence there is decoherence under the above conditions.

V. DECOHERENT-HISTORIES ANALYSIS FOR AN ARBITRARY SET OF TIME INTERVALS

We now turn to the more complicated question of much more refined histories, which may cross the origin at any one of a large number of times, during the time interval $[0,\tau]$. This corresponds more directly to the standard crossing probability $\Pi(t)dt$, the probability that the particle crosses during an infinitesimal time interval $[t,t+dt]$.

A. Class operators

We have defined class operators (4.1) and (4.2) describing crossing or not crossing during a time interval $[0,\tau]$. We now split this time interval into n equal parts of size ϵ, so $\tau=n\epsilon$, and we seek class operators describing crossing or not crossing during any one of the n intervals. We first note that

$$e^{-iH_0\epsilon} = C_{nc}(\epsilon) + C_c(\epsilon), \tag{5.1}$$

where $C_{nc}(\epsilon)$ and $C_c(\epsilon)$ are defined as in Eqs. (4.1) and (4.2) except that here they are for a time interval $[0,\epsilon]$. We now use this to decompose $e^{-iH_0\tau}$ into the desired class operators. We have

$$\begin{aligned} e^{-iH_0\tau} &= (e^{-iH_0\epsilon})^n \\ &= (e^{-iH_0\epsilon})^{n-1}[C_{nc}(\epsilon) + C_c(\epsilon)] \\ &= (e^{-iH_0\epsilon})^{n-1}C_{nc}(\epsilon) + e^{-iH_0(\tau-\epsilon)}C_c(\epsilon). \end{aligned} \tag{5.2}$$

Repeating the same steps on the first term, this yields

$$\begin{aligned} e^{-iH_0\tau} = {} & (e^{-iH_0\epsilon})^{n-2}C_{nc}(2\epsilon) + e^{-iH_0(\tau-2\epsilon)}C_c(\epsilon)C_{nc}(\epsilon) \\ & + e^{-iH_0(\tau-\epsilon)}C_c(\epsilon). \end{aligned} \tag{5.3}$$

Repeating more times eventually yields

$$e^{-iH_0\tau} = C_{nc}(\tau) + \sum_{k=0}^{n-1} e^{-iH_0[\tau-(k+1)\epsilon]}C_c(\epsilon)C_{nc}(k\epsilon). \tag{5.4}$$

From this expression, we see that the class operator for crossing $x=0$ for the first time during the time interval $[k\epsilon,(k+1)\epsilon]$ is given by the summand of the second term,

$$C_c[(k+1)\epsilon,k\epsilon] = e^{-iH_0[\tau-(k+1)\epsilon]}C_c(\epsilon)C_{nc}(k\epsilon). \tag{5.5}$$

We will not in fact work with class operator (5.5), since a more useful similar but alternative expression can also be found. Taking the continuum limit of Eq. (5.4) (and inserting the explicit expression for C_{nc}), we obtain

$$e^{-iH_0\tau} = e^{-iH_0\tau - V\tau} + \int_0^\tau dt e^{-iH_0(\tau-t)} V e^{-iH_0 t - Vt} \tag{5.6}$$

[where, recall, $V=V_0\theta(-\hat{x})$]. This indicates that the class operator for first crossing during the infinitesimal time interval $[t,t+dt]$ is

$$C_c(t) = e^{-iH_0(\tau-t)} V e^{-iH_0 t - Vt}. \tag{5.7}$$

We do not, however, expect histories characterized by such precise crossing time to be decoherent, so it is natural to consider coarser-grained class operators,

$$C_c^k = \int_{t_k}^{t_{k+1}} dt C_c(t), \tag{5.8}$$

which represents crossing during one of the N time intervals $[t_k,t_{k+1}]$ of size Δ, where $t_k=k\Delta$, with $k=0,1,\ldots,N-1$ and $\tau=N\Delta$. The complete set of class operators C_α for crossing and not crossing is the set of $N+1$ operators

$$C_\alpha = \{C_{nc}, C_c^k\}, \tag{5.9}$$

and Eq. (5.6) implies that they satisfy

$$e^{-iH_0\tau} = C_{nc} + \sum_{k=0}^{N-1} C_c^k. \tag{5.10}$$

To check for decoherence of histories we need to calculate two types of decoherence functional,

$$D_{kk'} = \mathrm{Tr}[C_c^k \rho (C_c^{k'})^\dagger], \tag{5.11}$$

$$D_{k,nc} = \mathrm{Tr}[C_c^k \rho (C_{nc})^\dagger], \tag{5.12}$$

and this will be carried out below.

B. Important simplification of the class operator

There turns out to be a very useful simplification in class operator (5.7). Consider the amplitude

$$\langle x| e^{iH_0\tau} C_c(t) |\psi\rangle = V_0 \langle x| e^{iH_0 t} \theta(-\hat{x}) e^{-iH_0 t - Vt} |\psi\rangle \tag{5.13}$$

for any x. The right-hand side is very similar to Eq. (3.1), except that there is no complex potential in one of the exponential terms and also the "final" state is $|x\rangle$, not $|\psi\rangle$. (There is also an overall factor-of-2 difference.) Despite these differences, we may once again make use of the details of the calculation in Sec. V, and we deduce from the analogous result (3.15) that

$$\langle x| e^{iH_0\tau} C_c(t) |\psi\rangle = V_0 \int_0^t ds e^{-V_0(t-s)} \frac{(-1)}{2m} \times \langle x| [\hat{p}\delta(\hat{x}_s) + \delta(\hat{x}_s)\hat{p}] |\psi\rangle. \tag{5.14}$$

Like the derivation of Eq. (3.15), this is valid under the conditions that all energy scales are much greater than V_0 and all time scales are much greater than $1/V_0$. Now we integrate this over time to obtain the coarse-grained crossing time operator [Eq. (5.8)], and again use approximations of forms (2.10)–(2.13) (again using the assumption of time scales much greater than $1/V_0$), to yield the remarkably simple and appealing form

$$e^{iH_0\tau} C_c^k = \int_{t_k}^{t_{k+1}} dt \frac{(-1)}{2m} [\hat{p}\delta(\hat{x}_t) + \delta(\hat{x}_t)\hat{p}]. \tag{5.15}$$

This may also be written even more simply as

$$e^{iH_0\tau} C_c^k = \theta(\hat{x}(t_k)) - \theta(\hat{x}(t_{k+1})). \tag{5.16}$$

C. Probabilities for crossing

The above expressions for the crossing time class operator are the most important results of the paper and provide an immediate connection to the standard expression for the arrival-time distribution. Supposing for the moment that there is decoherence of histories, we may assign probabilities to the histories. The probability for crossing during the time interval $[t_k,t_{k+1}]$ is

$$p(t_k,t_{k+1}) = \mathrm{Tr}[C_c^k \rho (C_c^k)^\dagger]. \tag{5.17}$$

However, as noted in Eqs. (1.19) and (1.20) when there is decoherence of histories, this expression for the probabilities for histories is equal to the simpler expression

$$\begin{aligned} q(t_k,t_{k+1}) &= \mathrm{Tr}(C_c^k \rho e^{iH_0\tau}) \\ &= \int_{t_k}^{t_{k+1}} dt \frac{(-1)}{2m} \langle\psi| [\hat{p}\delta(\hat{x}_t) + \delta(\hat{x}_t)\hat{p}] |\psi\rangle \\ &= \int_{t_k}^{t_{k+1}} dt J(t), \end{aligned} \tag{5.18}$$

which is precisely the standard result. The expression for the probability $q(t_k,t_{k+1})$ is not positive in general (although is real in this case, as it happens), but when there is decoherence, it is equal to $p(t_k,t_{k+1})$, which *is* positive. Hence the decoherent-histories result coincides with the standard result under the somewhat special conditions of decoherence of histories.

D. Decoherence of histories and the backflow problem

There is an interesting connection between decoherence of histories and backflow. To see this, consider the following simple case. We consider histories which either cross or do not cross the origin during the time interval $[t_1,t_2]$. So the crossing and not crossing class operators are C and $1-C$, where

$$C = \theta(\hat{x}_1) - \theta(\hat{x}_2), \tag{5.19}$$

where we have adopted the notation $\hat{x}_k=\hat{x}(t_k)$ (and for convenience we have dropped the exponential factor which is just a matter of definition and drops out of all expression of interest). The decoherence functional is

$$D = \langle C(1-C)\rangle = \langle C\rangle - \langle C^2\rangle. \tag{5.20}$$

This may also be written as

$$D = -\langle[\theta(-\hat{x}_1)\theta(\hat{x}_2) + \theta(\hat{x}_2)\theta(-\hat{x}_1)]\rangle, \tag{5.21}$$

a form we will use below to check decoherence. When there is decoherence, $D=0$ and the probability for crossing is

$$p(t_1,t_2) = \langle C^2\rangle = \langle C\rangle. \tag{5.22}$$

As noted above, $\langle C\rangle$ is the standard result [Eq. (5.18)] for the probability of crossing.

There is an interesting connection here between backflow and decoherence. If there is decoherence, D is zero so $\langle C\rangle$ must cancel $\langle C^2\rangle$ in Eq. (5.20), which means that $\langle C\rangle \geq 0$, so there is no backflow. Or we may make a logically equivalent statement: if there is backflow, $\langle C\rangle < 0$, then there cannot be decoherence, since $|D|$ is then greater than the probability $\langle C^2\rangle$. Hence, states with backflow do not permit decoherence of histories. (Note that absence of backflow, $\langle C\rangle \geq 0$, is not itself enough to guarantee decoherence—the stronger condition $D=0$ must be satisfied.)

This is an important result. The quantity $\langle C\rangle$ is regarded as the "standard" result for crossing time probability, and its possible negativity is disturbing. Here, the decoherent-histories approach sheds more light on this issue. In the decoherent-histories approach, the true probability for crossing is the manifestly positive quantity $\langle C^2\rangle$ and this is equal to $\langle C\rangle$ only when there is decoherence. In particular, when there is significant backflow, there cannot be decoherence, so probabilities cannot be assigned and $\langle C^2\rangle$ is not equal to $\langle C\rangle$.

E. Decoherence conditions

The crossing probabilities described above are only valid when all off-diagonal components of the decoherence functional, Eqs. (5.11) and (5.12), are zero. We therefore address the issue of finding those states for which there is negligible decoherence.

We consider first the simpler case, that of decoherence functional (5.12). The noncrossing class operator C_{nc} is given in general by Eq. (4.1). However, it simplifies considerably in the approximations used to derive Eq. (5.16), which we adopt here. In particular, Eq. (5.10) with Eq. (5.16) implies that

$$e^{iH_0\tau} = C_{nc} + e^{iH_0\tau}[\theta(\hat{x}) - \theta(\hat{x}(\tau))]. \tag{5.23}$$

Since we are interested only in initial states with support entirely in $x>0$, we have $\theta(\hat{x})|\psi\rangle = |\psi\rangle$, which means that effectively

$$C_{nc}(\tau) \approx \theta(\hat{x})e^{-iH_0\tau}. \tag{5.24}$$

The decoherence functional of interest is then

$$D_{k,nc} = \langle\psi|\theta(\hat{x}(\tau))[\theta(\hat{x}_k) - \theta(\hat{x}_{k+1})]|\psi\rangle. \tag{5.25}$$

This is conveniently rewritten as

$$D_{k,nc} = \langle\psi|\theta(\hat{x}(\tau))[\theta(-\hat{x}_{k+1}) - \theta(-\hat{x}_k)]|\psi\rangle. \tag{5.26}$$

We will take τ to be very large and it is pretty clear that this object will be approximately zero, since we expect all the initial state to end up in $x<0$ at large times. However, we will see this below in more detail.

The more important decoherence condition is that $D_{kk'}$ defined in Eq. (5.11) vanishes, so we now focus on that. We write class operator (5.16) for crossing during the time interval $[t_k,t_{k+1}]$ as

$$C_c^k = e^{-iH_0\tau}[\theta(\hat{x}_k) - \theta(\hat{x}_{k+1})]. \tag{5.27}$$

For an initial state $|\psi\rangle$, the quantity $C_c^k|\psi\rangle$ is a quantum state representing the property of crossing of the origin in the time interval $[t_k,t_{k+1}]$. The decoherence condition $D_{kk'}=0$ is simply the condition that the "crossing states" $C_c^k|\psi\rangle$ for different time intervals have negligible interference. The states $C_c^k|\psi\rangle$ consist of an initial state which has been localized to a range of time at $x=0$. This is closely related to the interesting question of diffraction in time [32] and this connection will be explored in more detail elsewhere [33].

The decoherence functional is given by

$$\begin{aligned} D_{kj} &= \langle[\theta(\hat{x}_k) - \theta(\hat{x}_{k+1})][\theta(\hat{x}_j) - \theta(\hat{x}_{j+1})]\rangle \\ &= \langle[\theta(\hat{x}_k) - \theta(\hat{x}_{k+1})][\theta(-\hat{x}_{j+1}) - \theta(-\hat{x}_j)]\rangle, \end{aligned} \tag{5.28}$$

where without loss of generality we take $t_{j+1}<t_k$. It is a sum of terms each of the form

$$d_{kj} = \langle\theta(-\hat{x}_k)\theta(\hat{x}_j)\rangle, \tag{5.29}$$

where $t_k<t_j$. Note that

$$|d_{kj}|^2 \leq d_m^2 = \langle\theta(-\hat{x}_k)\theta(\hat{x}_j)\theta(-\hat{x}_k)\rangle. \tag{5.30}$$

The key thing is that d_m^2 has the form of a probability—it is the probability of finding the particle in $x<0$ at t_k and then in $x>0$ at t_j. Semiclassical expectations suggest that this is small in general for the states considered here, which are left-moving wave packets, and indeed the classical limit of this probability is zero. So this is a useful object to calculate in terms of checking decoherence (although it may not be small for states with backflow). Note that it also implies that the other parts of decoherence functional (5.26) will also be small. In detailed calculations, upper bound (5.30) must be compared with the probabilities, as in Eq. (1.18).

F. Checking the decoherence condition for wave packets

We now consider the particular case of an initial state consisting of a wave packet

$$\psi(x) = \frac{1}{(2\pi\sigma^2)^{1/4}}\exp\left(-\frac{(x-q_0)^2}{4\sigma^2} + ip_0x\right), \tag{5.31}$$

where $q_0>0$ and $p_0<0$. We first consider a heuristic analysis of decoherence. In the simplest case, the wave packet crosses the origin almost entirely during the time interval $[t_k,t_{k+1}]$ (of size Δ) for some k, without any substantial overlap with any other time intervals. (See Fig. 1.) This means that

$$\begin{aligned} C_c^k|\psi\rangle &\approx |\psi\rangle, \\ C_c^{k'}|\psi\rangle &\approx 0 \quad \text{for } k' \neq k, \end{aligned} \tag{5.32}$$

and it follows that $D_{kk'}\approx 0$. The key time scales here are the classical arrival time for the center of the packet,

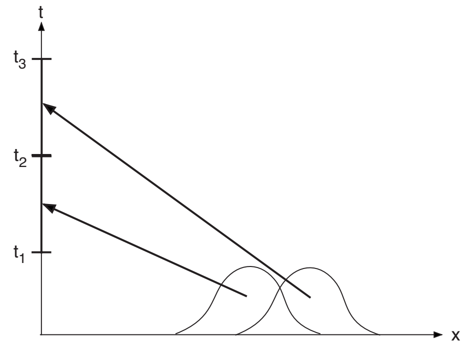

FIG. 2. An initial state consisting of a superposition of wave packets may have significant crossings in at least two different time intervals. If the initial packets are orthogonal and the time intervals are sufficiently large (greater than the Zeno time), the packets will remain orthogonal after passing through the time intervals and the corresponding histories will be decoherent.

$$t_a = \frac{mq_0}{|p_0|}, \tag{5.33}$$

and the Zeno time

$$t_Z = \frac{m\sigma}{|p_0|} \tag{5.34}$$

(which is also approximately equal to $1/\Delta H_0$). The Zeno time is the time taken for the wave packet to move a distance equal to its spatial width σ or, equivalently, it is the size of the packet's "temporal imprint" at the origin. Therefore, the above approximations work if, first,

$$t_Z \ll \Delta \tag{5.35}$$

and, second, if the classical arrival time t_a lies inside the interval $[t_k, t_{k+1}]$ and is at least one or two Zeno times away from the boundaries.

It is easy to see that similar conclusions hold for superpositions of initial states of form (5.31) as long as they are approximately orthogonal. Loosely, this is because under the above conditions, the class operators do not disturb the states and the only nonzero components of the off-diagonal terms of the decoherence functional will be proportional to the overlap of pairs of initial wave packets, and so will be approximately zero. (See Fig. 2.) More generally, nonorthogonal superpositions may, however, produce backflow, so there may be no decoherence.

G. Checking decoherence for a detailed model

Decoherence starts to become lost as the size Δ of the time intervals $[t_k, t_{k+1}]$ is reduced to close to the Zeno time. This is because the wave packet will split into parts that cross during different time intervals, and the effect of diffraction in time mentioned above [32] will cause these different parts to be nonorthogonal. These effects will be explored in more detail elsewhere [33]. Here, we give a more detailed calculation to check for decoherence.

For simplicity we work with the simple case considered in Sec. V D above. We take the initial state to be wave packet equation (5.31) and we note that decoherence functional (5.21) satisfies

$$|D|^2 \le 2d_m^2, \tag{5.36}$$

with d_m^2 given by Eq. (5.30) with $k=1$ and $j=2$. We need some probabilities to compare this with. We have that

$$|D|^2 \le \langle C^2\rangle\langle(1-C)^2\rangle. \tag{5.37}$$

The interesting case is that in which the crossing probability $\langle C^2\rangle$ is somewhat less than 1, say, less than about 1/2, in which case the noncrossing probability $\langle(1-C)^2\rangle$ will be of order 1. It is therefore sufficient to compare d_m^2 with $\langle C^2\rangle$. Now note that

$$\begin{aligned}\langle C^2\rangle &= \langle[\theta(\hat{x}_1) - \theta(\hat{x}_2)]^2\rangle \\ &= \langle\theta(\hat{x}_1)\theta(-\hat{x}_2)\theta(\hat{x}_1)\rangle + \langle\theta(\hat{x}_1)\theta(-\hat{x}_2)\theta(-\hat{x}_1)\rangle \\ &\quad + \langle\theta(\hat{x}_2)\theta(-\hat{x}_1)\rangle. \end{aligned} \tag{5.38}$$

This means that $\langle C^2\rangle$ is in fact equal to the probability

$$p_{12} = \langle\theta(\hat{x}_1)\theta(-\hat{x}_2)\theta(\hat{x}_1)\rangle \tag{5.39}$$

up to terms which vanish when $D=0$. This is useful since it is now identical in form to the expression for d_m^2 and our goal is to show that

$$d_m^2 \ll p_{12}. \tag{5.40}$$

It is useful to work in the Wigner representation [34], defined for a state $\rho(x,y)$ by

$$W(p,q) = \frac{1}{2\pi}\int d\xi e^{-ip\xi}\rho\left(q + \frac{1}{2}\xi, q - \frac{1}{2}\xi\right). \tag{5.41}$$

The probabilities p_{12} and d_m^2 are then given by

$$p_{12} = 2\pi\int dpdq W_{12}(p,q)W_0(p,q,t_1), \tag{5.42}$$

$$d_m^2 = 2\pi\int dpdq W_D(p,q)W_0(p,q,t_1). \tag{5.43}$$

Here, $W_0(p,q,t_1)$ is the Wigner function of the initial state, evolved in time to t_1,

$$W_0(p,q,t_1) = \frac{1}{\pi}\exp\left(-\frac{(q-q_0-p_0t_1/m)^2}{2\sigma^2} - 2\sigma^2(p-p_0)^2\right). \tag{5.44}$$

The objects W_P and W_D are the Wigner transforms of the θ-function combinations appearing in Eqs. (5.30) and (5.39) and are given by

$$W_P(p,q) = \frac{1}{2\pi^2}\theta(q)\int_{u(p,q)}^{\infty} dy\frac{\sin y}{y}, \tag{5.45}$$

$$W_D(p,q) = \frac{1}{2\pi^2}\theta(-q)\int_{u(p,q)}^{\infty} dy\frac{\sin y}{y}, \tag{5.46}$$

where

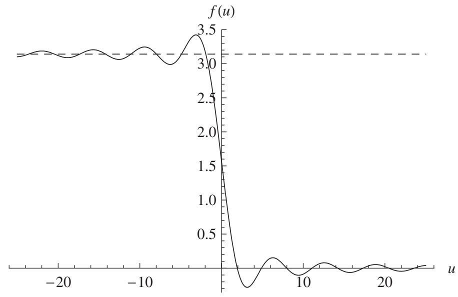

FIG. 3. The function $f(u)$. It oscillates around zero for $u \gg 0$ and oscillates around π for $u \ll 0$. As a function of q, $f(u(p_0,q))$ differs from 0 or π only in a region of size $1/|p_0|$ around $q=0$.

$$u(p,q) = 2q\left(p + \frac{mq}{(t_2 - t_1)}\right). \tag{5.47}$$

We see that the only difference between the expressions for p_{12} and d_m^2 is in the sign in the θ functions.

The integral

$$f(u) = \int_u^\infty dy \frac{\sin y}{y} \tag{5.48}$$

may be expressed in terms of the sine integral function $\mathrm{Si}(x)$,

$$f(u) = \frac{\pi}{2} - \mathrm{Si}(u), \tag{5.49}$$

but its properties are not hard to see directly. For large negative u, $f(u) \approx \pi$; at $u=0$, $f(0)=\pi/2$; and for large positive u, $f(u)$ goes to zero, oscillating around $1/u$. (See Fig. 3.)

We now compare the sizes of p_{12} and d_m^2.We assume that the wave packet is spatially broad, so σ is large and Wigner function (5.44) is therefore concentrated strongly about $p=p_0<0$. We therefore integrate out p and set $p=p_0$ throughout. The most important case to check is that in which the wave packet is reasonably evenly divided between $x>0$ and $x<0$ at time t_1 so that both p_{12} and d_m^2 have a chance of being reasonably large. This means that $q_0+p_0t_1/m$ should be close to zero (to within a few widths σ), so for simplicity we take it to be exactly zero.

With these simplifications, we have

$$d_m^2 = \frac{1}{(2\pi^3\sigma^2)^{1/2}} \int_{-\infty}^0 dq \exp\left(-\frac{q^2}{2\sigma^2}\right) f(u(p_0,q)). \tag{5.50}$$

Here, since $q \le 0$ and $p_0<0$, we have $u(p_0,q) \ge 0$. We can evaluate this expression by examining the comparative effects of $f(u)$ and the exponential term. From the plot of $f(u)$ (see Fig. 3), we see that it drops to zero at $u=u_0$ (which is of order 1) and oscillates rapidly around zero for $u>u_0$, so we expect the integral to be dominated by values of q for which $0 \le u \le u_0$. The value $u=u_0$ corresponds to $q=q_0$, where

$$q_0 = -\frac{|p_0|\Delta}{2m}\left(\left[1 + \frac{u_0}{E_0\Delta}\right]^{1/2} - 1\right), \tag{5.51}$$

where $E_0=p_0^2/2m$. In the complex potential calculations, we have assumed that $E_0 \gg V_0$ and we also assumed that all time scales are much greater than $1/V_0$, and these together imply that $E_0\Delta \gg 1$. We may therefore expand the square root to leading order and obtain

$$q_0 \approx -\frac{u_0}{2|p_0|}. \tag{5.52}$$

We have assumed that the wave packet is sufficiently broad that $\sigma p_0 \gg 1$, and this means that

$$|q_0| \ll \sigma. \tag{5.53}$$

This in turn means that $f(u)$ is significantly different from zero only in the range $|q| \ll \sigma$, and most importantly, in this range, the exponential term in Eq. (5.50) is approximately constant. We may therefore evaluate Eq. (5.50) by ignoring the exponential term, integrating from 0 to q_0 and approximating $f(u)$ as

$$f(u) \approx \frac{\pi}{2} - u + O(u^3). \tag{5.54}$$

We thus obtain the simple result

$$d_m^2 \approx \frac{1}{(2\pi^3)^{1/2}} \frac{1}{|p_0|\sigma} \ll 1. \tag{5.55}$$

In the expression for p_{12}, there is a key difference in that $q>0$, which means that $u(p_0,q)$ can be positive or negative. Introducing

$$q_Z = \frac{|p_0|}{m}\Delta = \sigma\frac{\Delta}{t_Z} \tag{5.56}$$

(where t_Z is the Zeno time), we see that $u<0$ for $q<q_Z$ and $u>0$ for $q>q_Z$. We therefore have

$$p_{12} = \frac{1}{(2\pi^3\sigma^2)^{1/2}} \int_0^{q_Z} dq \exp\left(-\frac{q^2}{2\sigma^2}\right) f(u(p_0,q)) + \frac{1}{(2\pi^3\sigma^2)^{1/2}} \int_{q_Z}^\infty dq \exp\left(-\frac{q^2}{2\sigma^2}\right) f(u(p_0,q)). \tag{5.57}$$

Here, $f(u(p_0,q)) \approx \pi$ in the first term, differing from this value only in a region of size $1/p_0$ close to $q=0$. In the second term $f(u)$ will tend to be small except for a small region of size $1/p_0$ around the origin.

If $q_Z \gg \sigma$ (that is, $\Delta \gg t_Z$), the second term in p_{12} is exponentially suppressed and in the first term the integration range is effectively 0 to ∞, so we obtain

$$p_{12} \approx \tfrac{1}{2}. \tag{5.58}$$

This is the expected result, since under the above assumptions on the wave packet, half of it will cross $x=0$ if the time interval is sufficiently large. Clearly $p_{12} \gg d_m^2$ in this case so there is decoherence.

If $q_Z \ll \sigma$, the first term in p_{12} is on the order of q_Z/σ and the second on the order of $1/(|p_0|\sigma)$, the same order of magnitude as d_m^2. Hence in this case we have decoherence if

$$\frac{\Delta}{t_Z} \gg \frac{1}{|p_0|\sigma}. \tag{5.59}$$

Since the right-hand side is already $\ll 1$, this is easily satisfied even for time intervals whose size is Δ is smaller than the Zeno time. In fact this condition is equivalent to the condition

$$E_0 \Delta \gg 1, \tag{5.60}$$

which is satisfied by the assumptions of the complex potential model, as stated above. Interestingly, this condition has also arisen previously as the limitation on the precision to which a quantum system may be used as a clock [35]. In brief, we therefore get decoherence of histories for a single wave packet under a wide variety of circumstances.

VI. SUMMARY AND CONCLUSIONS

This paper was initially motivated by a desire to analyze the arrival-time problem using the decoherent-histories approach to quantum theory. But along the way we have reconsidered and derived a number of other useful related results.

We considered the arrival-time problem using a complex potential to kill paths entering $x<0$. In Sec. II we gave a classical analysis of the problem. We derived a result of the expected form exposing the resolution function as an essentially classical effect summarizing the role of the complex potential. We also showed that coarse graining over time scales much greater than $1/V_0$ produces a formula for the arrival time of expected form and which is independent of the complex potential. This is an important result for the rest of the paper.

In Sec. III, we used the PDX to rederive the standard form of the arrival-time distribution with a complex potential, in the limit of weak potential. The form of this calculation turned out to be useful for the subsequent work on the decoherent-histories approach.

In Sec. IV, we considered the decoherent-histories analysis for the simple case of a particle crossing or not crossing $x=0$ during a large time interval $[0,\tau]$. We found the simple and expected result that the histories are decoherent as long as reflection by the complex potential is negligible. The resultant probabilities are consistent with the standard result for the arrival time.

The main part of the decoherent-histories analysis was given in Sec. V, where we first derived the class operators describing crossing $x=0$ for an arbitrary set of small time intervals. Here we obtained our most important result: crossing class operator (5.15) for time scales much greater than $1/V_0$. This form of the class operator gives an immediate connection with the standard result for probabilities when there is decoherence. Indeed, one may have *guessed* the form of the class operator from the standard form of the probabilities, and this is pursued in another paper [30]. However, it is also gratifying that it can be derived in some detail using the complex potential approach used here.

To assign probabilities, the decoherence functional must be diagonal and we considered this condition. We found a variety of states for which there is decoherence, under certain more detailed conditions, which we discussed.

We also noted an interesting and important relationship between decoherence and backflow: if there is decoherence, the probabilities for crossing must be positive so there cannot be any backflow. If there is no decoherence, the integrated current may still be positive, but one can say that if there is backflow there will definitely be no decoherence. This means that the decoherent-histories approach brings something genuinely different to the arrival-time problem: it establishes the conditions under which probabilities can be assigned and in particular forbids the assignment of probabilities in cases where there is backflow.

ACKNOWLEDGMENTS

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APPENDIX A: SOME PROPERTIES OF THE CURRENT

We have derived the expression

$$p(0,T) = \int_0^T dt J(t) \tag{A1}$$

as the approximate probability for crossing the origin during the time interval $[0,T]$, where $J(t)$ is the usual quantum-mechanical current. The current itself is not necessarily positive due to backflow. Here we explore the possibility that averaging it over time might improve the situation. On one hand, the results of Bracken and Melloy [3] show that there is always *some* state for which $p(0,T)$ defined above is negative, for any T. On the other hand, for a *given* state, one might hope that $p(0,T)$ will be positive for sufficiently large T. Here we give a brief argument for this, which also makes contact with the negativity of the Wigner function.

The current can be written in terms of the Wigner function $W(p,x)$ as

$$J(t) = -\int dp \frac{p}{m} W(p,0,t). \tag{A2}$$

The Wigner function evolves freely according to $W(p,x,t) = W(p,x-pt/m,0)$. We assume it has support only on negative-momentum states, with average momentum $p_0<0$ and momentum width σ_p.

Consider a time interval $0<t<T$ over which backflow occurs. It is clear that in order for this to occur the Wigner function must be negative for at least some of this interval. We can write

$$p(0,T) = -\int_0^T dt \int dp \frac{p}{m} W(p,-pt/m,0)$$
$$= \int dp \int_0^{|p|T/m} dx W(p,x,0). \quad \text{(A3)}$$

So $p(0,T)$ is given by the average of the Wigner function over a region of phase space. We now recall a standard property of the Wigner function which is that, broadly speaking, it will tend to be positive when averaged over a region of phase space of size greater than order 1 (in the units used here where $\hbar=1$). This region is of size on the order of $|p_0|T/m$ in the x direction but infinite in the p direction. However, the Wigner function has momentum spread σ_p, so the effective size averaged over is $\sigma_p p_0 T/m$, which is approximately the same as $\Delta H T$. This means that we expect that $p(0,T)$ will be positive as long as

$$T > \frac{1}{\Delta H}. \quad \text{(A4)}$$

Hence, as expected, the integrated current will be positive for T sufficiently large and the key time scale is the Zeno time. This heuristic argument will be revisited in more detail elsewhere.

APPENDIX B: THE PATH-DECOMPOSITION EXPANSION

In this appendix we review some useful path-integral techniques that will be useful in Sec. III and in Appendix C below. We wish to evaluate the propagator

$$g(x_1,\tau|x_0,0) = \langle x_1|\exp[-iH_0\tau - V_0\theta(-\hat{x})f(\hat{x})\tau]|x_0\rangle \quad \text{(B1)}$$

for arbitrary x_1 and $x_0>0$. This may be calculated using a sum over paths,

$$g(x_1,\tau|x_0,0) = \int \mathcal{D}x \exp(iS), \quad \text{(B2)}$$

where

$$S = \int_0^\tau dt\left[\frac{1}{2}m\dot{x}^2 + iV_0\theta(-x)f(x)\right] \quad \text{(B3)}$$

and the sum is over all paths $x(t)$ from $x(0)=x_0$ to $x(\tau)=x_1$.

To deal with the step-function form of the potential, we need to split off the sections of the paths lying entirely in $x>0$ or $x<0$. The way to do this is to use the PDX [36–38]. Consider first paths from $x_0>0$ to $x_1<0$. Each path from $x_0>0$ to $x_1<0$ will typically cross $x=0$ many times, but all paths have a first crossing, say, at time t_1. As a consequence of this, it is possible to derive the formula

$$g(x_1,\tau|x_0,0) = \frac{i}{2m}\int_0^\tau dt_1 g(x_1,\tau|0,t_1)\frac{\partial g_r}{\partial x}(x,t_1|x_0,0)|_{x=0}. \quad \text{(B4)}$$

Here, $g_r(x,t|x_0,0)$ is the restricted propagator given by a sum over paths of form (B2) but with all paths restricted to

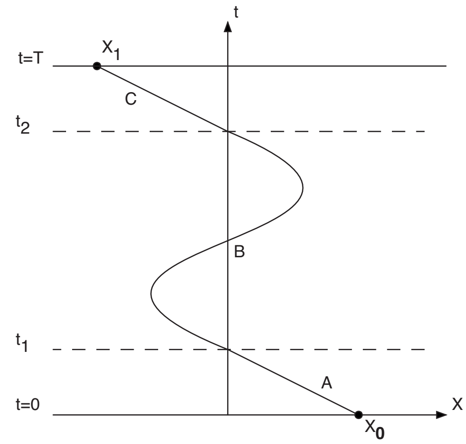

FIG. 4. The path-decomposition expansion. Any path from $x_0>0$ at $t=0$ to a final point $x_1<0$ at $t=T$ has a first crossing of $x=0$ at t_1 and a last crossing at t_2. The propagator from $(x_0,0)$ to (x_1,T) may be decomposed into three parts: (A) restricted propagation entirely in $x>0$, (B) free propagation starting and ending on $x=0$, and (C) restricted propagation entirely in $x<0$. The corresponding path-decomposition expansion formula is given in Eq. (B6).

$x(t)>0$. It vanishes when either end point is the origin but its derivative at $x=0$ is nonzero (and in fact the derivative of g_r corresponds to a sum over all paths in $x>0$ which end on $x=0$ [37]).

It is also useful to record a PDX formula involving the last crossing time t_2, for $x_0>0$ and $x_1<0$,

$$g(x_1,\tau|x_0,0) = -\frac{i}{2m}\int_0^\tau dt_2 \frac{\partial g_r}{\partial x}(x_1,\tau|x,t_2)|_{x=0} g(0,t_2|x_0,0). \quad \text{(B5)}$$

These two formulas may be combined to give a first and last crossing version of the PDX,

$$g(x_1,\tau|x_0,0) = \frac{1}{4m^2}\int_0^\tau dt_2 \int_0^{t_2} dt_1 \frac{\partial g_r}{\partial x}(x_1,\tau|x,t_2)|_{x=0} g(0,t_2|0,t_1) \times \frac{\partial g_r}{\partial x}(x,t_1|x_0,0)|_{x=0}. \quad \text{(B6)}$$

This is clearly very useful for a step potential since the propagator is decomposed in terms of propagation in $x<0$ and in $x>0$, essentially reducing the problem to that of computing the propagator along $x=0$, $g(0,t_2|0,t_1)$. (See Fig. 4.)

For paths with $x_0>0$ and $x_1>0$, Eq. (B4) is modified by the addition of a term $g_r(x_1,t|x_0,0)$, corresponding to a sum over paths which never cross $x=0$, so we have

$$g(x_1,\tau|x_0,0) = \frac{1}{2m}\int_0^\tau dt_1 g(x_1,\tau|0,t_1)\frac{\partial g_r}{\partial x}(x,t_1|x_0,0)|_{x=0} + g_r(x_1,t|x_0,0). \quad \text{(B7)}$$

Again a further decomposition involving the last crossing, as in Eq. (B6), can also be included.

The various elements of these expressions are easily calculated for a potential of simple step-function form $V(x)=V_0\theta(-x)$. The restricted propagator in $x>0$ is given by the method-of-images expression

$$g_r(x_1,\tau|x_0,0) = \theta(x_1)\theta(x_0)[g_f(x_1,\tau|x_0,0) - g_f(-x_1,\tau|x_0,0)], \tag{B8}$$

where g_f denotes the free-particle propagator

$$g_f(x_1,\tau|x_0,0) = \left(\frac{m}{2\pi i\tau}\right)^{1/2}\exp\left(\frac{im(x_1-x_0)^2}{2\tau}\right). \tag{B9}$$

It follows that

$$\frac{\partial g_r}{\partial x}(x,t_1|x_0,0)|_{x=0} = 2\frac{\partial g_f}{\partial x}(0,t_1|x_0,0)\theta(x_0). \tag{B10}$$

The restricted propagator in $x<0$ is given by Eq. (B8), multiplied by $\exp(-V_0\tau)$. The only complicated propagator to calculate is the propagation from the origin to itself along the edge of the potential. In the case $V(x)=V_0\theta(-x)$ this is given by [39]

$$g(0,t|0,0) = \left(\frac{m}{2\pi i}\right)^{1/2}\frac{[1-\exp(-V_0t)]}{V_0t^{3/2}}. \tag{B11}$$

Using these results we may write down the full solution to the evolution with a complex potential, described by Eq. (B1), for an initial state $\psi(x)$ with support only in $x>0$ and with negative momenta. It has the form

$$\psi(x,\tau) = \theta(-x)\psi_{tr}(x,\tau) + \theta(x)[\psi_{ref}(x,\tau) + \psi_f(x,\tau)]. \tag{B12}$$

Here, ψ_{tr} is the transmitted wave function and is given by the propagation of the initial state $\psi(x)$ using PDX equation (B4) and (B6).

The remaining part $\psi_{ref}+\psi_f$ is the wave function obtained by propagating using the PDX formula for initial and final points both in $x>0$, Eq. (B7) [rewritten using Eq. (B6) if necessary]. It is appropriate to break this into the two pieces ψ_{ref} and ψ_f defined as follows: the reflected wave function ψ_{ref} consists of that obtained using the first term in Eq. (B7) together with the reflected part $-g_f(-x_1,\tau|x_0,0)$ of the restricted propagator, g_r. This definition ensures that $\psi_{ref}\to 0$ as the complex potential in $x<0$ goes to zero. The remaining part, ψ_f, is the other part of the restricted propagator and so is simply free propagation in $x>0$. This corresponds to the part of the incoming wave packet that never reaches $x=0$ during the time interval $[0,\tau]$. This part clearly goes to zero for large τ.

APPENDIX C: SOLUTION THROUGH STATIONARY SCATTERING STATES

In this appendix we use the PDX to derive the standard representations of the scattering solutions to the Schrödinger equation with the simple complex potential (1.3). These are known results but this derivation confirms the validity of the PDX method and allows a certain semiclassical path-integral approximation (used in Sec. III) to be tested. The results will also be useful for the decoherent-histories analysis in Sec. IV.

The transmitted and reflected wave functions are defined above in Eq. (B12). For large τ, a freely evolving packet moves entirely into $x<0$ so that the free part $\psi_f(x,\tau)$ is zero, leaving just the transmitted and reflected parts. Following the above definition, the transmitted wave function is given by

$$\begin{aligned}\psi_{tr}(x,\tau) = \frac{1}{m^2}\int_0^\tau ds\int_0^{\tau-s} dv\langle x|\exp(-iH_0s)\hat{p}|0\rangle e^{-V_0s}\\ \times\langle 0|\exp(-iHv)|0\rangle\langle 0|\hat{p}\\ \times\exp[-iH_0(\tau-v-s)]|\psi\rangle,\end{aligned} \tag{C1}$$

where $|0\rangle$ denotes the position eigenstate $|x\rangle$ at $x=0$. Also, we have introduced $s=\tau-t_1$ and $v=t_2-t_1$, and $H=H_0-iV_0\theta(-x)$ is the total Hamiltonian. The scattering wave functions concern the regime of large τ, so we let the upper limit of the integration ranges extend to ∞.

Writing the initial state as a sum of momentum states $|p\rangle$, and introducing $E=p^2/2m$, we have

$$\begin{aligned}\psi_{tr}(x,\tau) = \frac{1}{m^2}\int dp\int_0^\infty ds\langle x|\exp(-iH_0s)\hat{p}|0\rangle e^{i(E+iV_0)s}\\ \times\int_0^\infty dv\langle 0|\exp(-iHv)|0\rangle e^{iEv}p\langle 0|p\rangle e^{-iE\tau}\psi(p).\end{aligned} \tag{C2}$$

To evaluate the s integral, we use the formula [40]

$$\int_0^\infty ds\left(\frac{m}{2\pi is}\right)^{1/2}\exp\left(i\left[\lambda s+\frac{mx^2}{2s}\right]\right) = \left(\frac{m}{2\lambda}\right)^{1/2}\exp(i|x|\sqrt{2m\lambda}), \tag{C3}$$

from which it follows by differentiation with respect to x and setting $\lambda=E+iV_0$ that

$$\int_0^\infty ds\langle x|\exp(-iH_0s)\hat{p}|0\rangle e^{i(E+iV_0)s} = m\exp\{i|x|[2m(E+iV_0)]^{1/2}\}. \tag{C4}$$

The v integral may be evaluated using the explicit expression for the propagator along the edge of the potential [Eq. (B11)], together with the formula

$$\left(\frac{m}{2\pi i}\right)^{1/2}\int_0^\infty dv\frac{(1-e^{-V_0v})}{V_0v^{3/2}}e^{iEv} = \frac{\sqrt{2m}}{(E+iV_0)^{1/2}+E^{1/2}}. \tag{C5}$$

We thus obtain the result

$$\psi_{tr}(x,\tau) = \int\frac{dp}{\sqrt{2\pi}}\exp\{-ix[2m(E+iV_0)]^{1/2} - iE\tau\}\psi_{tr}(p), \tag{C6}$$

where

$$\psi_{tr}(p) = \frac{2}{[1 + E^{-1/2}(E + iV_0)^{1/2}]}\psi(p). \quad \text{(C7)}$$

Note that in this final result, it is possible to identify the specific effects of the different sections of propagation: the propagation along the edge of the potential corresponds to the coefficient in transmission amplitude (C7) (which is equal to 1 when $V_0=0$), and the propagation from final crossing to the final point produces the V_0 dependence of the exponent. These observations will be useful below.

The reflected wave function ψ_{ref} is defined above using PDX equation (B7) [rewritten using Eq. (B6)]. The first term in Eq. (B7), the crossing part, is the same as the transmitted case [Eq. (C2)] except that $V_0=0$ in the last segment of propagation, from $x=0$ to the final point, and also the sign of x is reversed. We must also add the effects of the reflection part of the restricted propagator, and this simply subtracts the reflection of the incoming wave packet. The reflected wave function is therefore given by

$$\psi_{ref}(x,\tau) = \int \frac{dp}{\sqrt{2\pi}} \exp(ixp - iE\tau)\psi_{ref}(p), \quad \text{(C8)}$$

where

$$\psi_{ref}(p) = \psi_{tr}(p) - \psi(p) = \frac{[1 - E^{-1/2}(E + iV_0)^{1/2}]}{[1 + E^{-1/2}(E + iV_0)^{1/2}]}\psi(p). \quad \text{(C9)}$$

We thus see that the PDX very readily gives the standard stationary wave functions [8], without having to use the usual (somewhat cumbersome) technique of matching eigenfunctions at $x=0$. In fact, this procedure is in some sense already encoded in the PDX.

We now use these exact results to check the validity of a useful approximation in the path-integral representation of the propagator. In the PDX [Eq. (B4)], the part awkward to calculate (especially for more general potentials) is the propagation from $x=0$ to the final point $x_1<0$. The exact propagator for this section consists of propagation along the edge of the potential followed by restricted propagation from $x=0$ to x_1, as used in Eq. (B6). However, for sufficiently small V_0, one might expect that in the path-integral representation of the propagator, the dominant contribution will come from paths in the neighborhood of the straight-line path from $x=0$ to $x<0$. These paths lie almost entirely in $x<0$, and one might expect that the propagator is therefore given approximately by

$$\langle x|\exp(-iHs)|0\rangle \approx \langle x|\exp(-iH_0 s)|0\rangle \exp(-V_0 s). \quad \text{(C10)}$$

It is not entirely clear that this is the case, however. On the one hand, the usual semiclassical approximation indicates that paths close to the straight-line paths dominate, but on the other hand, paths in $x<0$ are suppressed, so maybe the wiggly paths that spend less time in $x<0$ make a significant contribution. Since this approximation is potentially a useful one, it is useful to compare with the exact result for the transmitted wave packet calculated above.

We therefore evaluate the following approximate expression for the transmitted wave function:

$$\psi_{tr}(x,\tau) = -\frac{1}{m}\int_0^\tau ds \langle x|e^{-iH_0 s}|0\rangle e^{-V_0 s}\langle 0|\hat{p}e^{-iH_0(\tau-s)}|\psi\rangle. \quad \text{(C11)}$$

This is the PDX [Eq. (B4)] in operator form with the semiclassical approximation described above and we have set $s=\tau-t_1$. We now take $\tau\to\infty$ in the integration and evaluate. The key integral is

$$\int_0^\infty ds\langle x|e^{-iH_0 s}|0\rangle e^{i(E+iV_0)s} = \left(\frac{m}{2(E+iV_0)}\right)^{1/2} \exp\{-ix[2m(E+iV_0)]^{1/2}\}, \quad \text{(C12)}$$

where we have used Eq. (C3) (and recall that $x<0$). The resulting expression for the transmitted wave function is of form (C6), with

$$\psi_{tr}(p) = \frac{1}{E^{-1/2}(E+iV_0)^{1/2}}\psi(p). \quad \text{(C13)}$$

This agrees with the exact expression for transmission coefficient (C7) only when $V_0=0$, with the difference on the order of V_0/E for small V_0. This establishes that the approximation is valid for V_0 much less than the energy scale of the initial state.

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