Uncontrollable quantum systems: A classification scheme based on Lie subalgebras

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(Received 22 January 2009; published 6 May 2009)

It is well known that a finite level quantum system is controllable if and only if the Lie algebra of its generators has full rank. When the rank of the Lie algebra is not full, there is a rich mathematical and physical structure to the subalgebra that to date has been analyzed only in special cases. We show that uncontrollable systems can be classified into reducible and irreducible ones. The irreducible class is the more subtle and can be related to a notion of generalized entanglement. We give a general prescription for revealing irreducible uncontrollable systems: the fundamental representation of su(N), where N is the number of levels, must remain irreducible in the subalgebra of su(N). We illustrate the concepts with a variety of physical examples.

DOI: 10.1103/PhysRevA.79.053403 PACS number(s): 32.80.Qk, 42.50.Hz, 02.20.Qs, 03.67.Mn

I. INTRODUCTION

Driving a quantum system through its interaction with a designed electric field is the essence of quantum coherent control. A question of both theoretical and practical importance is whether or not, given an external driving field that can be shaped at will, a system can be steered from one state to another. Controllability studies are designed to answer this question. In the general case, the answer is not obvious.

The mathematical tools of control theory have been harnessed in recent years to study quantum systems. A powerful approach known as geometrical control theory provides a general framework for determining controllability. In this approach, controllability is assessed using a criterion based on the Lie algebra structure of the control Hamiltonians [1–4]. This criterion, extended in [5], has been used to determine the controllability of few-level systems [6,7]. The advantages of this approach are its automation and complete generality. However, as is, it fails to provide a physical picture of the mechanisms at work when full controllability is available. Conversely, it fails to give insight into what is missing when full controllability is not present. In the latter case, we would like to be able to make precise statements about what limited control is achievable and to find a classification scheme for uncontrollable systems.

A more physically intuitive approach to controllability relies on the concept of connectivity [8,9]. This approach has also been reformulated within a Lie algebra framework [10]. It requires a coupling—possibly an indirect one—that connects any two basis states of the system to be controlled. If such coupling exists and if there are no degeneracies in the system (either level degeneracies or level-spacing degeneracies) full controllability is ensured. Within this approach, one has the intuitive result that noncontrollability results from the absence of coupling between states.

However, when the unperturbed Hamiltonian of the system possesses degeneracies, the previous intuitive result does not hold: connectivity does not guarantee controllability. This result can also be understood intuitively since level and transition degeneracies reduce the number of degrees of freedom available to manipulate a system. The reduction in control when degeneracy is present is well attested [7,11]. Moreover, it is known that in such cases, the interplay between the unperturbed and coupling Hamiltonian plays a critical role in determining controllability [6–8]. In addition, it has been shown that in some systems with level-spacing degeneracies, nontrivial conserved quantities can be an impediment to controllability [8]. Yet, a full understanding of controllability of degenerate systems and a classification scheme for uncontrollable systems has not been published to date. That is the topic of this paper.

Here we focus on finite-dimensional systems where the connectivity hypothesis holds and yet controllability fails. The centerpiece of our analysis is the notion of irreducibility. Conceptually, irreducibility is extremely close to connectivity. However, irreducibility provides a more precise criterion when analyzing degenerate cases. Moreover, irreducibility is a fundamental tool of group representation theory. It therefore allows an analysis of structure of the so-called dynamical group that captures the constraints imposed on the system.

Our analysis clearly separates two mutually exclusive sources of uncontrollability. One, intuitive and well known, is the lack of coupling between states which leads to "dark spaces." The dynamics then conserves the mean value of some Hermitian operators. The other, less well-known source of uncontrollability, results from a generalized entanglement symmetry [12,13]. We give a general procedure that reveals the building blocks of this symmetry and we give explicit examples of uncontrollable few-level systems that have not previously been identified.

II. CLASSIFYING UNCONTROLLABLE QUANTUM SYSTEMS

A. Controllability criteria

Consider an N-level system driven by a time-dependent Hamiltonian $H(t)=H_0+\sum_{k=1}^{K}u_k(t)H_k$, where the bare and cou-

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pling Hamiltonians H_0 and H_k are fixed and $u_k(t)$ are arbitrary real-valued functions. The evolution operator, $U(t)$, which gives the wave function of the system at time t as a function of an initial state $|\psi\rangle$, $|\psi(t)\rangle = U(t)|\psi\rangle$, evolves according to $i\hbar \frac{dU(t)}{dt} = H(t)U(t)$. Denote the Lie algebra generated by $iH_0, iH_1, \ldots, iH_K$ by $\mathfrak{g}$ and the corresponding Lie group by G. It is a well-known result that the set A of unitary transformations $U(t)$, obtained for $t \geq 0$ when the control functions u_k are arbitrarily varied, is a semigroup included in the Lie group G [2]. Without losing generality, one can set $\mathrm{Tr}(H_k) = 0$ for $0 \leq k \leq K$, so that the so-called dynamical group G sits in $\mathrm{SU}(N)$.

G and $\mathfrak{g}$ are central objects in controllability analysis. The definition for a system to be wave function controllable is that given any pair of states $|\psi\rangle$ and $|\phi\rangle$ there exists a U in A such that $|\langle\psi|U|\phi\rangle| = 1$. It has been shown that wave function controllability is equivalent to G being $\mathrm{SU}(N)$ when N is odd or G being $\mathrm{SU}(N)$ or $\mathrm{Sp}(N/2)$ (or anything isomorphic) when N is even [1,5]. This can be checked directly from the structure of the Lie algebra, $\mathfrak{g}$. Since we focus only on wave function controllability, we abbreviate it to controllability in the following.

B. Irreducibility

When the system is not controllable, the set $A|\psi\rangle$ of states reachable from any state $|\psi\rangle$ is not the full space of states. To analyze these cases, we introduce the notion of irreducibility. Consider a set Σ of linear operators operating on a finite-dimensional Hilbert space $\mathcal{H}$. A subspace $\mathcal{H}'$ of $\mathcal{H}$ is said to be invariant under Σ when $\sigma\mathcal{H}' \subset \mathcal{H}'$ for all σ in Σ. It is easy to check that the subspaces $\{0\}$ and $\mathcal{H}$ are always invariant. $\mathcal{H}$ is said to be irreducible for Σ when no subspaces other than these two are invariant [14–16] under Σ. Otherwise, $\mathcal{H}$ is said to be reducible.

Irreducibility is a fundamental notion in group representation theory. We now reformulate irreducibility in our context to highlight its relevance for our control problem. Let $\mathcal{H}$ be the Hilbert space supporting our N-level system. Thus $\mathcal{H} = \mathbb{C}^N$, and $\mathcal{H}$ is the vector space upon which the fundamental representation of $\mathrm{SU}(N)$ acts. We prove that $\mathcal{H}$ being irreducible for A is equivalent to (necessary and sufficient for) the following nonzero overlap condition: for any pair of states $|\psi\rangle$ and $|\phi\rangle$, a U in A exists such that $\langle\psi|U|\phi\rangle \neq 0$.

The proof of necessity is as follows. Assume the nonzero overlap condition is met. Consider a nontrivial subspace $\mathcal{H}'$ of $\mathcal{H}$ invariant under A. If its orthogonal complement, $\mathcal{H}'^{\perp}$, is nontrivial consider $|\phi\rangle$ in $\mathcal{H}'$ and $|\psi\rangle$ in $\mathcal{H}'^{\perp}$. A U in A then exists such that $\langle\psi|U|\phi\rangle \neq 0$. But this contradicts $A\mathcal{H}' \subset \mathcal{H}'$, so $\mathcal{H}'^{\perp} = \{0\}$, i.e., $\mathcal{H}' = \mathcal{H}$. Because any nontrivial A-invariant subspace of $\mathcal{H}$ is $\mathcal{H}$ itself, $\mathcal{H}$ is irreducible for A. □

We now prove sufficiency. Assume that two states $|\psi\rangle$ and $|\phi\rangle$ exist such that $\langle\psi|U|\phi\rangle = 0$ for all U in A. Consider the subspace $\mathcal{H}_{|\phi\rangle}$ spanned by elements of $A|\phi\rangle$. Any state $|\varphi\rangle$ in $\mathcal{H}_{|\phi\rangle}$ takes the form $|\varphi\rangle = \Sigma_j a_j U_j |\phi\rangle$ where the sum is finite; a_j are complex coefficients and the U_j are in A. Because A is a semigroup, $\mathcal{H}_{|\phi\rangle}$ is invariant under A. For any $|\varphi\rangle$ in $\mathcal{H}_{|\phi\rangle}$, $\langle\psi|\varphi\rangle = \Sigma_j a_j \langle\psi|U_j|\phi\rangle = 0$. So $\mathcal{H}_{|\phi\rangle}$ has a nonzero orthogonal

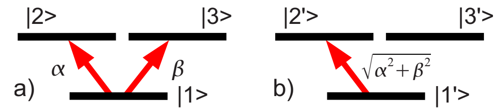

FIG. 1. (Color online) The three-level system shown in (a) is connected in the $\{|i\rangle\}$ basis. However, because of the degeneracy one cannot *a priori* say that the system is controllable. (b) The same system expressed in the primed $\{|i'\rangle\}$ basis shows that invariant subspaces exist that prevent controllability.

complement. $\mathcal{H}_{|\phi\rangle}$ is a nontrivial proper invariant subspace of $\mathcal{H}$. This shows that $\mathcal{H}$ is reducible for A. Equivalently, if $\mathcal{H}$ is irreducible for A, the nonzero overlap condition is met. □

When the system is reducible, $\mathcal{H}$ decomposes uniquely into irreducible orthogonal proper subspaces $\mathcal{H}_l$ invariant under A: $\mathcal{H} = \oplus_{l=1}^{L} \mathcal{H}_l$. This decomposition of $\mathcal{H}$ into invariant irreducible subspaces is identical for A, G, or $\mathfrak{g}$. This leads to an important second characterization of reducible systems: the system is reducible if and only if a nonzero traceless Hermitian operator commutes with the dynamical algebra $\mathfrak{g}$.

The proof is as follows. If the system is reducible, $\mathcal{H}$ admits an orthogonal decomposition $\mathcal{H} = \oplus_{l=1}^{L} \mathcal{H}_l$ into proper subspaces invariant under $\mathfrak{g}$. Any operator of the form $c = \oplus_{l=1}^{L} c_l Id_{\mathcal{H}_l}$ with $\Sigma_{l=1}^{L} c_l \dim(H_l) = 0$ is a traceless Hermitian operator commuting with $\mathfrak{g}$.

If the system is irreducible, by Shur's lemma, operators commuting with $\mathfrak{g}$ must be proportional to the identity and such nonzero traceless operators do not exist. □

C. Irreducibility and connectivity

The nonzero overlap condition shows that irreducibility is a necessary condition for controllability. It is instructive to compare irreducibility to the previously introduced notion of connectivity [8], another necessary criterion for controllability. If a basis $\{|i\rangle\}_{1 \leq i \leq N}$ of $\mathcal{H}$ is fixed, and if the basis is chosen so that H_0 is diagonal, connectivity requires the existence of a nonzero overlap $\langle i|U|j\rangle$ between any pair of states $|i\rangle$, $|j\rangle$ in the basis for some U in A [17]. The two notions are obviously very close. However, they differ.

Although irreducibility implies connectivity, the converse is not true: a connected system might be reducible. The difference between connectivity and irreducibility is relevant only when H_0 possesses degeneracies. Then, a basis change that does not affect H_0 can still possibly modify the structure of the couplings: connectivity then becomes basis dependent. Intuitively, irreducibility is a basis-independent criterion that requires connectivity in every possible basis. It is therefore more demanding than connectivity. As a result, it provides a more accurate criterion to characterize uncontrollability.

D. Examples of reducible vs irreducible uncontrollable systems

To illustrate the difference between connectivity, irreducibility, and controllability, we now consider some degenerate three-level systems.

Example 1: For the level-degenerate system shown in Fig. 1(a), the bare and coupling Hamiltonians read

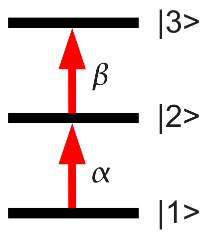

FIG. 2. (Color online) The transition-degenerate three-level system shown in the figure is irreducible for any nonzero values of the α and β coupling constants. The system is uncontrollable when $|\alpha|=|\beta|$.

$$H_0=\begin{pmatrix}-1 & 0 & 0\\ 0 & 1/2 & 0\\ 0 & 0 & 1/2\end{pmatrix}, \quad H_1=\begin{pmatrix}0 & \alpha & \beta\\ \alpha & 0 & 0\\ \beta & 0 & 0\end{pmatrix}.$$

This system is connected in the $\{|i\rangle\}_{1\le i\le 3}$ basis but another orthogonal basis exists where it is not. Using the primed basis $|1'\rangle=|1\rangle$, $|2'\rangle=\cos(\theta)|2\rangle+\sin(\theta)|3\rangle$ and $|3'\rangle=\sin(\theta)|2\rangle-\cos(\theta)|3\rangle$ with $\cos(\theta)=\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$ and $\sin(\theta)=\frac{\beta}{\sqrt{\alpha^2+\beta^2}}$, H_0 remains unchanged due to degeneracy but the coupling Hamiltonian now takes the form

$$H_1'=\sqrt{\alpha^2+\beta^2}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

as illustrated in Fig. 1(b). Connectivity does not hold in the primed basis: the system is reducible and therefore uncontrollable, independent of the coupling values α and β.

Uncontrollability could also have been inferred from the algebra $\mathfrak{g}$ generated by iH_0 and iH_1: $\mathfrak{g}=\mathrm{u}(1)\oplus\mathrm{su}(2)$ is a proper subalgebra of su(3). The su(2) part of $\mathfrak{g}$ corresponds to the generators steering the $|1'\rangle,|2'\rangle$ two-level system, while the one-dimensional u(1) factor commutes with it. Reducibility, which is already apparent in the block-diagonal form of H_1', can also be seen in the conservation of the population of state $|3'\rangle$.

Example 2: For the transition-degenerate system shown in Fig. 2, the bare and coupling Hamiltonians read

$$H_0=\begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}, \quad H_1=\begin{pmatrix}0 & \alpha & 0\\ \alpha & 0 & \beta\\ 0 & \beta & 0\end{pmatrix}.$$

This system is irreducible. This can be checked by looking for operators commuting with both H_0 and H_1. Because H_0 is nondegenerate, operators commuting with H_0 must be diagonal in the basis diagonalizing H_0. Such operators that furthermore commute with H_1 are necessarily proportional to the identity.

This system is not necessarily controllable. Although the algebra $\mathfrak{g}$ generated by iH_0 and iH_1 is the full su(3) algebra when $|\alpha|\neq|\beta|$, when $|\alpha|=|\beta|\mathfrak{g}$ is only a so(3) subalgebra of su(3), which leads to an uncontrollable system.

These two simple examples are known [8], and a detailed study of their density matrix decomposition has been given in [18]. The first example shows that when the system possesses level degeneracies, connectivity might be basis dependent. In such cases, irreducibility allows one to better characterize noncontrollable systems. The second example shows that irreducibility is not sufficient to ensure controllability.

III. GENERAL TREATMENT OF IRREDUCIBLE UNCONTROLLABLE SYSTEMS

When the system is reducible, uncontrollability follows from forbidden transitions between invariant subspaces; the signature of these invariant subspaces is the existence of operators that commute with the entire dynamical algebra. Irreducible cases do not satisfy such a constraint. Although irreducibility insures a light version of controllability—any initial state can be steered so as to partially overlap with any target state—it does not necessarily lead to a controllable system. Our second example above suggests that transition degeneracies combined with particular values of the coupling constants lead to irreducible but uncontrollable systems. We now turn to the task of identifying such irreducible and uncontrollable systems.

A. Irreducible dynamical algebras

The irreducibility of $\mathcal{H}$ under the algebra $\mathfrak{g}$ imposes a specific structure on both $\mathfrak{g}$ and $\mathcal{H}$. Recall that elements of $\mathfrak{g}\subset\mathrm{su}(N)$ are traceless skew-Hermitian operators on $\mathcal{H}$ having the form iH with H Hermitian. Because $\mathfrak{g}$ is invariant under Hermitian conjugation, it is reductive [19], i.e., it takes the form $\mathfrak{g}=\mathfrak{z}(\mathfrak{g})\oplus\mathfrak{g}_{ss}$, where $\mathfrak{z}(\mathfrak{g})$ is the center of $\mathfrak{g}$ and $\mathfrak{g}_{ss}=[\mathfrak{g},\mathfrak{g}]$ is a semisimple Lie algebra [20]. By definition, elements of the center commute with the whole algebra: $[\mathfrak{z}(\mathfrak{g}),\mathfrak{g}]=0$. From the second characterization of irreducibility, $\mathfrak{z}(\mathfrak{g})=0$. It follows that $\mathfrak{g}$ is semisimple, i.e., $\mathfrak{g}=[\mathfrak{g},\mathfrak{g}]$. This has two consequences. First, the semisimplicity of $\mathfrak{g}$ implies that G is compact [21] and this implies that $A=G$ [2]. Second, $\mathfrak{g}$ admits a unique decomposition $\mathfrak{g}=\oplus_{m=1}^{M}\mathfrak{g}_m$, where $\mathfrak{g}_m$'s are simple Lie algebras, orthogonal with respect to the trace induced Hermitian product, which furthermore satisfy $[\mathfrak{g}_m,\mathfrak{g}_m]=\mathfrak{g}_m$ and $[\mathfrak{g}_m,\mathfrak{g}_n]=0$ when $m\neq n$.

The structure of $\mathfrak{g}$, in turn, induces a parallel structure on $\mathcal{H}$. Because $\mathcal{H}$ carries an irreducible and faithful representation of $\mathfrak{g}=\oplus_{m=1}^{M}\mathfrak{g}_m$, it has the Hilbert space structure of a multipartite system, i.e., $\mathcal{H}=\otimes_{m=1}^{M}\mathcal{H}_m$, where each $\mathcal{H}_m$ carries an irreducible representation of $\mathfrak{g}_m$ [22] $\mathfrak{g}_m$ does not affect $\mathcal{H}_n$ for $n\neq m$. Such a situation, where a tensorial decomposition of $\mathcal{H}$ emerges from a given set of operators, has been described previously [13,23]. Because the $\mathfrak{g}_m$ algebras commute with each other, they appear as sets of independent degrees of freedom manipulating the system. The decomposition $\mathcal{H}=\otimes_{m=1}^{M}\mathcal{H}_m$ is especially relevant because it is the finest possible tensorial decomposition of $\mathcal{H}$ for which G transforms product states into product states (see Appendix). This follows from the simplicity of the $\mathfrak{g}_m$ algebras, i.e., the fact that they cannot be further decomposed into a direct sum of commuting algebras.

B. Algebraic criterion for irreducible uncontrollable systems

We now use the structure of $\mathfrak{g}$ to assess controllability. We separate the $M=1$ case, where $\mathfrak{g}$ is simple, from the case where $M \geq 2$, where $\mathfrak{g}$ is semisimple but not simple. When $M \geq 2$, independent of the nature of the $\mathfrak{g}_m$ algebras, the system is not controllable. Intuitively, entangled states relative to the decomposition of $\mathcal{H}$ cannot be reached from product states. When $M=1$, the nature of $\mathfrak{g}$ dictates controllability: the system is controllable if $\mathfrak{g}$ is su(N) or sp($N/2$) [5], and otherwise it is uncontrollable.

Proceeding further, we now identify the admissible dynamical algebras for the irreducible $M=1$ cases. This is fundamental since it also applies to the general case: if M is arbitrary, these algebras give the possible factors in the decomposition of $\mathfrak{g}=\oplus_{m=1}^{M}\mathfrak{g}_m$ into simple Lie algebras. The general criterion for an irreducible uncontrollable system is that *the fundamental representation spanned by* su(N) *remains irreducible in the subalgebra of* su(N) *spanned by the generators*. Limiting ourselves to $N \leq 9$, in the following table we tabulate simple subalgebras of su(N) for which the fundamental representation of su(N) remains irreducible [24]. These algebras organize into the following chains of simple Lie subalgebras:

$$N=2 \qquad \mathfrak{su}(2)$$

$$N=3 \qquad \mathfrak{so}(3) \subset \mathfrak{su}(3)$$

$$N=4 \qquad \mathfrak{su}(2) \subset \mathfrak{sp}(2) \subset \mathfrak{su}(4)$$

$$N=5 \qquad \mathfrak{so}(3) \subset \mathfrak{so}(5) \subset \mathfrak{su}(5)$$

$$N=6 \qquad \left\{ \begin{array}{l} \qquad\qquad\quad \mathfrak{so}(6) \subset \\ \mathfrak{su}(2) \subset \mathfrak{sp}(3) \subset \quad \mathfrak{su}(6) \\ \qquad\qquad\quad \mathfrak{su}(3) \subset \end{array} \right.$$

$$N=7 \qquad \mathfrak{so}(3) \subset \mathfrak{g}_2 \subset \mathfrak{so}(7) \subset \mathfrak{su}(7)$$

$$N=8 \qquad \left\{ \begin{array}{l} \mathfrak{su}(2) \subset \mathfrak{sp}(4) \subset \\ \mathfrak{so}(7) \subset \mathfrak{so}(8) \subset \quad \mathfrak{su}(8) \\ \mathfrak{su}(3) \subset \end{array} \right.$$

$$N=9 \qquad \mathfrak{so}(3) \subset \mathfrak{so}(9) \subset \mathfrak{su}(9).$$

If the system is irreducible and if its dynamical algebra is simple ($M=1$), then this algebra is to be found in the given list. The semisimple (but nonsimple) dynamical algebras of irreducible systems are not listed. For instance, for $N=4$, we do not include the su(2)$\oplus$su(2) subalgebra of su(4) that preserves irreducibility. This case is easy to see however because $N=4=N_1 \times N_2 = 2 \times 2$ and su(4) is an admissible irreducible algebra for both dimensions N_1 and N_2. [There is also a reducible imbedding of su(2)$\oplus$su(2) in su(4).] For N prime, $\mathcal{H}$ cannot be given a tensor product structure. Thus the algebra chains given for $N=3$, 5, and 7 completely list the dynamical algebras of irreducible systems.

Among the subalgebras on this list, only sp($N/2$) and su(N) give rise to controllable systems. The remainder of the subalgebras corresponds to irreducible uncontrollable systems. Their irreducibility implies that some nonzero overlap can be achieved between any initial state and any final state, but their uncontrollability implies that in general an overlap with absolute value of unity cannot be attained.

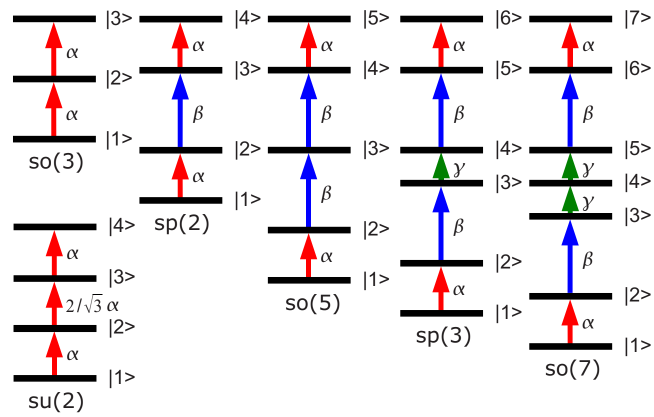

FIG. 3. (Color online) Energy levels and couplings for the series sp($N/2$)$\subset$su(N) (N even) and so(N)$\subset$su(N) (N odd). The coupling strengths and degeneracies are organized in a symmetric chain. These systems are controllable for N even and uncontrollable for N odd. sp(4) and so(9) symmetry cases are not shown; they have a similar structure. The su(2)$\subset$su(4) embedding shows up as a special case of the sp(2)$\subset$su(4) symmetry when an extra symmetry is present.

Although the dynamical algebra reflects the internal structure of the controls, this mere structure does not determine how the controls steer the system. Both the three-level system examples have a dynamical algebra based on su(2) or the isomorphic algebra so(3). However, one system is reducible and the other is not. One must therefore distinguish between dynamical algebras having the same structure but leading to inequivalent physical situations.

The following criterion can be used to distinguish equivalent from inequivalent physical systems. Two systems with dynamical algebras $\mathfrak{g}$ and $\mathfrak{g}'$ are equivalent if a T in SU(N) exists such that $\mathfrak{g}'=T\mathfrak{g}T^{\dagger}$. Then, up to a basis change given by T, the two algebras are the same set of matrices. Not only does this imply that $\mathfrak{g}$ and $\mathfrak{g}'$ have the same structure but also that $\mathfrak{g}$ and $\mathfrak{g}'$ have exactly the same irreducibility properties. This equivalence is well known and corresponds to the equivalence between representations of algebras. In particular, reducible and irreducible cases are inequivalent. Equivalence classes of irreducible representations of all simple compact Lie algebras are known and classified. From this classification, one can obtain $\mathfrak{g}$ explicitly as matrices.

C. Examples of building blocks of irreducible systems

In Figs. 3–6, the generators of the irreducible dynamical algebras of the list above are shown as energy diagrams. Degenerate frequency transitions are indicated by arrows of the same color. Their coupling strengths are in a fixed ratio; this is indicated by identical Greek letters. Transition frequencies corresponding to different colors are free parameters in the H_0 bare Hamiltonian. Each different Greek letter also corresponds to a nonzero free parameter in H_1. For each given diagram, the algebra generated by iH_0 and iH_1 is the same irreducible algebra for all values of the parameters, except for some exceptional values which correspond to an additional symmetry.

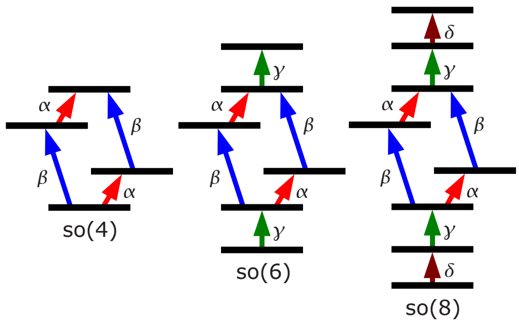

FIG. 4. (Color online) Energy levels and couplings for the series so(N)⊂su(N) for N even. These irreducible and uncontrollable systems appear for all even N. The nonsimple case so(4)=su(2) ⊕su(2) is included to show the similarity of its coupling structure. The so(N) (N≥6 even) symmetries thus appear as extensions of the so(4) case.

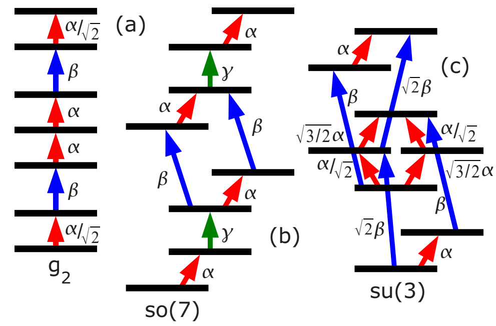

FIG. 6. (Color online) (a) The $\mathfrak{g}_2$ symmetry appears as a particular case of so(7) symmetry of the seven-level system (see Fig. 3) and contains a so(3) spin-3 symmetry as a particular case. [(b) and (c)] The so(7) and su(3) symmetries of the eight-level system appear as additional symmetries of the so(8) case (see Fig. 4)

The given diagrams therefore depict only the generic case of a given symmetry. In fact, all cases where an additional *irreducible* symmetry exists can be directly seen from the chain structure of our subalgebra list. But additional symmetries might also lead to reducible systems.

Because several coupling schemes can lead to the same dynamical algebra, presenting each case in the form of an energy diagram is necessarily arbitrary. As a rule, we give the minimum number of couplings such that if one of them is removed the system becomes reducible. Also, we choose the couplings to highlight connections between symmetries of the same chain and regularities in the coupling structure that appear for similar types of dynamical algebras.

Indeed, regardless of N, some subalgebra types always appear, for example, the embeddings sp($N/2$)⊂su(N) for N ≥4 (even) and so(N)⊂su(N) for N≥3 (odd). Although, unlike the so(N) embeddings, the sp($N/2$) embeddings lead to controllable systems, these two types of algebras lead to a similar coupling structure, as shown in Fig. 3. su(2) also appears for every N as a subsymmetry of the previous cases: su(2)⊂sp($N/2$) for N is even and so(3)⊂so(N) for N is odd.

so(N)⊂su(N) for N≥4 even is another series. It gives rise to an algebra chain different from the one of sp($N/2$). As shown in Fig. 4, their coupling structure extends the so(4) =su(2)⊕su(2)⊂su(4) symmetry type.

The embeddings in our list not falling into these schemes are su(3)⊂su(6), $\mathfrak{g}_2$⊂so(7), so(7)⊂su(8) and su(3)⊂su(8). They are shown in Figs. 5 and 6.

IV. CONCLUSION

Uncontrollable finite-dimensional quantum systems are scarce: among all systems, controllable systems are an open and dense subset [10]. This theoretical claim also has numerical support [17]. In other words, the generic N-level system is controllable. Such a result could make the investigation of singular uncontrollable systems endowed with exceptional symmetries seem pointless. There are several reasons why this is not the case:

(1) For systems lying close to these symmetric exceptions, the control may be theoretically attainable but experimentally demanding, possibly making these systems experimentally uncontrollable for all practical purposes.

(2) Important model systems, such as the harmonic oscillator, spin, or effective-spin systems [6] and multipartite systems, owe their specific properties to symmetry. This very same symmetry renders them uncontrollable, suggesting that symmetric singular cases are not minor exceptions but important cases.

(3) In practice, complete controllability might be too demanding. Sometimes, only limited control is desired, for example, driving a specific initial state to a specific target. In other cases, one might want to restrict the family of control fields to avoid reaching a particular intermediate state. This corresponds to deliberately seeking uncontrollability of a particular kind [25]. These examples motivate the study of partially controllable systems and the analysis of the physical basis for their uncontrollability.

In our investigation of uncontrollability, the notion of irreducibility plays a central role. We showed that when the system is degenerate or has degenerate transitions, irreducibility has an advantage over the previously introduced notion of connectivity: it distinguishes reducible systems that are always uncontrollable from irreducible ones that may or may not be controllable, without any hypothesis on degeneracy.

Because irreducible systems might also be uncontrollable, irreducibility allows one to distinguish between different types of uncontrollable systems. Using the fact that irreducibility is a fundamental notion of representation theory, we fully exploited the structure of the dynamical algebra and showed that irreducible systems are characterized by a semisimple dynamical algebra *and* a Hilbert space carrying an irreducible representation of this algebra. As a consequence, uncontrollable irreducible systems are generalized entangled. Among these systems, those having a simple dynamical al-

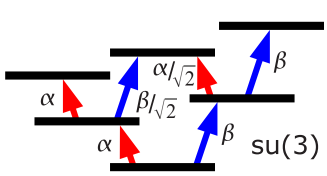

FIG. 5. (Color online) su(3)⊂su(6) not belonging to the sp(3) or so(6) subalgebra chains. It exhibits a different structure.

gebra appear as fundamental building blocks upon which all irreducible systems are based. We showed low-dimensional systems having these simple dynamical algebras.

ACKNOWLEDGMENTS

D.J.T. wishes to acknowledge funding from the German-Israel Foundation for Scientific Research and Development and the EMALI network of the EU under Contract No. MRTN-CT-2006-035369. This work is made possible in part by the historic generosity of the Harold Perlman Family. We thank Professor Alexander Kirillov, Jr. for helpful correspondence and Jeremy Schiff for a thorough reading of the paper and helpful conversations.

APPENDIX: $\mathcal{H}$ FINEST TENSORIAL DECOMPOSITION

We assume that $\mathfrak{g}=\oplus_{m=1}^{M}\mathfrak{g}_m$ is the unique decomposition of $\mathfrak{g}$ into simple Lie algebras. $\mathcal{H}$ is irreducible for $\mathfrak{g}$. It has a corresponding unique tensorial product decomposition $\mathcal{H}=\otimes_{m=1}^{M}\mathcal{H}_m$. We will show that $\mathcal{H}=\otimes_{m=1}^{M}\mathcal{H}_m$ is the finest tensorial decomposition of $\mathcal{H}$ for which G conserves product states. More precisely, we show that if G conserves the product states of some tensorial product decomposition $\mathcal{H}=\otimes_{p=1}^{P}\mathcal{H}'_p$, then necessarily, $\mathcal{H}'_p=\otimes_{m_p\in M_p}\mathcal{H}_{m_p}$ where $\{M_p\}_{1\le p\le P}$ is a partition of $\{1,\ldots,M\}$. The proof is as follows. $\mathfrak{g}$ is semisimple and G preserves product states, so $\mathfrak{g}\subset\oplus_{p=1}^{P}\mathrm{su}(\mathcal{H}'_p)$. We therefore have that $\mathfrak{g}=\oplus_{p=1}^{P}\mathfrak{g}'_p$ with $\mathfrak{g}'_p=\mathfrak{g}\cap\mathrm{su}(\mathcal{H}'_p)$. For each p, $\mathfrak{g}'_p$ is an ideal of $\mathfrak{g}$. Indeed, $[\mathfrak{g}'_p,\mathfrak{g}]=\{\mathfrak{g}\cap\mathrm{su}(\mathcal{H}'_p),\oplus_{p'=1}^{P}[\mathfrak{g}\cap\mathrm{su}(\mathcal{H}'_{p'})]\}$ but $[\mathrm{su}(\mathcal{H}'_p),\mathrm{su}(\mathcal{H}'_{p'})]=\delta_{p,p'}\mathrm{su}(\mathcal{H}'_p)$ so $[\mathfrak{g}'_p,\mathfrak{g}]\subset[\mathfrak{g}\cap\mathrm{su}(\mathcal{H}'_p),\mathfrak{g}\cap\mathrm{su}(\mathcal{H}'_p)]\subset\mathfrak{g}'_p$.

Ideals of $\mathfrak{g}$ are of the form $\oplus_{m\in\mathcal{M}}\mathfrak{g}_m$, where $\mathcal{M}$ is a subset of $\{1,\ldots,M\}$. For each p then, $\mathfrak{g}'_p=\oplus_{m_p\in M_p}\mathfrak{g}_{m_p}$ with M_p as a subset of $\{1,\ldots,M\}$. Because $\mathfrak{g}=\oplus_{m=1}^{M}\mathfrak{g}_m=\oplus_{p=1}^{P}\mathfrak{g}'_p=\oplus_{p=1}^{P}\oplus_{m_p\in M_p}\mathfrak{g}_{m_p}$, it follows from the uniqueness of the decomposition of $\mathfrak{g}$ into simple algebras that $\{M_p\}_{1\le p\le P}$ is a partition of $\{1,\ldots,M\}$.

$\mathcal{H}'_p$ is irreducible for $\mathfrak{g}'_p$. If not, this contradicts the fact that $\mathcal{H}$ is irreducible for $\mathfrak{g}$. Applying our theorem, $\mathcal{H}'_p=\otimes_{m_p\in M_p}\mathcal{H}''_{m_p}$, where $\mathcal{H}''_{m_p}$ is irreducible for $\mathfrak{g}_{m_p}$. Finally, $\mathcal{H}=\otimes_{m=1}^{M}\mathcal{H}_m=\otimes_{p=1}^{P}\mathcal{H}'_p=\otimes_{p=1}^{P}\otimes_{m_p\in M_p}\mathcal{H}''_{m_p}$ are two tensorial decompositions of $\mathcal{H}$ adapted to $\mathfrak{g}=\oplus_{m=1}^{M}\mathfrak{g}_m$. This decomposition is unique so $\mathcal{H}''_{m_p}=\mathcal{H}_{m_p}$ and $\mathcal{H}'_p=\otimes_{m_p\in M_p}\mathcal{H}_{m_p}$. □

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