Dynamical constants of structured photons with parabolic-cylindrical symmetry

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Electromagnetic modes with parabolic-cylindrical symmetry and their dynamical variables are studied both in the classical and quantum realms. As a result, a dynamical constant for the electromagnetic field is identified and linked to the symmetry operator which supports it.

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I. INTRODUCTION

The dynamical variables of the electromagnetic (EM) field define its mechanical identity and are essential for understanding the effects of the field on charged particles. As a consequence, finding the natural dynamical variables of the EM field and their relations to mode properties has been directly linked to the development of classical and quantum EM theories. Historically, the photon concept emerged from suggesting a definite relationship between the energy (linear momentum) of a photon and the frequency ω (wave vector $\vec{k}$) of plane waves. Similarly, the relationship between the angular momentum of EM waves and their polarization is very important for understanding, e.g., atomic processes mediated by photons.

Photons associated to EM modes with non-Cartesian symmetries are characterized by sets of dynamical constants different from those of plane waves. An example corresponds to circular-cylindrical EM waves known as Bessel modes [1] or their paraxial analog, i.e., Laguerre-Gaussian beams [2]. Bessel photons carry a well-defined orbital angular momentum [2,3] proportional to the winding number m of their vortices [4,5]. Another example corresponds to Mathieu modes which exhibit elliptical-cylindrical symmetry. Mathieu photons carry constant values for the balanced composition of the orbital angular momentum with respect to the foci of the elliptical coordinate system [6].

The purpose of this Brief Report is to analyze the mechanical properties of the EM modes for the last coordinate system with translational symmetry along an axis known to have separable analytical solutions, i.e., the parabolic-cylindrical coordinate system. The solutions of the corresponding wave equation can be expressed in terms of Weber functions, giving this name to the EM modes. We show that a balanced composition of a component of the lineal momentum with a component of the angular momentum is a natural dynamical variable for these modes. Weber photons carry a well-defined value of this variable.

Weber beams of zero order have already been experimentally generated by means of a thin annular slit modulated by the proper angular spectra [7]. This setup was conceived as a variation of that originally used by Durnin *et al.* [1] for generating Bessel beams. Higher-order Weber beams can also be produced by holograms encoded on plates [7] or in spatial light modulators [8].

A. Parabolic-cylindrical coordinates

The parabolic-cylindrical coordinate system (u,v,z) is defined by the transformations [9]

$$x + iy = \frac{1}{2}(u + iv)^2, \quad z = z, \tag{1}$$

where x, y, and z are the well-known Cartesian coordinates and $u \in (-\infty,\infty)$ and $v \in [0,\infty)$. Surfaces of constant u form half-confocal parabolic cylinders that open toward the negative x axis, while the surfaces of constant v form confocal parabolic cylinders that open in the opposite direction. The foci of all these parabolic cylinders are located at $x=0$ and $y=0$ for each z value. The scaling factors associated to u and v are $h_u = h_v = h = \sqrt{u^2+v^2}$. In the following, the notation $\hat{e}_x$ represents the unitary vector related to a given coordinate x, the shorthand notation ∂_x is used for partial derivatives with respect to the variable x, and $\partial_0 =: \partial_{ct} = \frac{1}{c}\partial_t$, with c the velocity of light in vacuum.

II. PARABOLIC SCALAR FIELD AND ITS DYNAMICAL VARIABLES

The scalar wave equation has separable solutions invariant under axial propagation,

$$\nabla^2\Psi = \partial_{ct}^2\Psi, \quad \Psi(\vec{r},t) = \psi(\vec{r}_\perp)e^{i(k_z z - \omega t)}/\sqrt{2\pi}, \tag{2}$$

in four coordinate systems: Cartesian, circular-, elliptic-, and parabolic-cylindrical coordinates. For parabolic-cylindrical symmetry Helmholtz equation reads

$$[h^{-2}(\partial_u^2 + \partial_v^2) + k_\perp^2]\psi(u,v) = 0, \quad k_\perp^2 = k^2 - k_z^2, \tag{3}$$

where the real constants $k=\omega/c$ are the magnitude of the wave vector for a given frequency ω, k_z its axial component, and $k_\perp$ its perpendicular component. If $\psi(u,v) = U(u)V(v)$,

$$(\partial_u^2 + k_\perp^2 u^2 - 2k_\perp a)U(u) = 0, \tag{4}$$

$$(\partial_v^2 + k_\perp^2 v^2 + 2k_\perp a)V(v) = 0, \tag{5}$$

with $2k_\perp a$ the separation constant. These equations are known as parabolic cylinder or Weber differential equations. Solutions for this differential set can be expressed as Frobenius series, parabolic cylinder functions, Whittaker functions, Hermite functions, and others [10–12]. Here, the solutions are expressed in terms of confluent hypergeometric functions of the first kind, ${}_1F_1$,

$$U_{p,k_\perp,a}(u) = s_p \zeta_u^{n_p - 1/4} e^{-i(\zeta_u/2)}\,{}_1F_1\left(\frac{n_p}{4} - i\frac{a}{2}, \frac{n_p}{2}; i\zeta_u\right), \tag{6}$$

$$V_{p,k_\perp,a}(v) = s_p \zeta_v^{n_p - 1/4} e^{-i(\zeta_v/2)} {}_1F_1\left(\frac{n_p}{4} + i\frac{a}{2}, \frac{n_p}{2}; i\zeta_v\right), \quad (7)$$

with $\zeta_u = k_\perp u^2$, $\zeta_v = k_\perp v^2$, and $n_e = 1$ $(n_o = 3)$ for even (odd) parity functions: $U_e(-u) = U_e(u)$ $[U_o(-u) = -U_o(u)]$. The normalization factors are taken as

$$s_e = \frac{\sqrt{\pi \sec(ia\pi)}}{|\Gamma(3/4 - ia/2)|}, \quad s_o = \frac{\sqrt{2\pi \sec(ia\pi)}}{|\Gamma(1/4 - ia/2)|}.$$

These expressions reduce directly to the Frobenius series, presented in Ref. [10] and used in Ref. [13], to introduce parabolic optical wave fields in the paraxial regime. In order to guarantee that these scalar fields vanish when the absolute values of the coordinate variables tend to infinity, a must be real [11]. The set $\{\Psi_{p,\kappa}, \kappa = (k_z, \omega, a)\}$ is complete and orthogonal. Each function $\Psi_{p,\kappa}$ satisfies the eigenvalue equations,

$$\mathfrak{P}_u \Psi_{p,\kappa}(u,v,z,t) \doteq \Psi_{p,\kappa}(-u,v,z,t) = (-1)^p \Psi_{p,\kappa}(u,v,z,t), \quad (8)$$

$$-i\partial_z \Psi_{p,\kappa}(u,v,z,t) = k_z \Psi_{p,\kappa}(u,v,z,t), \quad (9)$$

$$i\partial_t \Psi_{p,\kappa}(u,v,z,t) = \omega \Psi_{p,\kappa}(u,v,z,t), \quad (10)$$

$$2\mathbb{A}\Psi_{p,\kappa}(u,v,z,t) \doteq \left\{\frac{v^2}{h^2}\partial_u^2 - \frac{u^2}{h^2}\partial_v^2\right\}\Psi_{p,\kappa}(u,v,z,t) = 2k_\perp a \Psi_{p,\kappa}(u,v,z,t). \quad (11)$$

The operator $\mathbb{A}$ is directly identified as a generator of the balanced composition of a rotation around the z axis and translations along the y axis since

$$\mathbb{A} = (1/2)\partial_x + y\partial_{xy}^2 - x\partial_y^2 = (l_z p_y + p_y l_z)/2, \quad (12)$$

where $l_z = -i(\vec{r} \times \vec{\nabla})_z$ and $p_y = -i\partial_y$. For scalar fields and space-time continuous symmetries, the generators of infinitesimal transformations turn out to be good realizations of the corresponding dynamical operator. In that sense, $\mathbb{A}$ can be related to the product of the z component of the angular momentum and the y component of the linear momentum. Then, the eigenvalue equation Eq. (11) means that the scalar field $\Psi_{p,k_z,\omega,a}$ carries a well-defined value of that momenta product.

Traveling scalar Weber modes are defined by

$$\tilde{\Psi}_{\pm}\kappa(\vec{r},t) = [\psi_{e,\kappa}(u,v) \pm i\psi_{o,\kappa}(u,v)][e^{i(k_z z - \omega t)}]/\sqrt{2\pi}, \quad (13)$$

and these modes are orthonormal [12]

$$\int_{\mathbb{R}^3} \tilde{\Psi}^*_{\pm,\kappa'}(\vec{r})\tilde{\Psi}_{\pm,\kappa'}(\vec{r}) = \frac{2\pi}{k_\perp}\delta(k_z - k_z')\delta(k_\perp - k_\perp')\delta(a - a').$$

III. PARABOLIC-CYLINDRICAL EM MODES

In Coulomb gauge, any given solution for the vector electromagnetic field, $\vec{A}$, can be written as a superposition of modes related to a complete set of solutions for the scalar wave equation, Ψ, which are identified as Hertz potentials [14]. For EM fields in parabolic-cylindrical coordinates having a well-defined behavior under $\mathfrak{P}_u$,

$$\vec{A}_{p,\kappa} = \mathcal{A}_{p,\kappa}^{(\mathrm{TE})}\vec{\mathbb{M}}\Psi_{p,\kappa} + \mathcal{A}_{p,\kappa}^{(\mathrm{TM})}\vec{\mathbb{N}}\Psi_{p,\kappa}, \quad (14)$$

with the vector operators given by the expressions

$$\vec{\mathbb{M}} = \frac{\partial_{ct}}{h}(\hat{e}_u\partial_v - \hat{e}_v\partial_u), \quad \vec{\mathbb{N}} = \frac{\partial_z}{h}(\hat{e}_u\partial_u + \hat{e}_v\partial_v) - \hat{e}_z\nabla_\perp^2. \quad (15)$$

The constants $\mathcal{A}_{p,\kappa}^{(\mathrm{TE})}$ and $\mathcal{A}_{p,\kappa}^{(\mathrm{TM})}$ are proportional to the amplitudes of the transverse electric (TE) and transverse magnetic (TM) EM fields as can be directly seen from their connection with the associated electric and magnetic fields, $\vec{E} = -\partial_{ct}\vec{A}$ and $\vec{B} = \vec{\nabla} \times \vec{A}$, yielding

$$\vec{E}_{p,\kappa} = -\mathcal{A}_{p,\kappa}^{(\mathrm{TE})}\partial_{ct}\vec{\mathbb{M}}\Psi_{p,\kappa} - \mathcal{A}_{p,\kappa}^{(\mathrm{TM})}\partial_{ct}\vec{\mathbb{N}}\Psi_{p,\kappa},$$

$$\vec{B}_{p,\kappa} = \mathcal{A}_{p,\kappa}^{(\mathrm{TE})}\partial_{ct}\vec{\mathbb{N}}\Psi_{p,\kappa} - \mathcal{A}_{p,\kappa}^{(\mathrm{TM})}\partial_{ct}\vec{\mathbb{M}}\Psi_{p,\kappa}. \quad (16)$$

Similar expressions can be written for the traveling EM modes associated to Eq. (13). The intensity and polarization structure of an EM Weber beam are illustrated in Fig. 1. The instantaneous electric field orientation has a nontrivial structure and, as a function of time, **E** preserves its direction while its magnitude oscillates at each point.

IV. DYNAMICAL CONSTANTS FOR PARABOLIC-CYLINDRICAL EM MODES

Given a symmetry generator Noether theorem is usually applied to the field Lagrangian density

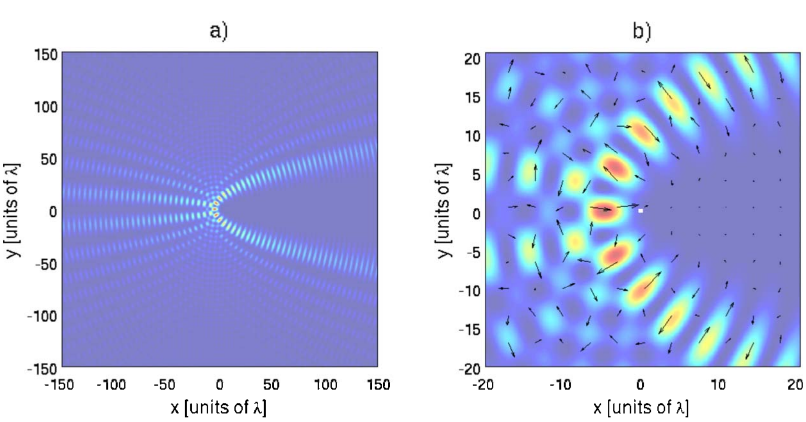

FIG. 1. (Color online) Sample of (a) transverse intensity and (b) polarization structure of an odd EM TE Weber field. They correspond to the eigenvalue $a = -2$, $k_z = 0.995\omega/c$, and the unit of length is taken as the wavelength.

$$\mathfrak{L} = \frac{1}{8\pi}\sum_{\mu,\nu=0}^{3}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \tag{17}$$

in order to find a dynamical constant for the electromagnetic field. This theorem states that [15] if under an infinitesimal transformation of the space coordinates $x_\mu \to x_\mu + \Sigma_\rho X^\rho_\mu \delta\omega_\rho$ and the field $A_\mu \to A_\mu + \Sigma_\rho \Phi^\rho_\mu \delta\omega_\rho$, the Lagrangian is left invariant, then the current,

$$\Theta^\nu_\rho = \Xi^\nu_\rho + \Lambda^\nu_\rho, \tag{18}$$

$$\Xi^\nu_\rho = -\sum_\lambda \frac{\partial\mathfrak{L}}{\partial(\partial_\nu A_\lambda)}\Phi_{\lambda\rho}, \tag{19}$$

$$\Lambda^\nu_\rho = \sum_{\lambda,\sigma}\frac{\partial\mathfrak{L}}{\partial(\partial_\nu A_\lambda)}X^\sigma_\rho A_{\lambda,\sigma} - \mathfrak{L}X^\nu_\rho, \tag{20}$$

has a null divergence. As a consequence, Θ^0_ρ defines the density of a dynamical variable whose integrated value over a volume can change only due to the flux of the current Θ^i_ρ through the surface that delimits the volume.

Since we are working with EM modes that have a particular symmetry the subset of all possible transformations whose generators are directly identified from the eigenvalue Eqs. (8)–(11) is particularly relevant. Under an infinitesimal translation along the main direction of propagation δz or a time translation δt, the EM field changes according to the expressions $A_\mu \to A_\mu + \partial_z A_\mu \delta z$ or $A_\mu \to A_\mu + \partial_t A_\mu \delta t$, respectively. This is reflected in the fact that the field-momentum-like variable

$$P_z^{(i,p,\kappa,p',\kappa')} = \frac{1}{4\pi c}\int d^3x(\vec{E}^{(i)}_{p,\kappa}\times\vec{B}^{(i)}_{p',\kappa'})_z, \quad i=\text{TE},\text{TM} \tag{21}$$

is independent of time if the integration is taken over the whole space. Similarly, the energylike integral

$$\mathcal{E}^{(i,p,\kappa,p',\kappa')} = \frac{1}{4\pi}\int d^3x[\vec{E}^{(i)}_{p,\kappa}\cdot\vec{E}^{(i)}_{p',\kappa'} + \vec{B}^{(i)}_{p,\kappa}\cdot\vec{B}^{(i)}_{p',\kappa'}] \tag{22}$$

is also constant. In fact, $P_z^{(i,p,\kappa,p',\kappa')}$ and $\mathcal{E}^{(i,p,\kappa,p',\kappa')}$ are proportional to each other with k_z/ω the constant of proportionality. Notice that, in both cases, the dynamical constant can be inferred from the factor Ξ^ν_ρ defined in Eq. (19) up to a term proportional to the divergence of a vector field, for instance,

$$\Xi^0_j = \frac{1}{4\pi}\sum_i E_i\partial_j A_i = \frac{1}{4\pi}(\vec{E}\times\vec{B})_j - \sum_i \partial_i(E_iA_j), \tag{23}$$

where Ξ^ν_ρ contains just the transformation of the field A_μ.

Under an infinitesimal rotation around the z axis with an angle $\delta\omega$, the electromagnetic field $\vec{A}$ has a well-defined transformation rule, $A_i = A_i \to \epsilon_{ij3}A_j\delta\omega$, which is independent of the origin of space coordinates. If one considers the Noether term Ξ^0_ρ, Eq. (19), an expression for the helicity is found

$$S_z^{(i,i',p,\kappa,p',\kappa')} = (1/4\pi c)\int_{\mathcal{V}} d^3x(\vec{E}^{(i)}_{p,\kappa}\times\vec{A}^{i'}_{p',\kappa'})_z. \tag{24}$$

This is another dynamical constant for Weber EM modes as can be directly verified by substituting the general expression for the vectors $\vec{E}$ and $\vec{A}$ in terms of the Hertz parabolic modes. As expected, it turns out that for a given mode p, κ, the helicity S_z is different from zero only if the amplitudes $\mathcal{A}^{(\text{TM})}$ and $\mathcal{A}^{(\text{TE})}$ that define the polarization of a mode are complex. The Noether term Λ^0_ρ, Eq. (20), defines the density of the z component of the orbital angular momentum

$$\mathcal{L}_z^{(i,p,\kappa,p',\kappa')} = \frac{1}{8\pi c}\sum_j (\vec{E}^{(i)}_{p,\kappa})_j(u\partial_v - v\partial_u)(\vec{A}^{(i)}_{p',\kappa'})_j. \tag{25}$$

For Weber modes, similarly to periodic plane waves [16], $\mathcal{L}_z = \vec{\nabla}\cdot\vec{G} + \delta\mathcal{L}_z$ with the latter term becoming zero for $p=p'$, $\kappa=\kappa'$. Explicitly,

$$\vec{G} = \frac{kh^2}{8\pi k' c}\sum_j (h\vec{A}^{(i)}_{p,\kappa})_j\vec{\mathbb{M}}(h\vec{A}^{(i)}_{p',\kappa'})_j,$$

$$\delta\mathcal{L}_z^{(\text{TE})} = \frac{k^2 i}{8\pi k_z c}\sum_{j=u,v}(\partial_j h\vec{\mathbb{N}}\Psi_{p,\kappa})(\partial_j h\vec{\mathbb{M}}\Psi_{p',\kappa'}),$$

$$\delta\mathcal{L}_z^{(\text{TM})} = \frac{k_z k'_z}{kk'}\delta\mathcal{L}_z^{(\text{TE})} - \frac{k_\perp^2 k'^2_\perp h^4 i}{8\pi k_z c}(\vec{\mathbb{M}}\Psi_{p,\kappa})(\vec{\mathbb{N}}\Psi_{p',\kappa'}). \tag{26}$$

Thus, a not null orbital angular momentum for a given Weber mode (p,κ) in a volume $\mathcal{V}$ can be due just to a flux of the vector field $\vec{G}$ through the boundary surface.

Let us consider $\mathbb{A}$ as generator of a transformation for the vector field $\vec{A}$. Noether theorem as described above concerns first-order differential operators as generators of continuous symmetries while $\mathbb{A}$ contains second-order operators. Nevertheless, the Ξ^0_ρ term associated to this transformation gives rise to

$$\mathfrak{A}^{(i,p,\kappa,p',\kappa')} = \frac{1}{4\pi c}\int_{\mathcal{V}}\sum_j (\vec{E}^{(i)}_{p,\kappa})_j\mathbb{A}(A^{(i)}_{p',\kappa'})_j d^3x,$$

$$\sum_j (\vec{E}^{(i)}_{p,\kappa})_j\mathbb{A}(\vec{A}^{(i)}_{p',\kappa'})_j = k_\perp a\vec{E}_{p,\kappa}\cdot\vec{A}_{p',\kappa'} + \nabla\cdot\vec{C}^{(i)},$$

$$\vec{C}^{(\text{TE})} = \vec{C}, \vec{C}^{(\text{TM})} = \frac{k_z k'_z}{kk'}\vec{C}, \vec{C} = -(\partial_{ct}\Psi_{p,\kappa})\vec{\mathbb{M}}\Psi_{p',\kappa'}. \tag{27}$$

The first resulting term in Eq. (27) is proportional to the integrand that defines the energy, Eq. (22). Thus, for integrations over a finite volume $\mathcal{V}$, $\vec{C}^{(i)}$ defines the flux of $\mathfrak{A}$ through the surface around the integration volume. Equation (27) supports the identification of $\mathfrak{A}$ as the electromagnetic dynamical variable related to the generator $\mathbb{A}$.

As for the discrete symmetry, using the properties of the scalar function U under the reflection of u and the expression of the EM modes in terms of Hertz potentials, it is straightforward to find the reflection properties of the electric field $\vec{E}_{p,\kappa}$ for each mode

$$\mathfrak{P}_u\vec{E}^{(\text{TE})}_{p,\kappa} = (-1)^p(-E^{(\text{TE})}_{p,\kappa,x}, E^{(\text{TE})}_{p,\kappa,y}, E^{(\text{TE})}_{p,\kappa,z}),$$

$$\mathfrak{P}_u\vec{E}^{(\text{TM})}_{p,\kappa} = (-1)^p(E^{(\text{TM})}_{p,\kappa,x}, -E^{(\text{TM})}_{p,\kappa,y}, E^{(\text{TM})}_{p,\kappa,z}). \tag{28}$$

V. QUANTIZATION OF THE EM FIELD IN TERMS OF WEBER MODES

Standard quantization rules require a proper normalization of the EM modes so that each traveling photon carries an energy $\hbar\omega$. The classical electric field amplitude is substituted by the electric field per photon times the creation operator for the given traveling mode, $\mathcal{A}_\kappa^{(i)}=\varepsilon_\kappa \hat{a}_\kappa^{(i)}$, $|\varepsilon_\kappa|^2=\hbar/k^2k_\perp^2$. The quantum energy and the momentum along z operators take the form

$$\hat{\mathcal{E}}=\sum_{i,\kappa}\hbar\omega\hat{N}_\kappa^{(i)},\quad \hat{\mathcal{P}}_z=\sum_{i,\kappa}\hbar k_z\hat{N}_\kappa^{(i)}, \tag{29}$$

in terms of the number operator

$$\hat{N}_\kappa^{(i)}=\frac{1}{2}(\hat{a}_\kappa^{(i)\dagger}\hat{a}_\kappa^{(i)}+a_\kappa^{(i)}\hat{a}_\kappa^{(i)\dagger}),\quad [a_\kappa^{(i)},a_{\kappa'}^{(j)\dagger}]=\delta_{i,j}\delta_{\kappa,\kappa'}, \tag{30}$$

allowing the identification of $\hbar k_z$ and $\hbar\omega$ with the photon momentum along z and the photon energy, respectively. As for the helicity,

$$\hat{S}_z=\sum_\kappa\frac{i\hbar k_z c}{2\omega}(\hat{a}_\kappa^{(\mathrm{TE})\dagger}\hat{a}_\kappa^{(\mathrm{TM})}-\hat{a}_\kappa^{(\mathrm{TE})}\hat{a}_\kappa^{(\mathrm{TM})\dagger}). \tag{31}$$

A quantum analysis of the relation between polarization and helicity can be carried out in analogy with the study in Ref. [5] for Bessel fields. Finally, in the quantum realm the field operator associated to $\mathbb{A}$ is

$$\hat{\mathfrak{A}}=\sum_{i,\kappa}\hbar^2k_\perp a\hat{N}_\kappa^{(i)}. \tag{32}$$

An overall factor $\hbar$ was introduced so that the dynamical variable $\hat{\mathfrak{A}}$ for a photon has units of linear momentum times angular momentum as expected for the quantum variable associated to $p_y l_z$.

VI. DISCUSSION

The parabolic-cylindrical modes differ from other separable cylindrical modes by having $(l_zp_y+p_yl_z)/2$ as a symmetry operator. We showed that the quantum numbers of the EM modes $\{k_z,\omega,a\}$ are related to their linear momentum along z, the energy, and the symmetrized product of the angular momentum along z and the momentum along y. The helicity, a property intrinsic to the vector nature of the EM field, was shown to be diagonal in the circular basis resulting from the complex superposition of TE and TM modes. The dynamical variable $\mathfrak{A}$ is gauge dependent although it can be written in a gauge-independent-looking form for monochromatic modes for which $\vec{E}_{p,\kappa}=i\omega\vec{A}_{p,\kappa}$. Contrary to standard EM dynamical variables which depend on products of $\vec{E}$ and $\vec{B}$, $\mathfrak{A}$ depends on the products of $\vec{E}$, $\vec{B}$, and their derivatives. Since Weber EM modes form a complete set, $\mathfrak{A}=(1/4\pi c)\Sigma_j\int_V d^3xE_j\mathbb{A}A_j$ will be a conserved quantity for any EM wave $\vec{A}$ whenever the flux of $\mathfrak{A}$ through asymptotic parabolic cylinder surfaces at infinity is null. This flux can be evaluated writing the given EM field $\vec{A}$ as a superposition of Weber modes and applying Eq. (27).

The mechanical effects of Weber beams on cold atoms deserve a detailed study both quantum mechanically and semiclassically (in complete analogy to that already done for Mathieu beams [6]). However, there are some qualitative features that can be expected without performing such an analysis. For instance, since under stationary conditions cold noninteracting atoms in a red-detuned light beam have a higher probability of being located in the higher intensity regions of the beam, the corresponding squared atomic wave function mimics the intensity pattern of the light field. Thus, for a Weber lattice, the atomic wave function is expected to have a geometrical structure similar to that of the scalar Weber function, Eq. (2). This structure gives rise to the eigenvalue equations Eqs. (8)–(11). Thus, necessarily $\mathfrak{A}$ defines a natural dynamical variable for the mechanical description of the *atomic* cloud in a Weber lattice. A careful analysis concerning this idea is in progress.

[1] J. Durnin, J. Opt. Soc. Am. A **4**, 651 (1987); J. Durnin, J. J. Miceli, and J. H. Eberly, Phys. Rev. Lett. **58**, 1499 (1987).

[2] L. Allen, M. W. Beijersbergen, R. J. C. Spreeuw, and J. P. Woerdman, Phys. Rev. A **45**, 8185 (1992).

[3] C. G. Darwin, Proc. R. Soc. London, Ser. A **136**, 36 (1932); J. Humblet, Physica (Amsterdam) **10**, 585 (1943); L. de Broglie, *Mécanique Ondulatoire du Photon et Théorie Quantique des Champs* (Gauthier-Villars, Paris, 1949), p. 65.

[4] R. Horák, Z. Bouchal, and J. Bajer, Opt. Commun. **133**, 315 (1997).

[5] R. Jáuregui and S. Hacyan, Phys. Rev. A **71**, 033411 (2005).

[6] B. M. Rodríguez-Lara and R. Jáuregui, Phys. Rev. A **78**, 033813 (2008).

[7] C. López-Mariscal, M. A. Bandres, and J. C. Gutiérrez-Vega, Opt. Express **13**, 2364 (2005).

[8] J. A. Davis, M. J. Mintry, M. A. Bandres, and D. M. Cottrell, Opt. Express **16**, 12866 (2008).

[9] N. N. Lebedev, I. P. Skal'skaya, and Y. S. Uflyand, *Problems in Mathematical Physics* (Prentice-Hall, Englewood Cliffs, NJ, 1966).

[10] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover Publications, New York, 1972).

[11] N. N. Lebedev and R. A. Silverman, *Special Functions and their Applications* (Dover Publications, New York, 1965).

[12] W. Miller, Jr., SIAM J. Math. Anal. **5**, 626 (1974).

[13] M. A. Bandres, J. C. Gutiérrez-Vega, and S. Chávez-Cerda, Opt. Lett. **29**, 44 (2004).

[14] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941); A. Nisbet, Proc. R. Soc. London, Ser. A **231**, 250 (1955); **240**, 375 (1957).

[15] N. N. Bogolubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (John Wiley and Sons, New York, 1980).

[16] D. Lenstra and L. Mandel, Phys. Rev. A **26**, 3428 (1982).