Dynamics of SU(1,1) coherent states for the damped harmonic oscillator

Jeong Ryeol Choi[1,*] and Kyu Hwang Yeon[2]

[1]*School of Electrical Engineering and Computer Science, Kyungpook National University, 1370 Sankyuk-dong, Buk-gu, Daegu 702–701, Republic of Korea*

[2]*Department of Physics and BK21 Physics Program, College of Natural Science, Chungbuk National University, Cheongju, Chungbuk 361-763, Republic of Korea*

(Received 7 March 2009; published 29 May 2009; publisher error corrected 3 June 2009)

Gerry, Ma, and Vrscay [Phys. Rev. A **39**, 668 (1989)] studied the time evolution of SU(1,1) coherent states for the damped harmonic oscillator by introducing the Kanai-Caldirola Hamiltonian. The purposes of this Brief Report are to demonstrate that there are somewhat serious errors on their results and to correct them. Most of the figures given in their work are reproduced with correction in order to facilitate our explanation of results.

DOI: 10.1103/PhysRevA.79.054103 PACS number(s): 03.65.Ca, 03.65.Fd

The dynamics of SU(1,1) coherent states for the damped harmonic oscillator is investigated in Ref. [1] by Gerry *et al.* about two decades ago. (Hereafter, we call Ref. [1] as paper I for convenience.) When we study paper I, a somewhat nontrivial error is found concerning the representation of Hamiltonian describing the system. As a matter of course, this led to wrong consequences for their subsequent evaluations associated to SU(1,1) coherent states, such as coherent state parameter, variances, and quantum energy expectation value. Therefore, the relevant interpretations and figures of the paper include inherent misleadings.

Nevertheless, paper I is cited by quite a few researchers until now [2–10] without pointing out any errors of the paper. Moreover, one of these researchers recently made similar mistakes in his/her research paper [4] which employed the same method of paper I for a somewhat different system. For these reasons, it may worth revisiting the same subject of paper I. The corrected results will be presented in this Brief Report. Figures 1–3 of paper I will be reproduced on the basis of exact development.

The Caldirola-Kanai Hamiltonian which describes the damped harmonic oscillator is given by

$$\hat{H} = e^{-\lambda t}\frac{\hat{p}^2}{2m} + e^{\lambda t}\frac{1}{2}m\omega^2\hat{q}^2. \tag{1}$$

Although we follow the method of paper I in this investigation, the relevant evaluations will be made with correction of their mistakes. The conventional type of annihilation operator which is associated to simple harmonic oscillator is

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{q} + i\frac{\hat{p}}{\sqrt{2m\omega\hbar}}. \tag{2}$$

Of course, the Hermitian adjoint of the above equation is creation operator, $\hat{a}^\dagger$. Using these operators, the SU(1,1) generators $\hat{K}_0$, $\hat{K}_+$, and $\hat{K}_-$ are represented as

$$\hat{K}_0 = \tfrac{1}{2}(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}), \tag{3}$$

$$\hat{K}_+ = \tfrac{1}{2}(\hat{a}^\dagger)^2, \tag{4}$$

$$\hat{K}_- = \tfrac{1}{2}\hat{a}^2. \tag{5}$$

Then, the canonical variables can be written in terms of these generators to be

$$\hat{q}^2 = \frac{\hbar}{m\omega}(2\hat{K}_0 + \hat{K}_+ + \hat{K}_-), \tag{6}$$

$$\hat{p}^2 = m\omega\hbar(2\hat{K}_0 - \hat{K}_+ - \hat{K}_-). \tag{7}$$

After a straightforward calculation, the Hamiltonian given in Eq. (1) is expressed in terms of SU(1,1) generators,

$$\hat{H} = \hbar\omega(e^{\lambda t} + e^{-\lambda t})\hat{K}_0 + \tfrac{1}{2}\hbar\omega(e^{\lambda t} - e^{-\lambda t})(\hat{K}_+ + \hat{K}_-). \tag{8}$$

This is somewhat different from Eq. (3.7) of paper I, even when one neglect $\beta(t)$ in their equation. In fact, their overall mistakes stem from the miscalculation of the Hamiltonian.

Among well-defined classes of SU(1,1) coherent states, paper I is concerned with Perelomov coherent state. If we let the eigenvalue of $\hat{a}$ as α, the Perelomov coherent state is constructed from [11]

$$|\xi;k\rangle = \hat{D}(\alpha)|0\rangle_k, \tag{9}$$

where $\hat{D}(\alpha)$ is a displacement operator of the form

$$\hat{D}(\alpha) = \exp[\tfrac{1}{2}(\alpha^2\hat{K}_+ - \alpha^{*2}\hat{K}_-)] = e^{\xi\hat{K}_+}\exp[-\Gamma\hat{K}_0]e^{-\xi^*\hat{K}_-}, \tag{10}$$

with

$$\xi = \frac{\alpha^2}{|\alpha|^2}\tanh(|\alpha|^2/2), \tag{11}$$

$$\Gamma = 2\ \ln[\cosh(|\alpha|^2/2)]. \tag{12}$$

The classical equation of motion for the parameter ξ is determined from [12,13]

*choiardor@hanmail.net

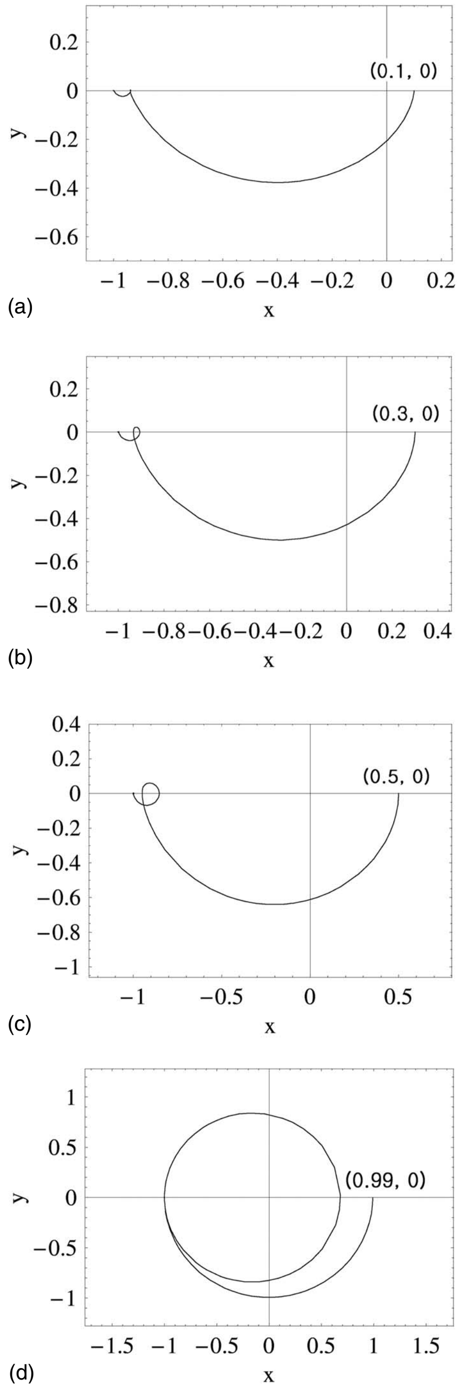

FIG. 1. Evolution of the SU(1,1) coherent state parameter $\xi(t) = x(t) + iy(t)$ given in Eqs. (16) and (17). The initial conditions $(x(0), y(0))$ are (a) (0.1,0), (b) (0.3,0), (c) (0.5,0), and (d) (0.99,0). We used $\omega = 1$ and $\lambda = 1$.

$$\dot{\xi} = \frac{1}{i\hbar}[\xi, H_k], \tag{13}$$

where $[X_1, X_2]$ is a generalized Poisson bracket defined as

$$[X_1, X_2] = \frac{(1 - |\xi|^2)^2}{2k}\left(\frac{\partial X_1}{\partial \xi}\frac{\partial X_2}{\partial \xi^*} - \frac{\partial X_1}{\partial \xi^*}\frac{\partial X_2}{\partial \xi}\right). \tag{14}$$

Thus, from the Hamiltonian dynamics, the classical equation of motion for the damped harmonic oscillator yields

$$\dot{\xi} = -i\omega[2\xi \cosh \lambda t + (1 + \xi^2)\sinh \lambda t]. \tag{15}$$

The real and imaginary parts of the above equation are given by

$$\dot{x} = 2\omega(y \cosh \lambda t + xy \sinh \lambda t), \tag{16}$$

$$\dot{y} = -\omega[2x \cosh \lambda t + (1 + x^2 - y^2)\sinh \lambda t]. \tag{17}$$

Apparently, these equations are different from those of paper I, the two equations in Eq. (3.12) of Ref. [1]. This difference is taken place from misevaluation of the Hamiltonian in their work. Figure 1 of paper I shows that all trajectories starting inside the unit circle asymptotically approaches to $(x, y) = (1.0)$ as t increases. However, our counterpart results displayed in Fig. 1 follow quite different trajectories and end up

$$(x, y) = (-1.0) \quad \text{for large } t. \tag{18}$$

It is also possible to prove the asymptotic behavior of Eq. (18) analytically. For sufficiently large t, Eq. (15) reduces to

$$\dot{\xi} \simeq -i\frac{\omega}{2}(1 + \xi)^2 e^{\lambda t}. \tag{19}$$

Through the method of separation of variables, we can easily solve the above equation to be

$$\xi \simeq -\left(1 + i\frac{2\lambda}{\omega}e^{-\lambda t}\right). \tag{20}$$

Since the case $\lambda t \gg 1$ is considered, this actually approaches to $\xi \to -1$, restoring relation (18).

For any quantum variable $\hat{X}$, the variance is defined as

$$V(X) = \langle \xi; k|\hat{X}^2|\xi; k\rangle - \langle \xi; k|\hat{X}|\xi; k\rangle^2. \tag{21}$$

The variances of canonical variables $\hat{q}$ and $\hat{p}$ are therefore obtainable using Eqs. (6) and (7) and with the help of Eq. (9). Thus we have

$$V(q) = \frac{2\hbar k}{m\omega(1 - |\xi|^2)}[1 + |\xi|^2 + 2\,\mathrm{Re}(\xi)], \tag{22}$$

$$V(p) = \frac{2m\omega\hbar k}{1 - |\xi|^2}[1 + |\xi|^2 - 2\,\mathrm{Re}(\xi)]. \tag{23}$$

From Figs. 2(a)–2(c), we can confirm that the uncertainty product (UP) obeys

$$V(q)V(p) \geq \hbar^2/4. \tag{24}$$

As you can see, this is always above the physically acceptable minimum uncertainty relation for the harmonic oscillator, which is $\hbar^2/4$. On the other hand, it is hard to know whether the uncertainty principle equally holds for the case of Fig. 2(d). But, from the enlarged graph of the UP displayed in Fig. 2(d′), it is clear that the uncertainty principle

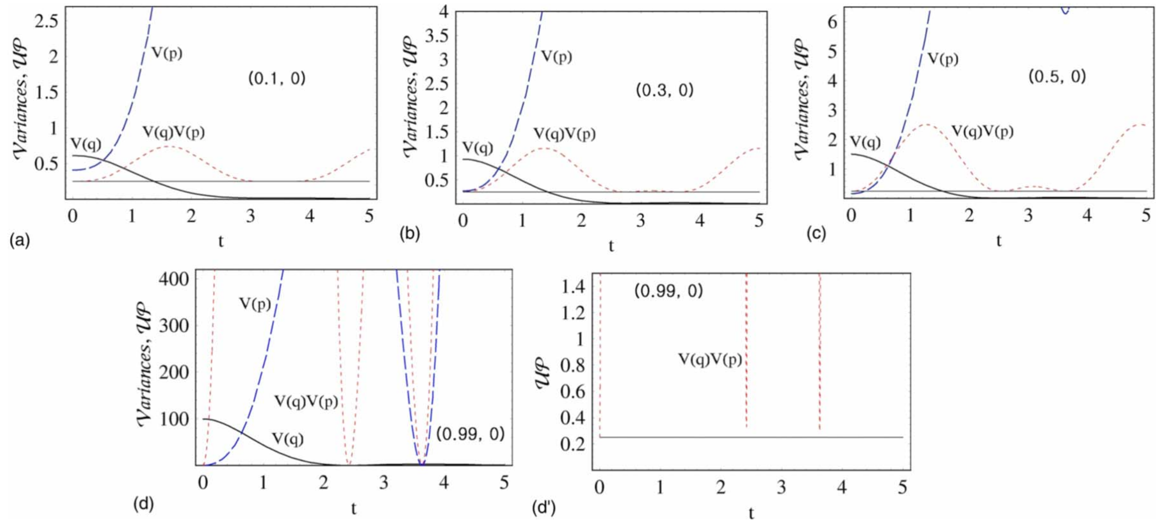

FIG. 2. (Color online) Variances $V(q)$ and $V(p)$ given in Eqs. (22) and (23), respectively, and UP. (a)–(d) are connected with the numerical values of the SU(1,1) coherent state parameter $\xi(t)=x(t)+iy(t)$ depicted in Figs. 1(a)–1(d). Thus, the initial conditions $(x(0),y(0))$ of (a)–(d) are the same as those of Figs. 1(a)–1(d) in turn. The straight horizontal reference line is $\hbar^2/4$ which is minimally allowed UP for the harmonic oscillator. (d′) is a highly enlarged graph of the lower part of (d) (but only for UP), displayed on the purpose of checking distinctly whether the uncertainty principle also holds for the case of (d). We used $m=1$, $\hbar=1$, $k=1/4$, $\omega=1$, and $\lambda=1$.

also holds even for the case of Fig. 2(d). However, the counterpart relation given in paper I is $V(q)V(p)\geq\hbar^2/16$. Thus, strictly speaking, Fig. 2 of paper I does not satisfy uncertainty principle since their counterpart UP is allowed to be lower below $\hbar^2/4$.

For this nonstationary system, the energy expectation value is somewhat different from that of the Hamiltonian. The relation between Hamiltonian and energy is [14,15]

$$E_k = e^{-\lambda t}\langle\xi;k|\hat{H}|\xi;k\rangle. \tag{25}$$

Hence, through a little calculation using the Hamiltonian of Eq. (8), we obtain the quantum energy such that

$$E_k = \frac{2\hbar\omega k}{1-|\xi|^2}e^{-\lambda t}[(1+|\xi|^2)\cosh\lambda t + (\xi+\xi^*)\sinh\lambda t]. \tag{26}$$

Figure 3 is the evolution of several variables in the SU(1,1) coherent state with no damping force ($\lambda=0$). The parametric plot of $\xi(t)=x(t)+iy(t)$ displayed in Fig. 3(a) is identical to that of paper I [see Fig. 3(a) of Ref. [1]]. At a glance, Fig. 3(b) looks similar to the counterpart plot presented in Fig. 3(b) of paper I. However, it is possible to find their difference from a careful comparison of them. Though the UP shown in Fig. 3(b) is always larger than $\hbar^2/4$, the UP associated to Fig. 3(b) of paper I does not satisfy uncertainty principle in the strict sense.

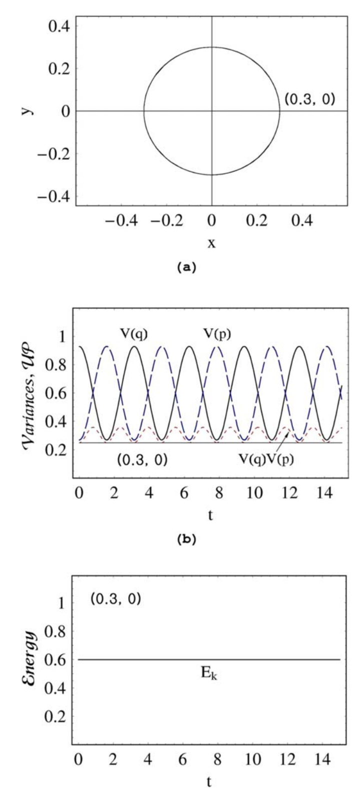

FIG. 3. (Color online) Evolution of the SU(1,1) coherent state when disappearing the damping force, i.e., $\lambda=0$, with the initial condition $(x(0),y(0))=(0.3,0)$. (a) Plot of the SU(1,1) coherent state parameter $\xi(t)=x(t)+iy(t)$. (b) Variances $V(q)$ and $V(p)$. (c) Expectation value of the quantum energy E_k. The straight horizontal reference line exhibited in (b) is $\hbar^2/4$. We used $m=1$, $\hbar=1$, $k=1/4$, and $\omega=1$.

For λ=0, the system becomes just a simple harmonic oscillator and Eq. (26) reduces to

$$E_k = 2\hbar\omega k\frac{1+|\xi|^2}{1-|\xi|^2}. \tag{27}$$

Besides, for λ=0, the solution of Eq. (15) is nothing but

$$\xi = \xi_0 e^{-2i(\omega t+\phi)}, \tag{28}$$

where ξ_0 and ϕ are constants. In this case, Eq. (27) is constant since $|\xi|$ does not vary with time. However, paper I shows that the time behavior of energy expectation value for λ=0 oscillates with time [see Fig. 3(c) of Ref. [1]] on the contrary to our counterpart plot shown in Fig. 3(c).

In summary, we made a rigorous study for the dynamical feature of SU(1,1) coherent states with correction of the errors of the previous work. The results are carefully compared with those of paper I and the differences between them are designated in detail.

This work was supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD) (Grant No. KRF-2007-313-C00162).

[1] C. C. Gerry, P. K. Ma, and E. R. Vrscay, Phys. Rev. A **39**, 668 (1989).
[2] J.-W. Wu, C.-W. Li, T.-J. Tarn, and J. Zhang, Phys. Rev. A **76**, 053403 (2007).
[3] H. K. Kim and S. P. Kim, J. Korean Phys. Soc. **48**, 119 (2006).
[4] S. F. Özeren, Physica A **337**, 81 (2004).
[5] F. A. A. El-Orany, S. S. Hassan, and M. S. Abdalla, J. Opt. B: Quantum Semiclassical Opt. **5**, 396 (2003).
[6] K. H. Yeon, S. S. Kim, C. I. Um, and T. F. George, Int. J. Theor. Phys. **42**, 2043 (2003).
[7] D.-Y. Song, Phys. Rev. A **68**, 012108 (2003).
[8] J. Zaleśny, Acta Phys. Pol. A **98**, 11 (2000).
[9] C. F. Lo and D. Kiang, Int. J. Mod. Phys. B **14**, 993 (2000).
[10] J.-B. Xu, X.-S. Zhang, and S.-B. Li, Phys. Scr. **56**, 225 (1997).
[11] A. M. Perelomov, Commun. Math. Phys. **26**, 222 (1972).
[12] C. C. Gerry and S. Silverman, J. Math. Phys. **23**, 1995 (1982).
[13] C. C. Gerry, Phys. Lett. **119B**, 381 (1982).
[14] M. A. Marchiolli and S. S. Mizrahi, J. Phys. A **30**, 2619 (1997).
[15] J. R. Choi and K. H. Yeon, Int. J. Mod. Phys. B **19**, 2213 (2005).