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Local controllability of quantum networks

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We give a sufficient criterion that guarantees that a many-body quantum system can be controlled by properly manipulating the (local) Hamiltonian of one of its subsystems. The method can be applied to a wide range of systems: it does not depend on the details of the couplings but only on their associated topology. As a special case, we prove that Heisenberg and Affleck-Kennedy-Lieb-Tasaki chains can be controlled by operating on one of the spins at their ends. In principle, arbitrary quantum algorithms can be performed on such chains by acting on a single qubit.

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I. INTRODUCTION

The main obstacle in developing an efficient quantum information technology is posed by the difficulties one faces in achieving coherent control of quantum mechanical systems, i.e., in externally manipulating them while preserving their quantum coherence. There are three aspects that make control hard: first, quantum systems are often rather small, so local addressing is difficult. Second, control turns the quantum computer into an open system, which introduces noise. Third, due to shielding and off-resonance problems, in general there are "invisible" components of an extended controlled system (say the qubits of a quantum computer) that one cannot directly address. It is well known that quantum control can be simplified by properly exploiting the free Hamiltonian evolution of the controlled system [1–5]. Using this idea the problem of achieving "complete control everywhere" on an extended quantum system can be reduced to "*some* control everywhere." In this approach *each* component of the extended system is individually [1,3,5,6] or jointly [7–9] addressed by the controlling setup, but the latter is assumed to perform only a *limited* set of allowed transformations. While this partially solves the problem of local addressing (at least from a theoretical perspective) and reduces some "harder" quantum transformations to easier ones (for example, two-qubit gates to one-qubit gates), the problem of coupling the quantum system to the external world and the problem of invisible qubits remain. One way to cope with these issues is to replace the "some control everywhere" approach with a "complete control *somewhere*" approach, where "somewhere" is ideally a small portion C of a larger system $V=C\cup\bar{C}$ that we want to control ($\bar{C}$ being the part of V on which we do not have direct access). In this scenario two alternative control techniques have been proposed so far: an *algebraic control* (AC) method [2,10–12] and a *control by relaxation* (RC) method [13,14] (see Fig. 1). In the former case one assumes unlimited direct access on C by means of time-dependent local Hamiltonians which are properly modulated—see below for details. In the latter case instead one assumes to operate on C by means of a limited set of quantum gates that couple it with some external, completely controlled, quantum memory M. Here the control is realized by transferring the states of V into M where they are manipulated (e.g., transformed or measured) and then transferred back to V.

In this Rapid Communication we show that an easy-to-check graph infection criterion that has been developed for control by relaxation [13] can also be used for algebraic control. This is a major improvement with respect to previous works on the subject [2,10–12] since it allows us to check AC controllability of *large* many-body systems. As a special case we prove that Heisenberg spin chains of *arbitrary length* admit algebraic control when acted upon at one end spin only. This gives a nontrivial example of such mediated control and has important consequences for quantum computation: in principle, arbitrary quantum algorithms can be performed on such chains by acting on a single qubit. The controllability even holds when magnetic fields spoil the conservation of excitations (in this case the criterion is no longer applicable to RC). Since the criterion developed here is of topological nature, it does not depend on the details of the couplings and can therefore be applied to a wide range of experimental realizations of many-body quantum informa-

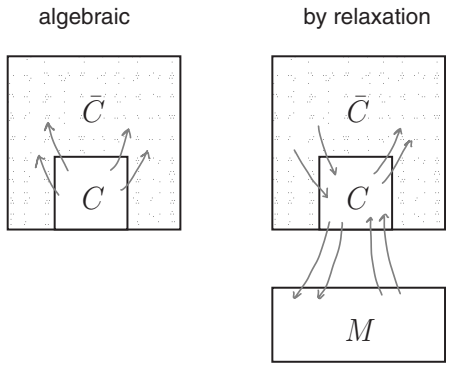

FIG. 1. Schematic comparison of algebraic control and control by relaxation. Left panel: controlling a subsystem C of a larger system $V=C\cup\bar{C}$ is sufficient to control the whole system (algebraic control). Right panel: the control is performed on a controlled memory. States on $\bar{C}$ can be transferred from/to M *through* C.

tion processing, ranging from optical lattices [15], to arrays of coupled cavities [16,17], to solid-state qubits [1,18]. Finally, the issue of how local time-dependent terms in the Hamiltonian can influence the global dynamics is also interesting from a purely theoretical perspective.

II. ALGEBRAIC CONTROL

We start by reviewing the basic properties of AC. In algebraic control [2–5] the composite system $V=C\cup\bar{C}$ is described by a global Hamiltonian of the form $H_V+\Sigma_k f_k(t)h_C^{(k)}\otimes\mathbb{1}_{\bar{C}}$. Here H_V is some fixed coupling Hamiltonian on V while $h_C^{(k)}$ are a set of local controlling Hamiltonians operating on C that can be activated through the (time-dependent) modulating parameters $f_k(t)$. At the mathematical level, a general necessary and sufficient criterion for this scheme has been derived [2–5]. It states that V is AC controllable by properly tuning the functions $f_k(t)$ if and only if (iff) iH_V and $ih_C^{(k)}$ are *generators* of the Lie algebra $\mathcal{L}(V)$ of the composite system V (the set of all skew-Hermitian operators of V), i.e.,

$$\langle iH_V,\mathcal{L}(C)\rangle=\mathcal{L}(V), \tag{1}$$

where, for the sake of simplicity, we have assumed the $ih_C^{(k)}$'s to be generators of the local Lie algebra $\mathcal{L}(C)$ of C and where we use the symbol $\langle\mathcal{A},\mathcal{B}\rangle$ to represent the algebraic closure of the operator sets $\mathcal{A}$ and $\mathcal{B}$. In simpler terms this implies that any possible quantum transformation on V can be operated by acting on C iff all elements of $\mathcal{L}(V)$ can be obtained as a linear combinations of iH_V, $\mathcal{L}(C)$, and iterated commutators of these operators.

Although the general arguments in [2] suggest that *most* quantum systems satisfy criterion (1), up to now only few examples have been presented [10–12]. Indeed condition (1) can be tested numerically only for relatively small systems (say maximally ten qubits). It becomes impractical instead when applied to large many-body systems where V is a collection of quantum sites (e.g., spins) whose Hamiltonian is described as a summation of two-site terms. The main result of this Rapid Communication is the derivation of an *inductive* easy-to-check method to test the AC controllability condition (1) for such configurations.

III. GRAPH CRITERION

The proposed method exploits the topological properties of the graph defined by the coupling terms entering the many-body Hamiltonian H_V. This allows us to translate the AC controllability problem into a simple graph infection property which can be easily tested. We start reviewing the latter for the most general setup, which will show more clearly where the topological properties come from.

The graph infection property was introduced in [13] and analyzed from a purely graph theoretical perspective in [19]. In words, the infection process can be described as follows: an initial set of nodes of the graph is "infected." The infection then spreads by the following rule: an infected node infects a "healthy" neighbor if and only if it is its *only* healthy neighbor. If eventually all nodes are infected, the initial set is called *infecting*. More formally, we consider an undirected graph $G=(V,E)$ characterized by a set of nodes V and by a set of edges E, and a subset $C\subseteq V$. We call C *infecting* G if there exists an ordered sequence $\{P_k;k=1,2,\ldots,K\}$ of K subsets of V,

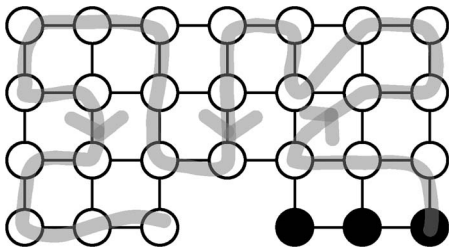

FIG. 2. Example for a graph that fulfills the infection criterion. The set C of "infecting nodes" is given by the black nodes and the gray line shows the order in which new nodes can be added to the set P_k such that Eqs. (3) and (4) hold.

$$C=P_1\subseteq P_2\subseteq\cdots\subseteq P_k\subseteq\cdots\subseteq P_K=V, \tag{2}$$

such that each set is exactly one node larger than the previous one,

$$P_{k+1}\setminus P_k=\{m_k\}, \tag{3}$$

and there exists an $n_k\in P_k$ such that m_k is its unique neighbor *outside* P_k:

$$N_G(n_k)\cap V\setminus P_k=\{m_k\}, \tag{4}$$

with $N_G(n_k)\equiv\{n\in V|(n,n_k)\in E\}$ being the set of nodes of V which are connected to n_k through an element of E. The sequence P_k provides a natural structure (Fig. 2) on the graph which allows us to treat it almost as a chain (although the graphs can be very much different from chains, see also the examples given in [13]). In particular, it gives us an index k over which we will be able to perform inductive proofs.

The link to quantum mechanics is that each node n of the graph has a quantum degree of freedom associated with the Hilbert space $\mathcal{H}_n$, which describes the nth site of the many-body system V we wish to control. The coupling Hamiltonian determines the edges through

$$H_V=\sum_{(n,m)\in E}H_{nm}, \tag{5}$$

where $H_{nm}=H_{mn}$ are some arbitrary Hermitian operators acting on $\mathcal{H}_n\otimes\mathcal{H}_m$. Within this context we call Hamiltonian (5) *algebraically propagating* if for all $n\in V$ and $(n,m)\in E$ one has

$$\langle[iH_{nm},\mathcal{L}(n)],\mathcal{L}(n)\rangle=\mathcal{L}(n,m), \tag{6}$$

where for a generic set of nodes $P\subseteq V$, $\mathcal{L}(P)$ is the Lie algebra associated with the Hilbert space $\otimes_{n\in P}\mathcal{H}_n$ [20]. The graph criterion can then be expressed as follows:

Theorem. Assume that Hamiltonian (5) of the composed system V is algebraically propagating and that $C\subseteq V$ infects V. Then V is algebraically controllable acting on its subset C.

Proof. To prove the theorem we have to show that Eq. (1) holds, or equivalently that $\mathcal{L}(V)\subseteq\langle iH_V,\mathcal{L}(C)\rangle$ (the opposite inclusion being always verified). To do so we

proceed by induction over $k=1,\ldots,K$, showing that $\mathcal{L}(P_k)\subseteq\langle iH_V,\mathcal{L}(C)\rangle$.

Basis: by Eq. (2) we have $\mathcal{L}(P_1)=\mathcal{L}(C)\subseteq\langle iH_V,\mathcal{L}(C)\rangle$. Inductive step: assume that for some $k<K$

$$\mathcal{L}(P_k)\subseteq\langle iH_V,\mathcal{L}(C)\rangle. \tag{7}$$

We now consider n_k from Eq. (4). We have $\mathcal{L}(n_k)\subset\mathcal{L}(P_k)\subseteq\langle iH_V,\mathcal{L}(C)\rangle$ and

$$[iH_{n_k,m_k},\mathcal{L}(n_k)]=[iH_V,\mathcal{L}(n_k)]-\sum_m[iH_{n_k,m},\mathcal{L}(n_k)],$$

where the sum on the right-hand side contains only nodes from P_k by Eq. (4). It is therefore an element of $\mathcal{L}(P_k)$. The first term on the right-hand side is a commutator of an element of $\mathcal{L}(P_k)$ and iH_V and thus an element of $\langle iH_V,\mathcal{L}(C)\rangle$ by Eq. (7). Therefore $[iH_{n_k,m_k},\mathcal{L}(n_k)]\subseteq\langle iH_V,\mathcal{L}(C)\rangle$ and by algebraic propagation Eq. (6) we have

$$\langle[iH_{n_k,m_k},\mathcal{L}(n_k)],\mathcal{L}(n_k)\rangle=\mathcal{L}(n_k,m_k)\subseteq\langle iH_V,\mathcal{L}(C)\rangle.$$

But $\langle\mathcal{L}(P_k),\mathcal{L}(n_k,m_k)\rangle=\mathcal{L}(P_{k+1})$ by Eq. (3) so $\mathcal{L}(P_{k+1})\subseteq\langle iH_V,\mathcal{L}(C)\rangle$. Thus by induction

$$\mathcal{L}(P_K)=\mathcal{L}(V)\subseteq\langle iH_V,\mathcal{L}(C)\rangle\subseteq\mathcal{L}(V). \tag{8}$$

■

The above theorem has split the question of algebraic control into two separate aspects. The first part, the algebraic propagation Eq. (6), is a property of the coupling that lives on a small Hilbert space $\mathcal{H}_n\otimes\mathcal{H}_m$ and can therefore be checked easily numerically—we have for instance verified this property for Heisenberg-type (see below), Affleck-Kennedy-Lieb-Tasaki (AKLT) [21], and for SU(3) Hamiltonians [22]. The second part is a topological property of the (classical) graph. An important question arises here if this may not only be a sufficient but also a necessary criterion. As we will see below, there are systems where C does not infect V but the system is controllable *for specific coupling strengths*. However the topological stability with respect to the choice of coupling strengths is no longer given.

IV. APPLICATION TO SPIN NETWORKS

An important example of the above theorem is systems of coupled spin-1/2 systems (qubits). We consider the two-body Hamiltonian given by the following Heisenberg-type coupling:

$$H_{nm}=c_{nm}(X_nX_m+Y_nY_m+\Delta Z_nZ_m), \tag{9}$$

where the c_{nm} are arbitrary coupling constants, Δ is an anisotropy parameter, and X, Y, and Z are the standard Pauli matrices. The edges of the graph are those (n,m) for which $c_{nm}\neq 0$. The relaxation controllability of this model was extensively analyzed in Refs. [13,14] while, in the restricted case of the single excitation subspace, its algebraic controllability was exactly solved in Refs. [12,23].

To apply our method we have first shown that the Heisenberg interaction is algebraically propagating. In this case the Lie algebra $\mathcal{L}(n)$ is associated to the group su(2) and it is generated by the operators $\{iX_n,iY_n,iZ_n\}$. Similarly the algebra $\mathcal{L}(n,m)$ is associated with su(4) and it is generated by the operators $\{iX_nI_m,iX_nX_m,iX_nY_m,\ldots,iZ_nZ_m\}$. The identity (6) can thus be verified by observing that

$$[X_n,H_{nm}]=Z_nY_m-Y_nZ_m,$$

$$[Z_n,Z_nY_m-Y_nZ_m]=X_nZ_m,$$

$$[Y_n,X_nZ_m]=Z_nZ_m,$$

$$[X_n,Z_nZ_m]=Y_nZ_m,$$

where for the sake of simplicity irrelevant constants have been removed. Similarly using the cyclicity $X\to Y\to Z\to X$ of the Pauli matrices we get

$$X_nZ_m\to Y_nX_m\to Z_nY_m,$$

$$Z_nZ_m\to X_nX_m\to Y_nY_m,$$

$$Y_nZ_m\to Z_nX_m\to X_nY_m.$$

Finally, using

$$[Z_nZ_m,Z_nY_m]=X_m,$$

and cyclicity, we obtain all 15 basis elements of $\mathcal{L}(n,m)$ concluding the proof. According to our theorem we can thus conclude that *any* network of spins coupled through Heisenberg-type interaction is AC controllable when operating on the subset C, if the associated graph can be infected. In particular, this shows that Heisenberg-type chains with arbitrary coupling strengths admit AC controllability when operated at one end (or, borrowing from [2], that the extreme of such chains are universal quantum interfaces for the whole system). We remark that in this case, knowledge about the coupling parameters of the Hamiltonian can be obtained by controlling one end qubit only [24]. The case $\Delta=0$ on the other hand is an interesting example where relaxation control is possible but our theorem cannot be applied. Using the numerical method from [5] we found that already a chain of length $N=2$ cannot be controlled by acting with arbitrary Pauli operators on one end—see also Ref. [10]. For the case $\Delta\neq 0$, a star with $N=4$ provides a good example that property (6) without graph infection does not suffice to provide controllability (Fig. 3). Another interesting example is an Ising chain with a magnetic field in a generic direction, which is controllable for $N=2,3$ but, perhaps surprisingly, not for longer chains. Finally, we have confirmed that SU(3) and AKLT Hamiltonians are algebraically propagating. These interactions have the form

$$H_{nm}=c_{nm}(A(S_nS_m)^2+BS_nS_m), \tag{10}$$

where S_n is the spin operator of particle n. The analytical method sketched above for the Heisenberg chain turns out to be quite cumbersome, so we used the numerical methods given in [5] to check that Eq. (6) holds. Since Eq. (6) lives in a small Hilbert space, this computation is efficient and fast. It then follows by our theorem that these systems are controllable for arbitrary length.

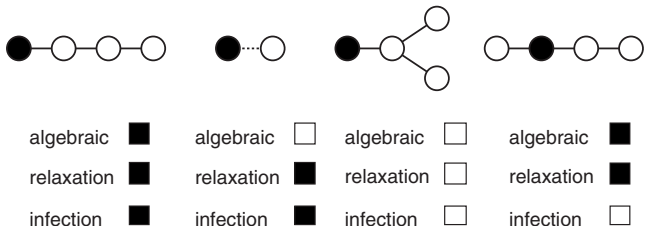

FIG. 3. Examples of small spin networks and their controllability. The black circles are the controlled parts C and the white ones the uncontrolled part $\bar{C}$. The dashed line stands for a coupling of the $X_nX_m+Y_nY_m$ type (which is not algebraically propagating) and the solid line stands for the full (possibly anisotropic) Heisenberg coupling (9) with $\Delta \neq 0$. The filled/empty boxes below each graph indicate whether or not it can be controlled by algebraic methods, by relaxation, and whether the graph infection property holds. For the last two examples the graph is not infected and controllability depends explicitly on the coupling strengths (all of which are assumed to be equal).

V. CONCLUSIONS

In this Rapid Communication we have presented a criterion to determine if a many-body quantum system allows for algebraic control by operating on a proper subset of it. In contrast to previous proposals the method does not require the knowledge of the spectrum of the system Hamiltonian. Instead it exploits some topological properties of the graph associated with its coupling terms. As a special case, we have proven that Heisenberg and AKLT chains can be controlled by operating on one of the spins at their ends. In principle, arbitrary quantum algorithms can be performed on such chains by acting on a single qubit. This motivates the search for further explicit specific and efficient control schemes on spin chains.

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