# Quantum corrections to the semiclassical collective dynamics in the Tavis-Cummings model

Jonathan Keeling*
*Cavendish Laboratory, University of Cambridge, J. J. Thomson Avenue, Cambridge CB3 0HE, United Kingdom*
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The Tavis-Cummings model (the Dicke model treated in the rotating-wave approximation), describing many two-level systems coupled to a single bosonic mode, has been long known to show collective semiclassical oscillations when prepared in an inverted state, with all two-level systems excited and the bosonic mode is empty. This paper discusses how the quantum dynamics approaches this semiclassical result for large numbers of two-level systems, focusing on how the eigenvalues approach their semiclassical limit. The approach to the semiclassical result is found to be slow, scaling like a power of the logarithm of the system size. Considering also the effect of weak detuning between the two-level system and the bosonic field, quantum corrections are again found to decay slowly with system size, such that for a fixed detuning, the quantum effects of detuning are greater than the classical effect.

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## I. INTRODUCTION

The Dicke model [1], describing interaction between a number of two-level systems and a single bosonic mode, has long been studied as a simple model of cavity quantum electrodynamics, which despite its simplicity can show quite intricate behavior. When the coupling between the two-level system and the bosonic mode is treated in the rotating-wave approximation, the Dicke model reduces to the Tavis-Cummings model [2,3]. One long studied feature of this model is collective oscillations that arise when the initial state is inverted; the simplest such case concerns an initial state with an empty bosonic mode and all two-level systems in their excited state. In a semiclassical approximation [4] the number of bosons describes a train of hyperbolic secant pulses. Interest in these collective oscillations has recently been revived both by the connection to atom-molecule interconversion in cold atomic gases [5–8], as well as potential experiments studying coupling between quantum dots and cavity photon modes [9,10]. The possibility of coupling between a radiation mode and multiple two-level systems is also being pursued in circuit-QED experiments [11], in which the two-level systems are superconducting qubits. Both these latter examples are closely related to the original context of this problem [4]: two-level atoms coupled to a photon mode in a cavity. This problem is also closely related to collective superfluorescence [12] for initially inverted atoms but without a cavity. Without the cavity there is only a single hyberbolic secant pulse since the dense spectrum of photon modes prevents recurrence.

The aim of this paper is to study the quantum dynamics of the Tavis-Cummings model with a finite number of two-level systems $N$, starting from a fully inverted state, in order to see how the quantum dynamics differ from the semiclassical dynamics. In particular, considering the case of a completely symmetric Tavis-Cummings model (where all two-level systems are identical), one finds that the semiclassical results are recovered in the limit of an infinite number of two-level systems but that the approach to this semiclassical limit is slow, scaling as a power of the logarithm of the number of two-level systems. In addition to the sequence of hyberbolic secant pulses that exist in the semiclassical dynamics, the quantum dynamics is found to have an additional slow envelope. As the number of two-level systems increases, the period of this envelope increases compared to the period of the train of hyberbolic secant pulses and so its effects become negligible; however this trend is very slow, with $T_{\mathrm{envelope}}/T_{\mathrm{pulse}} \sim [\ln(\sqrt{N})]^3$.

A closely related question has been addressed in [13] for the case of the Richardson model rather than the Tavis-Cummings model. Their work focused on how the integrability of the quantum model allows one to calculate overlaps between the initial state and the eigenstates, as well as matrix elements of the physical observables. The current work addresses a complementary question on how the eigenvalues approach the semiclassical results for large system sizes.

Aspects of the behavior of the Dicke model in the absence of the rotating-wave approximation have also been studied; in particular, features of the finite size system associated with the quantum phase transition in the infinite size system have been considered. These include the following: changes in statistics of excited state energies [14,15]; perturbative approaches in the limit of large coupling strengths [16]; entanglement between the bosonic mode and the two-level systems [17], including how that entanglement scales with the system size; and the scaling of other quantities, such as ground state energies or excitation gap with system size [18,19]. The results of these previous studies differ from the question addressed in the current work for two reasons. First, without the rotating-wave approximation, the number of excitations in the system is no longer conserved and so the classical problem is no longer integrable. Second, the results in this paper relate to the collection of eigenstates near $|E_q| = 0$, while the ground state, or the thermodynamics at low temperatures [20], involves eigenstates with much lower energy.

An alternate method to include quantum corrections to the semiclassical dynamics is by accounting for the dynamics of higher cumulants (as well as expectations) of the collective

*jmjk2@cam.ac.uk

 

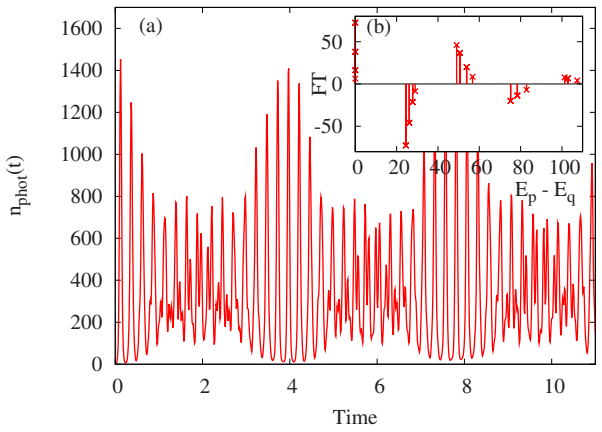


FIG. 1. (Color online) Quantum dynamics of the number of photons for 2000 spins. Panel (a): time dependence of photon amplitude. Panel (b): principal Fourier components of the time dependence. The lowest seven eigenvalues are sufficient to describe the time dependence to better than visible resolution on this scale. All energies and times are in units of the coupling between two-level systems and the bosonic mode.

operators, which has been discussed by Vardi *et al.* [21,22] in related but different models. A similar idea has also been discussed in the context of the BCS model [8,23] but in cases where the semiclassical dynamics is more complicated. Another related problem concerns quantum dynamics in the central spin model, where a large nuclear spin (analogous to the bosonic field here) is coupled to a sea of electronic spins (analogous to two-level systems); Tsyplyatyev and Loss [24] considered the quantum dynamics of this model with initial conditions such that there is only a single excitation in the system. Other related work concerns the dynamics of the Tavis-Cummings model in the opposite limit of a small number of two-level systems, starting from an initial coherent boson state, studied in Refs. [25–27], in connection to the collapse and revival of Rabi oscillations in the case of a single two-level system.

The results presented in this paper refer mainly to the completely symmetric Tavis-Cummings model, for which all two-level systems are identical. This restriction allows one to extract simple analytical formulas for the scaling of quantum corrections with system size. In addition, this symmetric case can be expected to be the "most classical" limit of the Tavis-Cummings model, as one may derive the semiclassical dynamics in such a case by replacing large quantum spins by classical spins. The fact that quantum corrections exist in this most classical case suggests that important corrections may exist in the nonsymmetric Tavis-Cummings model. One specialized limit of this is discussed in Sec. IV, supporting this idea.

The rest of the paper is organized as follows: Sec. II introduces the Hamiltonian and discusses the previously known results of the semiclassical approximation; these are compared to the results of exact diagonalization in Fig. 1. Section III then shows how the quantum corrections can be extracted from a WKB approach to the problem, focusing on the case where the two-level system and boson energies match; the effect of detuning is discussed in Sec. IV, and concluding remarks are given in Sec. V.

## II. MODEL AND COMPARISON OF SEMICLASSICAL AND NUMERICAL RESULTS

The model studied in this paper can be written as

$$H = \sum_{i=1}^{N} (\epsilon_i s_i^z + s_i^+ a + s_i^- a^\dagger), \tag{1}$$

where the spin operators obey $[s_i^z, s_i^\pm] = \pm s_i^\pm$ and $[s_i^+, s_i^-] = 2s_i^z$, and $a$, $a^\dagger$ are bosonic operators. The coupling between the two-level systems and the bosonic mode has been scaled to 1, hence all other energies and times are measured in units of this coupling. The initial state of the system is taken to be $|n=0, \uparrow\uparrow\uparrow\cdots\rangle$, where all the two-level systems are excited, and the bosonic mode is empty. In the subsequent dynamics there are collective oscillations, transferring excitations between the two-level systems and the bosonic mode.

For comparison to the exact dynamics, the following briefly summarizes the semiclassical solution, described in Refs. [5,6]. The semiclassical equations correspond to writing the Heisenberg equations of motion for the operators $s_i^-$, $s_i^z$, and $a$ and then replacing these operators by commuting classical variables. After transforming to a frame rotating at a rate $\mu$, the resultant equations can be solved by the ansatz

$$s_i^- = \frac{(\epsilon_i - \mu)a + i\dot{a}}{(\epsilon_i - \mu)^2 + \lambda}, \quad s_i^z = \frac{1}{2} - \frac{a^2}{(\epsilon_i - \mu)^2 + \lambda} \tag{2}$$

along with the equation of motion for $a$, $\dot{a}^2 = a^2(\lambda - a^2)$, and the self consistency conditions for $\mu$ and $\lambda$,

$$1 = \sum_i \frac{1}{(\epsilon_i - \mu)^2 + \lambda}, \quad \mu = \sum_i \frac{\epsilon_i - \mu}{(\epsilon_i - \mu)^2 + \lambda}. \tag{3}$$

These equations are analogous to the BCS gap equation, with $\mu$ acting as a generalized chemical potential (i.e., common oscillation frequency) and $\lambda$ acting as the square of the gap (i.e., pairing field). The solution of the equation for $a$ gives a train of hyperbolic secant pulses with a period $T_{\text{pulse}} = 2\ln(\sqrt{\lambda})/\sqrt{\lambda}$; such pulses can be seen in the time dependence of physical observables such as the occupation of the bosonic mode $\langle n_{\text{phot}} \rangle = a^2$. For the special case of $\epsilon_i = 0$, Eq. (3) has the solution $\mu = 0$ and $\lambda = N$.

The problem this paper addresses can be seen most clearly in Fig. 1, which shows the dynamics of the population of the bosonic mode $n_{\text{phot}}$ according to Eq. (1) with $\epsilon_i = 0$. The semiclassical train of hyperbolic pulses is seen in Fig. 1, but there is in addition a slow envelope not predicted by the semiclassical equations, and it is this slow envelope which is discussed below.

By writing the photon number as a sum over eigenstates,

$$n(t) = \sum_{pq} \langle 0|X_p\rangle\langle X_p|\hat{n}|X_q\rangle\langle X_q|0\rangle e^{i(E_p - E_q)t}, \tag{4}$$

one may note that the semiclassical result, with its perfectly periodic train of pulses, corresponds (for $\epsilon = 0$) to

$$E_q = q\Omega, \quad \Omega = \pi\frac{\sqrt{N}}{\ln(\sqrt{N})}. \tag{5}$$

The inset of Fig. 1 shows the Fourier transform of $n_{\text{phot}}(t)$. In contrast to Eq. (5), the eigenvalues of the full quantum problem are not equally spaced, and so the sidebands seen in Fig. 1(b) arise. The period of the slow envelope is given by the splitting of these sidebands. Hence, to describe the approach to the semiclassical result, one must consider how the deviation from regular spacing $E_q=q\Omega$ (i.e., the anharmonicity of the exact eigenvalues) collapses as $N\to\infty$.

Describing the quantum dynamics by finding the eigenstates is considerably simplified since the initial state overlaps with only a small number of eigenstates. As discussed in Ref. [13], the initial condition chosen here satisfies $\langle 0|H|0\rangle=0$, $\langle 0|H^2|0\rangle=N$, while the semiclassical eigenstates have $E_q=q\pi\sqrt{N}/\ln(\sqrt{N})$, which implies that only $O(\ln(\sqrt{N})^2)$ eigenstates near $|E_q|=0$ can have any significant overlap with the initial state. The validity of this is confirmed by noting that the result of exact diagonalization and the result restricting the summation in Eq. (4) to the seven smallest values of $|E_q|$ are indistinguishable by eye on the scale of Fig. 1.

## III. WKB APPROXIMATION AND SCALING OF CORRECTION

To find the eigenvalues of the quantum problem, one may approach the problem by a method closely related to that in [4]. As discussed above, this paper considers the symmetric case, $\epsilon_i=\epsilon$, for which the Hamiltonian becomes $H=\epsilon S^z+S^-a+S^+a^\dagger$, where $\vec{S}=\Sigma_i\vec{s}_i$, and as a result, the quantum state may be written as a wave function in the one-dimensional space of occupation, as used in Ref. [4] to show how the semiclassical limit can arise. To find not only the semiclassical limit but also the corrections to it, one may solve this one-dimensional problem using a discrete WKB approximation [28]. Using the basis $|n\rangle=|n_{\text{phot}}=n,|\vec{S}|=N/2,S_z=N/2-n\rangle$, the equation $(E-\epsilon N/2)\Psi=H\Psi$ becomes

$$(E+\epsilon n)\psi_n = n\sqrt{N+1-n}\psi_{n-1} + (n+1)\sqrt{N-n}\psi_{n+1}. \tag{6}$$

The WKB approach consists of two parts: finding the WKB form of the wave function for $(n,N-n)\gg 1$ and then matching this wave function to appropriate forms for $n\simeq 0$ and $n\simeq N$. In the following, this matching is referred to as matching the "boundary conditions" for the wave function at $n=0$, $n=N$, but these boundary conditions are just the Schrödinger equation in Eq. (6), evaluated at $n=0$, $n=N$, for which those approximations valid for $(n,N-n)\gg 1$ no longer hold. This section will consider the case $\epsilon=0$; the effect of nonzero $\epsilon$ is discussed in Sec. IV.

The WKB form of the wave function can be derived by considering a transformation $\psi_n\to\tilde{\psi}_n(i)^n$ which gives the right-hand side of Eq. (6) as a discrete derivative with a variable prefactor, i.e., $E\tilde{\psi}=-iv(x)\partial_x\tilde{\psi}(x)$. Written this way suggests a WKB wave function of the form [28] $\tilde{\psi}\approx[1/\sqrt{v(x)}]\exp[i\int dzE/v(z)]$ or the discrete analog of this wave function.

To write the wave function compactly, it is useful to introduce the function defined in Ref. [4],

$$g_k \equiv \frac{\sqrt{2}\Gamma(1+k/2)}{\Gamma[1/2+k/2]} \approx \begin{cases} \sqrt{2/\pi}, & k=0 \\ \sqrt{\pi/2}, & k=1 \\ \sqrt{k+1/2}, & k\gg 1, \end{cases} \tag{7}$$

which is chosen such that $g_kg_{k+1}=k+1$. With this notation, the WKB wave function can be written as

$$\psi_n^{(\text{WKB})} = \frac{\cos(E\Phi_n+\phi+n\pi/2)}{\sqrt{g_n^2g_{N-n}}}, \tag{8}$$

where $\Phi_N=\Sigma_{m=n}^N 1/2g_m^2g_{N-m}$ and $\phi$ is an arbitrary phase set by boundary conditions. This wave function is valid while the change in phase between two successive values of $n$ is much less than 1; this condition for the validity of the WKB wave function can formally be written as $g_n^2g_{N-n}\gg E$. Under the same conditions, one may approximate the summation in the definition of $\Phi_n$ by integration, and hence to leading order one has

$$\Phi_n \approx \frac{1}{\sqrt{N+1}}\cosh^{-1}\left[\sqrt{\frac{N+1}{n+1/2}}\right]. \tag{9}$$

It is worth noting that for even $N$, the solution in Eq. (8) with $E=0$ can be seen to be an exact eigenstate of Eq. (6).

### A. Matching WKB solution at boundaries

#### 1. Boundary condition at $n=N$

When considering the boundary conditions, the boundaries at $n=0$ and $n=N$ behave differently. At $n=N$ first note that $g_N^2g_0\simeq(N+1/2)\sqrt{2/\pi}\gg E\simeq\pi\sqrt{N}/\ln(\sqrt{N})$, meaning the WKB wave function is valid right up to this boundary. In addition, the Schrödinger equation at the boundary becomes $E\psi_N=N\psi_{N-1}$, and so to order $1/\sqrt{N}$, this boundary condition is satisfied by $\psi_{N-1}=0$. Since the corrections to the semiclassical energies found below are of order $1/\ln(\sqrt{N})$, the much smaller corrections of order $1/\sqrt{N}$ can be neglected, and so the boundary condition at $n=N$ requires $\phi=(2q+2-N)\pi/2$, where $q$ is any integer; for even $N$ one may thus choose $\phi=0$. From herein, even $N$ is assumed; for odd $N$ similar results with $\phi=\pi/2$ follow straightforwardly.

#### 2. Boundary condition at $n=0$

The boundary condition at $n=0$ is more complicated. For this boundary, $g_0^2g_N\simeq(2/\pi)\sqrt{N+1/2}$, and so the condition $g_0^2g_N\gg E\simeq\pi\sqrt{N}/\ln(\sqrt{N})$ is not necessarily satisfied—the validity of the WKB wave function depends on taking $\ln(\sqrt{N})\gg 1$, which requires very large $N$; these corrections due to finite $\ln(\sqrt{N})$ are the quantum corrections of interest in this paper.

Because the WKB wave function breaks down for small values of $n$, it is necessary to match the WKB wave function to the exact wave function at a nonzero value of $n=n_0$ rather than at $n=0$; i.e., the exact wave function should be calculated for $n\le n_0$ and the WKB wave function made to match it at $n=n_0$. The larger the value of $n_0$ one takes, the better the

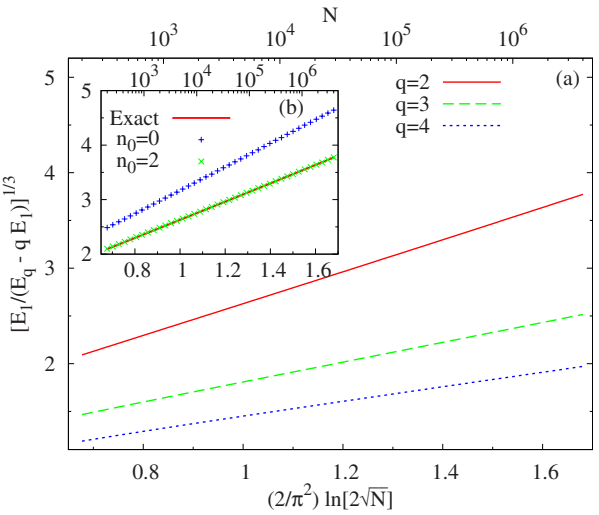


FIG. 2. (Color online) Anharmonicity, $E_q - qE_q$, plotted against system size $N$ with variables transformed to demonstrate agreement with the scaling expected from Eq. (13). Panel (a) compares $E_q$ for $q=2$, 3, and 4. Panel (b) compares WKB approximations to $E_2 - 2E_1$ on the same transformed axes, with $n_0$ terms near the $n=0$ boundary.

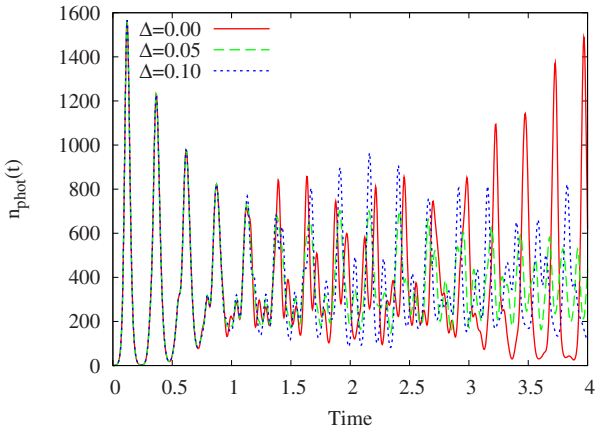


FIG. 3. (Color online) Time dependence of photon number for differing values of detuning $\epsilon$, showing modification to the slow envelope.

calculated energies will match the exact solution. Surprisingly, even matching the solution at $n=0$ turns out to provide useful information and correctly reproduces the scaling of energy with system size; this is discussed below in Sec. III B.

The procedure of matching is straightforward; for a given value $n_0$, one finds the exact solution $\psi^{(0)}_{n\leq n_0}(E)$ by solving Eq. (6), with $\psi_{n_0+1}=\psi^{(\mathrm{WKB})}_{n_0+1}$; i.e.,

$$E\psi^{(0)}_n = H^{(n_0)}_{nm}\psi^{(0)}_m + \delta_{n,n_0}(n_0+1)\sqrt{N-n_0}\,\psi^{(\mathrm{WKB})}_{n_0+1}, \quad (10)$$

where $H^{(n_0)}$ is the Hamiltonian restricted to $n\leq n_0$; the matching condition then becomes $\psi^{(0)}_{n_0}=\psi^{(\mathrm{WKB})}_{n_0}$. By diagonalizing the $(n_0+1)\times(n_0+1)$ matrix $H^{(n_0)}$ to give $H^{(n_0)}=\lambda_\alpha|\chi_\alpha\rangle\langle\chi_\alpha|$, the boundary condition equation can be written as

$$\cos\left(\frac{n_0\pi}{2}+\Phi_{n_0}\right) = -\sin\left(\frac{n_0\pi}{2}+\Phi_{n_0+1}\right) \times g^2_{n_0}g_{N-n_0}\sum_\alpha \frac{|\langle\chi_\alpha|n_0\rangle|^2}{E-\lambda_\alpha}. \quad (11)$$

This gives a nonlinear equation for $E$, the complexity of which increases with increasing $n_0$. An example of solving this equation, with $n_0=2$, is shown in Fig. 2(b), which clearly accurately reproduces the results of exact diagonalization.

### B. Simplified boundary condition and scaling with system size

To understand how the corrections to $E_q$ depend on number of two-level systems $N$, the equation with general $n_0$ is rather complicated, but as seen in Fig. 2(b), matching at $n=0$ gives the correct scaling with system size but an incorrect numerical coefficient. Considering this matching at $n=0$, the boundary condition becomes, from Eqs. (6) and (8),

$$E\cos(E\Phi_0) = -\frac{2}{\pi}\sqrt{N}\sin(E\Phi_1). \quad (12)$$

In Fig. 2(b), the solution of this equation is represented by the blue crosses, labeled $n_0=0$.

To the same level of approximation made so far, one may set $\Phi_0\simeq\Phi_1\equiv\ln(\sqrt{N})/\sqrt{N}$. If $\ln(\sqrt{N})$ were large, the solution of Eq. (12) would be $E_q=q\pi/\Phi_1=q\pi\sqrt{N}/\ln(\sqrt{N})$, i.e., the semiclassical result. For finite $\ln(\sqrt{N})$, Eq. (12) produces corrections $O(q^3)$ that break the harmonicity; these terms look like

$$\delta E_q = Cq^3\frac{\sqrt{N}}{[\ln(\sqrt{N})]^4}. \quad (13)$$

The coefficient $C$ as calculated by expanding Eq. (12) in the small parameter $1/\ln(\sqrt{N})$ is $C=\pi^6/24\simeq 40.1$ while the best fit coefficient over the range shown in Fig. 2(a) is $C=13.5$; however this gradient may should be treated with caution, as $1/\ln(\sqrt{10^6})$ is not a small number. The form in Eq. (13) is the first main result of this paper—as shown in Fig. 2, this dependence on $\ln(\sqrt{N})$ well describes the scaling of the anharmonicity; because the dependence on $N$ is so slow, quantum corrections to the semiclassical result remain relevant even for $10^6$ two-level systems.

## IV. DETUNING

The remainder of this paper discusses the effect of detuning, i.e., of $\epsilon_i=\epsilon\neq 0$; this again reveals a distinction between the semiclassical dynamics and the full quantum mechanical problem and provides some insight into the case where not all values of $\epsilon_i$ are identical. The time dependence in the case of $\epsilon_i=\epsilon\neq 0$ is shown in Fig. 3, showing a change to the slow envelope. Semiclassically, the effect of nonzero $\epsilon$ in Eq. (3) is to give $\mu=\epsilon/2$, $\lambda=N-(\epsilon/2)^2$. Since it is $\lambda$ that controls the collective oscillations, the leading correction to the frequency of pulses, i.e., the semiclassical energy splitting $\Omega$ should be quadratic in $\epsilon$,

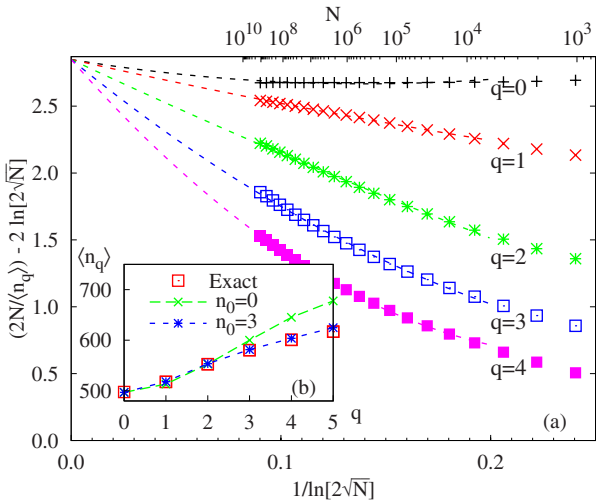


FIG. 4. (Color online) Panel (a): scaling of $\langle n_q\rangle$ with system size calculated using the WKB wave function matched to the exact solution for $n<n_0=40$ and transformed following Eq. (20). Dotted lines are a quadratic best fit to the data for $N>10^4$, assuming all lines tend to a common asymptote. Panel (b): comparison of exact diagonalization and WKB approximations for $\langle n_q\rangle$ for $N=3000$, showing that the WKB approximation should be valid for the panel (a).

$$E_q = q\Omega, \quad \delta\Omega \simeq \frac{\pi\epsilon^2}{8\sqrt{N}\ln(\sqrt{N})}. \tag{14}$$

The following discussion focuses on the case of small $\epsilon$ and shows that quantum corrections give shifts of energies linear in $\epsilon$ for small $\epsilon$, such that for any finite $N$, the leading correction to $E_q$ is quantum not semiclassical.

It is straightforward to see from Eq. (6) that for small $\epsilon$, the linear peturbative correction to eigenstate $q$ would be

$$E_q \to E_q - \epsilon\langle X_q|\hat{n}|X_q\rangle, \tag{15}$$

and so the aim of this section is to find how $\langle n_q\rangle \equiv \langle X_q|\hat{n}|X_q\rangle$ depends on system size $N$ and eigenstate label $q$. One should note that a $q$ independent shift of $E_q$ has no effect on the time dependence of photon number. In the semiclassical picture, such a constant shift just corresponds to a change in $\mu$, hence our interest is in the quantity $\delta E_q - \delta E_0$.

Figure 4(b) shows that the WKB eigenstates matched to the exact solution for $n<n_0=3$ reproduce the result of exact diagonalization for $N=3000$, giving confidence that the WKB wave function can be used to determine $\langle n_q\rangle$. Using the WKB wave function of Eq. (8) one has

$$\langle n_q\rangle \simeq \frac{\sum_{n=0}^{N} n\dfrac{[1+(-1)^n\cos(2E_q\Phi_n)]}{g_n^2 g_{N-n}}}{\sum_{n=0}^{N}\dfrac{[1+(-1)^n\cos(2E_q\Phi_n)]}{g_n^2 g_{N-n}}}. \tag{16}$$

To analyze how the above expression depends on the number of two-level systems $N$, it is convenient to write it as $\langle n_q\rangle = (A_0+\delta A_q)/(B_0+\delta B_q)$, where $\delta A_q$ and $\delta B_q$ come from the term proportional to $(-1)^n$ in the summand and $A_0$ and $B_0$ from the other term. One may then show that

$$A_0 = \sum_{n=0}^{N}\frac{n}{g_n^2 g_{N-n}} \approx \int_0^n dn\frac{1}{\sqrt{N-n}} = 2\sqrt{N} \tag{17}$$

and from Eq. (9), one sees that

$$B_0 \approx \frac{2}{\sqrt{N}}\cosh^{-1}(\sqrt{2N}) \approx \frac{2}{\sqrt{N}}\ln(2\sqrt{2N}). \tag{18}$$

Because $A_0 \gg B_0$, one may anticipate that $\delta B_q$ has a more significant effect on the answer than does $\delta A_q$; this can be made more firm by noting that for even $N$,

$$\delta A_{q=0} \approx \sum_{n=0}^{N}\frac{(-1)^n}{\sqrt{N-n+1/2}} \approx \sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+1/2}} \approx 0.944,$$

and numerically one may confirm that $\delta A_q$ is a factor $\sqrt{N}$ smaller than $A_0$ in general. For $\delta B_q$ one may observe that the main contribution to the sum comes from values of $n \ll N$, where the denominator is smallest, and so approximate

$$\delta B_q \approx \frac{1}{\sqrt{N}}\sum_{n=0}^{\infty}\frac{(-1)^n\cos(E_q\Phi_n)}{(n+1/2)}. \tag{19}$$

This expression for $\delta B_q$ has the form of $1/\sqrt{N}$ multiplied by a function that should depend only on $\ln(\sqrt{N})$—by considering the forms of $E$ in Eq. (5) and $\Phi_n$ in Eq. (9), it is clear that only logarithmic dependence on system size enters the product $E\Phi_n$.

Putting all these considerations together one may write

$$\langle n_q\rangle \approx \frac{2N}{2\ln(\sqrt{N}) + \sqrt{N}\delta B_q}$$

and thus one may expand

$$\frac{2N}{\langle n_q\rangle} - 2\ln(\sqrt{N}) \approx \sqrt{N}\delta B_q = \beta_q^0 + \frac{\beta_q^1}{\ln(\sqrt{N})} + \frac{\beta_q^2}{\ln(\sqrt{N})^2} + \cdots \tag{20}$$

for large $N$. This scaling indeed seems to occur in Fig. 4(a) and strongly suggests that $\beta_q^0$ is independent of $q$, while $\beta_q^1$ depends on $q$. One may thus write the $q$ dependent part of the energy shift as

$$\delta E_q - \delta E_0 \approx -\frac{\epsilon N}{\ln(\sqrt{N})}\left(\frac{\beta_0^1 - \beta_q^1}{2\ln(\sqrt{N})^2}\right) + \cdots, \tag{21}$$

where only the leading order term in powers of the logarithm has been retained. Comparing Eq. (21) to the semiclassical correction of Eq. (14), one may note that in both the limit fixed $N$, $\epsilon \to 0$ and also for fixed $\epsilon$, $N \to \infty$, the quantum correction is larger than the semiclassical correction.

### Relating detuning to disordered $\epsilon_i$

The above results on treating $\epsilon$ perturbatively can also describe properties of the model where $\epsilon_{i=1,\ldots,N-1}=0$, $\epsilon_N=\epsilon$; i.e., $\delta H = \epsilon s_N^z$, again considering $\epsilon$ perturbatively. The fact this case can be treated by first-order nondegenerate pertur-

bation theory is not trivial; it arises because although degenerate states do exist (particularly states with zero energy), the asymmetry introduced by altering a single spin energy $\epsilon_i$ does not mix these degenerate states. With a greater number of different $\epsilon_i$ this simplifying condition fails, and degenerate perturbation theory is instead required.

For the special form of $\delta H$ above for which nondegenerate perturbation theory is relevant, one may consider the expectation of perturbation Hamiltonian between the boson number states, for which one may show that

$$\langle n|s_i^z|n'\rangle = \delta_{nn'}\left(\frac{1}{2}-\frac{n}{N}\right), \tag{22}$$

which reproduces the results of Sec. IV with $\epsilon \to \epsilon/N$. Such a result hints that quantum corrections in the nonsymmetric model may become more significant since for weak disorder of energies $\epsilon_i$, such linear corrections due to quantum corrections will win over the quadratic semiclassical effects of disorder. However, direct calculation of the quantum dynamics is challenging, as the size of the Hilbert space explored in such a model is exponential in $N$, and if quantum corrections still vanish as powers of $\ln(\sqrt{N})$, one requires very large systems to study the asymptotic behavior at large $N$.

## V. CONCLUSIONS

In conclusion, the quantum dynamics of the symmetric Tavis-Cummings model starting from an initially inverted state describes a train of hyberbolic secant pulses (as in the semiclassical result) but with an additional slow envelope. The slow envelope arises due to the anharmonicity of the eigenvalues of the quantum problem; this anharmonicity reduces as the system size increases but only logarithmically with the number of two-level systems. In the absence of detuning, i.e., for $\epsilon_i=0$, the eigenvalues take the form $E_q \simeq q\pi\sqrt{N}/\ln(\sqrt{N})+Cq^3\sqrt{N}/[\ln(\sqrt{N})]^4$.

With a small detuning $\epsilon$, the quantum problem has a perturbative correction linear in the size of detuning $\delta E \propto \epsilon N/[\ln(\sqrt{N})]^3$, while the semiclassical result has only corrections quadratic in the detuning $\delta E \propto \epsilon^2/[\sqrt{N}\ln(\sqrt{N})]$. As a result, for either finite detuning and $N\to\infty$ or finite $N$ and $\epsilon\to 0$, the classical effects vanish faster than quantum corrections. This result may also be indicative of the effects of disorder, i.e., of nonidentical values of $\epsilon_i$. For a fixed distribution of $\epsilon_i$, as $N\to\infty$ the effects of this disorder vanish compared to the energy scale of the common coupling $\sim\sqrt{N}$. It seems likely that in such a limit, quantum corrections may vanish more slowly than the classical effects of the distribution of $\epsilon_i$.

In the general case of disordered $\epsilon_i$, the quantum problem is significantly harder to solve as the size of the Hilbert space grows exponentially with the number of two-level systems, whereas for $\epsilon_i=\epsilon$, it grows only linearly. However, the integrability of the problem may simplify the problem, as discussed in [13]. In the symmetric case, the integrability of the semiclassical problem means that while conservation laws restrict the system to exploring an $N$-dimensional Hilbert space, in the limit of large $N$, the system in fact only explores a one-dimensional path through this space. Similarly for the disordered model, the problem is integrable [29,30], which in turn leads to the semiclassical dynamics [5–7] following a one-dimensional path, suggesting that a simple description of the quantum corrections even for large values of $N$ might be possible.

*Note added.* Recently, a similar treatment of this model has been undertaken in [31].

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