

Correlation matrices of two-mode bosonic systems

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We present a detailed analysis of all the algebraic conditions an arbitrary 4×4 symmetric matrix must satisfy in order to represent the correlation matrix of a two-mode bosonic system. Then, we completely clarify when this arbitrary matrix can represent the correlation matrix of a separable or entangled Gaussian state. In this analysis, we introduce alternative sets of conditions, which are expressed in terms of local symplectic invariants.

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I. INTRODUCTION

Statistical moments of second order represent a key element of the quantum mechanical paradigm. Besides providing the “language” in which the uncertainty principles are expressed, they serve as indicators in a number of applications of the theory, with both applied and fundamental interest. In particular, second moments are central to the description of bosonic fields in second quantization (as is the case, for instance, in quantum optics, where second order coherence is characterized in terms of second moments) and of nonrelativistic particles in first quantization. Such systems do, in fact, share the same formal description [1], which hence extends its domain to a variety of fields ranging from atomic physics to quantum optics, from superconductors’ physics to nanomechanical systems.

During the last decade, the rise of quantum information science has renewed the focus on these areas because of their potential for coherent quantum manipulations, and has concomitantly brought new problems and questions to the attention of theorists, which resulted in the birth of the field of “continuous variable” (CV) quantum information [2] (see Refs. [3–5] for some literature on CV quantum computation, CV quantum teleportation, and CV quantum key distribution, respectively). A systematic analysis of the properties of continuous variable quantum states inferred from the structure of their second moments has been thereby carried out, which lead to a well-established, extensive theoretical picture (see, for instance [6–9]). Such an analysis proved to be most relevant also in view of the experimental prominence of the class of *Gaussian states* [10,11], which are completely determined by their first and second moments. In first, seminal endeavors [6,7], the qualitative characterization of the quantum correlations (“entanglement”) of Gaussian states of two degrees of freedom has been successfully achieved. Now, while very well established and relatively simple, this result is still often expressed in a nonrigorous or incomplete manner, which is prone to confound the unacquainted reader. Because it constitutes one of the basic building blocks on which the theoretical characterization of Gaussian states has been constructed, it seems to us extremely important for it to be re-derived and re-expressed in a rigorous manner and full detail: this is one of the motifs and central aims of the present paper.

In general, the main question we will address and rigorously answer is the following: (i) What are the algebraic conditions that a 4×4 real symmetric matrix \mathbf{V} must satisfy in order to represent the correlation matrix of a two-mode bosonic system [12]? Then, we shall move on to answer a closely connected question: (ii) What are the algebraic conditions to be satisfied by \mathbf{V} in order to represent the correlation matrix of a separable (or entangled) Gaussian state of two bosonic modes?

Both these questions are thoroughly answered providing complete sets of conditions, which are expressed in terms of global or local symplectic invariants. In particular, the set of local conditions, i.e., given in terms of local invariants, is completely new in literature. Then, by specifying some of these algebraic results in the case of positive-definite matrices, we can make a direct comparison with the previous work of Ref. [6] and provide a rigorous and correct interpretation of its seminal results.

The paper is organized as follows. In Sec. II we review basic notions about bosonic states, correlation matrices, and symplectic transformations. In Sec. III we present the mathematical tools to be used in the derivations of Secs. IV and V. In Sec. IV we provide two sets of algebraic conditions for the physical genuinity of the correlation matrix of two bosonic modes. These conditions are expressed in terms of global or local symplectic invariants. In Sec. V we provide similar conditions for separability. Next, in Sec. VI, we specify some of our results for making a direct comparison with the previous achievements of Ref. [6]. Finally, Sec. VII is for conclusions. Notice that in the paper we will denote by $\mathcal{M}(n, \mathbb{R})$ the set of $n \times n$ real matrices. Then, we use the compact notation

$$\mathcal{S}(n, \mathbb{R}) = \{\mathbf{M} \in \mathcal{M}(n, \mathbb{R}) : \mathbf{M} = \mathbf{M}^T\}, \quad (1)$$

for the set of the $n \times n$ symmetric real matrices, and

$$\mathcal{P}(n, \mathbb{R}) = \{\mathbf{M} \in \mathcal{S}(n, \mathbb{R}) : \mathbf{M} > 0\}, \quad (2)$$

for the set of the $n \times n$ positive-definite real matrices. We will also consider the set (group) of proper rotations

$$\text{SO}(n) = \{\mathbf{M} \in \mathcal{M}(n, \mathbb{R}) : \mathbf{M}^T \mathbf{M} = \mathbf{I}, \det \mathbf{M} = 1\}. \quad (3)$$

II. BOSONIC SYSTEMS AND SYMPLECTIC TRANSFORMATIONS

A. Correlation matrix of a bosonic system

Let us consider a bosonic system composed by n modes, labeled by an index k . Such a system can be described by an infinite-dimensional Hilbert space $\mathcal{H} = \otimes_{k=1}^n \mathcal{H}_k$ and a vector of quadrature operators $\hat{\mathbf{x}}^T := (\hat{q}_1, \hat{p}_1, \dots, \hat{q}_n, \hat{p}_n)$. In particular, these operators satisfy the commutation relations

$$[\hat{x}_l, \hat{x}_m] = 2i\Omega_{lm}, \tag{4}$$

where $l, m = 1, \dots, 2n$ and Ω_{lm} are the entries of the symplectic form

$$\mathbf{\Omega} = \bigoplus_{k=1}^n \boldsymbol{\omega}, \quad \boldsymbol{\omega} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{5}$$

Let us denote by $\mathcal{D}(\mathcal{H})$ the space of density operators acting on \mathcal{H} . It is known that an arbitrary density operator $\rho \in \mathcal{D}(\mathcal{H})$ has an equivalent representation in a real symplectic space $\mathcal{K} = \mathcal{K}(\mathbb{R}^{2n}, \mathbf{\Omega})$ called the *phase space*. This is a real vector space which is spanned by the singular eigenvalues $\mathbf{x}^T = (q_1, p_1, \dots, q_n, p_n)$ of $\hat{\mathbf{x}}^T$ (representing the ‘‘continuous variables’’ of the system) and associated to a symplectic product $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{\Omega} \mathbf{v}$. In this space, ρ corresponds to a quasiprobability distribution known as the Wigner function $W = W(\mathbf{x})$. In general, such a function is fully characterized by the entire set of its statistical moments [13]. However, in the particular case of Gaussian states, the Wigner function is Gaussian and, therefore, fully characterized by the first and second moments only. These two moments are also known as the *displacement vector* $\mathbf{d} := \langle \hat{\mathbf{x}} \rangle$ and the *correlation matrix* (CM) \mathbf{V} , whose generic entry is defined by

$$\mathbf{V}_{lm} := \frac{1}{2} \langle \Delta \hat{x}_l \Delta \hat{x}_m + \Delta \hat{x}_m \Delta \hat{x}_l \rangle, \tag{6}$$

where $\Delta \hat{x}_l := \hat{x}_l - \langle \hat{x}_l \rangle$. According to the definition of Eq. (6), the CM of n bosonic modes is a real and symmetric matrix in $2n$ dimension, i.e., $\mathbf{V} \in \mathcal{S}(2n, \mathbb{R})$. As a direct consequence of Eq. (4), such a matrix must also satisfy the uncertainty principle [9,14]

$$\mathbf{V} + i\mathbf{\Omega} \geq 0. \tag{7}$$

In other words, an arbitrary $\mathbf{V} \in \mathcal{S}(2n, \mathbb{R})$ is a *bona fide* quantum CM if and only if Eq. (7) holds. Equivalently, we can introduce the set of n -mode quantum CM’s to be defined as

$$q\mathcal{CM}(n) := \{\mathbf{V} \in \mathcal{S}(2n, \mathbb{R}) : \mathbf{V} + i\mathbf{\Omega} \geq 0\}. \tag{8}$$

Notice that the condition of Eq. (7) implies a first relevant constraint on the matrix \mathbf{V} :

Lemma 1 (Definite positivity of \mathbf{V}). For every $\mathbf{V} \in \mathcal{S}(2n, \mathbb{R})$ satisfying $\mathbf{V} + i\mathbf{\Omega} \geq 0$, one has

$$\mathbf{V} > 0. \tag{9}$$

Proof. Let $\mathbf{u} \in \mathbb{R}^{2n}$, then $0 \leq \mathbf{u}^T \mathbf{V} \mathbf{u} + i\mathbf{u}^T \mathbf{\Omega} \mathbf{u} = \mathbf{u}^T \mathbf{V} \mathbf{u}$ because $\mathbf{\Omega}$ is antisymmetric. Hence $\mathbf{V} \geq 0$. To prove definite positivity suppose, *ad absurdum*, that a nontrivial real vector \mathbf{u}_0 exists such that $\mathbf{u}_0^T \mathbf{V} \mathbf{u}_0 = 0$. Another vector $\mathbf{u}_1 \in \mathbb{R}^{2n}$ such that $\mathbf{u}_1^T \mathbf{\Omega} \mathbf{u}_0 \neq 0$ always exists as $\text{nul}(\mathbf{\Omega}) = 0$. As a consequence,

one can always construct a set of complex vectors $\mathbf{z} = \mathbf{u}_0 + i\mathbf{u}_1$, for $a \in \mathbb{R}$, such that

$$0 \leq (\mathbf{z}^*)^T (\mathbf{V} + i\mathbf{\Omega}) \mathbf{z} = 2a\mathbf{u}_1^T \mathbf{\Omega} \mathbf{u}_0 + a^2 \mathbf{u}_1^T \mathbf{V} \mathbf{u}_1. \tag{10}$$

Values of a such that the inequality above is violated can always be found regardless of the values of $\mathbf{u}_1^T \mathbf{\Omega} \mathbf{u}_0$ and $\mathbf{u}_1^T \mathbf{V} \mathbf{u}_1$. This implies $\mathbf{u}^T \mathbf{V} \mathbf{u} \neq 0$ for every $\mathbf{u} \in \mathbb{R}^{2n}$ and, therefore, $\mathbf{V} > 0$. ■

According to lemma 1, we then have

$$q\mathcal{CM}(n) \subseteq \mathcal{P}(2n, \mathbb{R}). \tag{11}$$

Furthermore, it is trivial to show positive-definite matrices which violate Eq. (7), so that we actually have

$$q\mathcal{CM}(n) \subset \mathcal{P}(2n, \mathbb{R}). \tag{12}$$

Notice that definite positivity is the only requirement for a real symmetric matrix to be a *classical* correlation matrix.

B. Symplectic transformations

The most general real linear transformation of the quadratures

$$\mathbf{S} : \hat{\mathbf{x}} \rightarrow \hat{\mathbf{x}}' := \mathbf{S} \hat{\mathbf{x}}, \tag{13}$$

must preserve Eq. (4) in order to be a physical operation. This happens when the matrix $\mathbf{S} \in \mathcal{M}(2n, \mathbb{R})$ preserves the symplectic form of Eq. (5), i.e.,

$$\mathbf{S} \mathbf{\Omega} \mathbf{S}^T = \mathbf{\Omega}. \tag{14}$$

The set of all the matrices $\mathbf{S} \in \mathcal{M}(2n, \mathbb{R})$ satisfying Eq. (14) forms the so-called *real symplectic group*

$$\mathcal{S}_p(2n, \mathbb{R}) := \{\mathbf{S} \in \mathcal{M}(2n, \mathbb{R}) : \mathbf{S} \mathbf{\Omega} \mathbf{S}^T = \mathbf{\Omega}\}, \tag{15}$$

whose elements are called *symplectic* or *canonical transformations*. As a consequence, the most general real linear transformation in phase space $\mathbf{S} : \mathbf{x} \rightarrow \mathbf{x}' := \mathbf{S} \mathbf{x}$ must be symplectic. Its action on the Wigner function is simply given by $W(\mathbf{x}) \rightarrow W(\mathbf{S}^{-1} \mathbf{x})$, so that the displacement is linearly modified while the CM is transformed according to the congruence

$$\mathbf{V} \rightarrow \mathbf{S} \mathbf{V} \mathbf{S}^T. \tag{16}$$

Symplectic transformations are very important since every \mathbf{S} acting in the phase space \mathcal{K} corresponds to a *Gaussian* unitary $\hat{U}(\mathbf{S})$ acting on the Hilbert space \mathcal{H} , i.e., a unitary operator preserving the Gaussian statistics of the quantum states. These unitaries are the ones generated by bilinear Hamiltonians and can always be decomposed into single-mode squeezers and multimode interferometers [10,15]. In particular, local symplectic transformations

$$\mathbf{S} = \bigoplus_{k=1}^n \mathbf{S}_k \in \mathcal{S}_p(2, \mathbb{R}) \oplus \dots \oplus \mathcal{S}_p(2, \mathbb{R}) \tag{17}$$

correspond to local Gaussian unitaries

$$\hat{U}(\mathbf{S}) = \bigotimes_{k=1}^n \hat{U}_k. \quad (18)$$

Local symplectic transformations can always be decomposed as products of local rotations and local squeezings. In fact, thanks to the following characterization

$$\mathcal{S}_p(2, \mathbb{R}) = \{\mathbf{S} \in \mathcal{M}(2, \mathbb{R}) : \det \mathbf{S} = 1\}, \quad (19)$$

we have that every $\mathbf{S} \in \mathcal{S}_p(2, \mathbb{R})$ can be expressed as a product of proper rotations

$$\mathbf{R}(\varphi) := \begin{pmatrix} \sin \varphi & -\cos \varphi \\ \cos \varphi & \sin \varphi \end{pmatrix}, \quad (20)$$

and squeezing matrices

$$\mathbf{S}(\xi) := \begin{pmatrix} \xi^{1/2} & 0 \\ 0 & \xi^{-1/2} \end{pmatrix}, \quad \xi > 0. \quad (21)$$

Besides, it is also important to identify which quantities of a CM are preserved under the application of symplectic transformations. In general, for a given CM \mathbf{V} , we say that a functional

$$f: \mathbf{V} \rightarrow f(\mathbf{V}) \in \mathbb{R} \quad (22)$$

is a (global) symplectic invariant if

$$f(\mathbf{V}) = f(\mathbf{S}\mathbf{V}\mathbf{S}^T), \quad (23)$$

for every $\mathbf{S} \in \mathcal{S}_p(2n, \mathbb{R})$. Then, we say that $f(\mathbf{V})$ is a local symplectic invariant if Eq. (23) holds for every $\mathbf{S} \in \mathcal{S}_p(2, \mathbb{R}) \oplus \cdots \oplus \mathcal{S}_p(2, \mathbb{R})$ [16]. Notice that we can extend the notion of symplectic invariance also to a property of a matrix. For instance, the definite positivity of \mathbf{V} is a global symplectic invariant since $\mathbf{V} > \mathbf{0} \Rightarrow \mathbf{S}\mathbf{V}\mathbf{S}^T > \mathbf{0}$.

III. SYMPLECTIC ANALYSIS

Here, we review some basic tools that can be used for the symplectic manipulation of the CM's. In particular, the central tool in this trade is Williamson's theorem [17], which ensures the possibility of carrying out the symplectic diagonalization of real matrices in even dimension *under the definite positivity constraint* (as in the case of the CM's).

Lemma 2 (Williamson's theorem). For every $\mathbf{V} \in \mathcal{P}(2n, \mathbb{R})$, there exists a symplectic matrix $\mathbf{S} \in \mathcal{S}_p(2n, \mathbb{R})$ such that

$$\mathbf{S}\mathbf{V}\mathbf{S}^T = \begin{pmatrix} \nu_1 & & & & & \\ & \nu_1 & & & & \\ & & \ddots & & & \\ & & & \nu_n & & \\ & & & & \nu_n & \\ & & & & & \nu_n \end{pmatrix} := \mathbf{W} > \mathbf{0}, \quad (24)$$

where the n positive quantities $\{\nu_1, \dots, \nu_n\}$ are the ‘‘symplectic eigenvalues’’ of \mathbf{V} , and the diagonal matrix \mathbf{W} is its ‘‘Williamson form’’ (or ‘‘normal form’’).

The symplectic spectrum $\{\nu_1, \dots, \nu_n\}$ can be computed as the standard eigenspectrum of the matrix $|i\mathbf{Q}\mathbf{V}|$ where the modulus must be understood in the operatorial sense [18].

The corresponding Williamson form \mathbf{W} is unique up to a permutation of the symplectic spectrum (i.e., of the bosonic modes). Fixing this permutation, the diagonalizing symplectic matrix \mathbf{S} of Eq. (24) is defined up to local rotations $\bigoplus_{k=1}^n \mathbf{R}_k$ with $\mathbf{R}_k \in \text{SO}(2)$. For the sake of completeness, we report in Appendix A a simple proof of Williamson's theorem, originally presented in Ref. [19]. Using this proof, we show in Appendix B an algorithm which finds the diagonalizing symplectic matrix \mathbf{S} of Eq. (24). This algorithm is not the fastest but it can be helpful in studying problems like the optimal discrimination of Gaussian states [20] and the quantum illumination [21].

Let us consider the case of a 4×4 positive-definite matrix $\mathbf{V} \in \mathcal{P}(4, \mathbb{R})$, as in the case of CM's describing two bosonic modes. This matrix can be expressed in the block form

$$\mathbf{V} = \begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^T & \mathbf{B} \end{pmatrix}, \quad (25)$$

where $\mathbf{A}, \mathbf{B} \in \mathcal{S}(2, \mathbb{R})$, and $\mathbf{C} \in \mathcal{M}(2, \mathbb{R})$. In this case the symplectic spectrum $\{\nu_1, \nu_2\} := \{\nu_-, \nu_+\}$ can be computed via the simple formula [22]

$$\nu_{\pm} = \sqrt{\frac{\Delta(\mathbf{V}) \pm \sqrt{\Delta(\mathbf{V})^2 - 4 \det \mathbf{V}}}{2}}, \quad (26)$$

where

$$\Delta(\mathbf{V}) := \det \mathbf{A} + \det \mathbf{B} + 2 \det \mathbf{C}. \quad (27)$$

Here, the quantities $\det \mathbf{A}$, $\det \mathbf{B}$, and $\det \mathbf{C}$ are local symplectic invariants, while $\det \mathbf{V}$ and $\Delta(\mathbf{V})$ are global symplectic invariants, which can also be written as

$$\det \mathbf{V} = \nu_-^2 \nu_+^2, \quad \Delta(\mathbf{V}) = \nu_-^2 + \nu_+^2. \quad (28)$$

Another important tool in the symplectic analysis is the reduction to *standard form* by local symplectic transformations [6,7]. In general, such a reduction holds for symmetric matrices $\mathbf{V} \in \mathcal{S}(4, \mathbb{R})$ with positive diagonal blocks, as we easily show in the following. In particular, it can be applied to positive-definite matrices $\mathbf{V} \in \mathcal{P}(4, \mathbb{R})$ and, therefore, to CMs $\mathbf{V} \in q\mathcal{CM}(2)$.

Lemma 3 (Standard form). For every

$$\mathbf{V} = \begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^T & \mathbf{B} \end{pmatrix} \in \mathcal{S}(4, \mathbb{R}), \quad (29)$$

with $\mathbf{A}, \mathbf{B} > \mathbf{0}$, there exists some $\mathbf{S} \in \mathcal{S}_p(2, \mathbb{R}) \oplus \mathcal{S}_p(2, \mathbb{R})$ such that

$$\mathbf{S}\mathbf{V}\mathbf{S}^T = \begin{pmatrix} a & & c_+ \\ & a & c_- \\ c_+ & & b \\ & c_- & b \end{pmatrix} := \mathbf{V}^I, \quad (30)$$

where the real parameters a, b, c_+, c_- satisfy

$$\det \mathbf{A} = a^2, \quad \det \mathbf{B} = b^2, \quad \det \mathbf{C} = c_+ c_-, \quad (31)$$

and

$$\det \mathbf{V} = \det \mathbf{V}^I = (ab - c_+^2)(ab - c_-^2). \quad (32)$$

Proof. Let us consider a pair of single-mode symplectic transformations $\mathbf{S}_A, \mathbf{S}_B \in \mathcal{S}_p(2, \mathbb{R})$ and a pair of single-mode proper rotations $\mathbf{R}(\theta_A), \mathbf{R}(\theta_B) \in \text{SO}(2)$. By applying the local symplectic transformation

$$\mathbf{S} = \mathbf{R}(\theta_A)\mathbf{S}_A \oplus \mathbf{R}(\theta_B)\mathbf{S}_B \quad (33)$$

to the matrix \mathbf{V} , we get

$$\mathbf{S}\mathbf{V}\mathbf{S}^T = \begin{pmatrix} \mathbf{A}' & \mathbf{C}' \\ \mathbf{C}'^T & \mathbf{B}' \end{pmatrix}, \quad (34)$$

where

$$\mathbf{A}' := \mathbf{R}(\theta_A)(\mathbf{S}_A\mathbf{A}\mathbf{S}_A^T)\mathbf{R}^T(\theta_A), \quad (35)$$

$$\mathbf{B}' := \mathbf{R}(\theta_B)(\mathbf{S}_B\mathbf{B}\mathbf{S}_B^T)\mathbf{R}^T(\theta_B), \quad (36)$$

$$\mathbf{C}' := \mathbf{R}(\theta_A)(\mathbf{S}_A\mathbf{C}\mathbf{S}_B^T)\mathbf{R}^T(\theta_B). \quad (37)$$

Since $\mathbf{A}, \mathbf{B} \in \mathcal{P}(2, \mathbb{R})$, we can apply Williamson's theorem. This means that we can choose \mathbf{S}_A and \mathbf{S}_B such that

$$\mathbf{A}' = \mathbf{R}(\theta_A)a\mathbf{I} \quad \mathbf{R}^T(\theta_A) = a\mathbf{I}, \quad (38)$$

$$\mathbf{B}' = \mathbf{R}(\theta_B)b\mathbf{I} \quad \mathbf{R}^T(\theta_B) = b\mathbf{I}, \quad (39)$$

where $a(b)$ is the symplectic eigenvalue of $\mathbf{A}(\mathbf{B})$ while the angle $\theta_A(\theta_B)$ is arbitrary. Since the pair $\{\theta_A, \theta_B\}$ can be chosen freely, we can always choose a pair $\{\bar{\theta}_A, \bar{\theta}_B\}$ in Eq. (37) such that $\mathbf{C}' = \text{diag}(c_+, c_-)$ [23]. As a consequence, Eq. (34) is globally equal to Eq. (30). Finally, since the transformation of Eq. (33) is local and symplectic, all the determinants relative to the blocks and the global matrix are preserved, so that Eqs. (31) and (32) are trivially implied. ■

IV. GENUINENESS OF A TWO-MODE CORRELATION MATRIX

By applying the symplectic tools of Sec. III, we can now derive very simple algebraic conditions for characterizing the genuineness of a two-mode CM. In other words, starting from a generic 4×4 real and symmetric matrix $[\mathbf{V} \in \mathcal{S}(4, \mathbb{R})]$, we give the algebraic conditions that such a matrix must satisfy in order to represent the CM of two bosonic modes, i.e., a bona fide two-mode quantum CM $[\mathbf{V} \in q\mathcal{CM}(2)]$. As a consequence of Williamson's theorem, we have the following algebraic conditions in terms of *global* symplectic invariants [9].

Theorem 4. An arbitrary $\mathbf{V} \in \mathcal{S}(4, \mathbb{R})$ is a quantum CM if and only if it satisfies

$$\mathbf{V} > 0, \quad \nu_- \geq 1, \quad (40)$$

or, equivalently,

$$\mathbf{V} > 0, \quad \det \mathbf{V} \geq 1, \quad \Delta(\mathbf{V}) \leq 1 + \det \mathbf{V}. \quad (41)$$

Proof. For every $\mathbf{V} \in \mathcal{P}(4, \mathbb{R})$, the application of Williamson's theorem to Eq. (7) gives $\mathbf{V} + i\mathbf{\Omega} \geq 0 \Leftrightarrow \nu_- \geq 1$ (recalling that ν_- is the *smallest* symplectic eigenvalue). Since $\mathbf{V} + i\mathbf{\Omega} \geq 0 \Rightarrow \mathbf{V} > 0$ (see lemma 1), we can write $\mathbf{V} + i\mathbf{\Omega} \geq 0 \Leftrightarrow (\mathbf{V} > 0 \wedge \nu_- \geq 1)$ which proves the bona fide condition of

Eq. (40) for a generic $\mathbf{V} \in \mathcal{S}(4, \mathbb{R})$. Under the definite positivity assumption $\mathbf{V} > 0$, one can also use Eq. (26) to prove the equivalences

$$\begin{aligned} \nu_- \geq 1 &\Leftrightarrow \Delta(\mathbf{V}) - 2 \geq \sqrt{\Delta(\mathbf{V})^2 - 4 \det \mathbf{V}} \Leftrightarrow \max\{2, 2\sqrt{\det \mathbf{V}}\} \\ &\leq \Delta(\mathbf{V}) \leq 1 + \det \mathbf{V} \Leftrightarrow \begin{cases} \det \mathbf{V} \geq 1 \\ 2\sqrt{\det \mathbf{V}} \leq \Delta(\mathbf{V}) \leq 1 + \det \mathbf{V}. \end{cases} \end{aligned} \quad (42)$$

According to Eq. (28) the condition $2\sqrt{\det \mathbf{V}} \leq \Delta(\mathbf{V})$ in Eq. (42) corresponds to $2\nu_- \nu_+ \leq \nu_-^2 + \nu_+^2$, which is trivially satisfied. Then, we have

$$\nu_- \geq 1 \Leftrightarrow \begin{cases} \det \mathbf{V} \geq 1 \\ \Delta(\mathbf{V}) \leq 1 + \det \mathbf{V}, \end{cases} \quad (43)$$

which states the equivalence between to Eqs. (40) and (41), where the underlying assumption $\mathbf{V} > 0$ is also shown. ■

Notice that, crucially, Williamson's theorem could be applied to \mathbf{V} *because of its definite positivity*, which thus implies the existence of well-defined symplectic eigenvalues [24]. The condition $\nu_- \geq 1$ alone is therefore not, by itself, fully equivalent to the uncertainty principle $\mathbf{V} + i\mathbf{\Omega} \geq 0$, unless definite positivity is also assumed. The essential role of the prescription $\mathbf{V} > 0$, often neglected in the literature, is especially clear in the formulation of Eq. (41): in fact, the other two inequalities in Eq. (41) only depend on the squared symplectic eigenvalues and cannot thus distinguish between positive and negative eigenvalues.

Besides the algebraic requirements of the previous theorem, we can derive an alternative set of conditions by applying the reduction to *standard form*. These conditions are expressed in terms of *local* symplectic invariants.

Theorem 5. An arbitrary

$$\mathbf{V} = \begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^T & \mathbf{B} \end{pmatrix} \in \mathcal{S}(4, \mathbb{R}) \quad (44)$$

is a quantum CM if and only if it satisfies

$$\mathbf{A}, \mathbf{B} > 0, \quad (45)$$

$$\Delta(\mathbf{V}) \leq 1 + \det \mathbf{V}, \quad (46)$$

$$2\sqrt{\det \mathbf{A} \det \mathbf{B}} + (\det \mathbf{C})^2 \leq \det \mathbf{V} + \det \mathbf{A} \det \mathbf{B}. \quad (47)$$

Proof. Under the assumption $\mathbf{A}, \mathbf{B} > 0$, the matrix \mathbf{V} can be reduced to the standard form \mathbf{V}^l of Eq. (30) via a local symplectic transformation \mathbf{S} . Since $\mathbf{S}\mathbf{\Omega}\mathbf{S}^T = \mathbf{\Omega}$, the Heisenberg principle can be written in the equivalent form [25]

$$\mathbf{V} + i\mathbf{\Omega} \geq 0 \Leftrightarrow \mathbf{V}^l + i\mathbf{\Omega} \geq 0. \quad (48)$$

Since the matrix $\mathbf{V}^l + i\mathbf{\Omega}$ is Hermitian, its four eigenvalues $\lambda_+^+, \lambda_-^+, \lambda_+^-, \lambda_-^-$ are real. It is then easy to show that

$$2\lambda_{\pm}^+ = a + b + \sqrt{\mu \pm 2\sqrt{\nu}}, \quad (49)$$

$$2\lambda_{\pm}^- = a + b - \sqrt{\mu \pm 2\sqrt{\nu}}, \quad (50)$$

where

$$\mu := 4 + (a - b)^2 + 2(c_+^2 + c_-^2) \geq 4, \quad (51)$$

and

$$\nu := 4(a - b)^2 + (c_+ + c_-)^2[4 + (c_+ - c_-)^2] \geq 0. \quad (52)$$

Since λ_+^- is the minimum eigenvalue, we have that

$$\begin{aligned} \mathbf{V}^l + i\boldsymbol{\Omega} \geq 0 &\Leftrightarrow \lambda_+^- \geq 0 \Leftrightarrow a + b - \sqrt{\mu + 2\nu} \geq 0 \\ &\Leftrightarrow \begin{cases} (a + b)^2 \geq \mu + 2\nu \\ a + b \geq 0 \end{cases} \\ &\Leftrightarrow \begin{cases} [(a + b)^2 - \mu]^2 \geq 4\nu \\ (a + b)^2 - \mu \geq 0 \\ a + b \geq 0. \end{cases} \end{aligned} \quad (53)$$

Last condition $a + b \geq 0$ in Eq. (53) is trivially included in $\mathbf{A} > 0$ and $\mathbf{B} > 0$ which gives $a > 0$ and $b > 0$ (for congruence with the diagonal matrices $a\mathbf{I}$ and $b\mathbf{I}$). The other two conditions

$$[(a + b)^2 - \mu]^2 \geq 4\nu, \quad (54)$$

and

$$(a + b)^2 - \mu \geq 0, \quad (55)$$

can be recast in terms of the local symplectic invariants. In fact, by inserting Eqs. (51) and (52) in Eq. (54), we get

$$a^2 + b^2 + 2c_+c_- \leq (ab - c_+^2)(ab - c_-^2) + 1, \quad (56)$$

which is equivalent to Eq. (46) by using Eq. (32) and $\Delta(\mathbf{V}) = \Delta(\mathbf{V}^l) = a^2 + b^2 + 2c_+c_-$. Finally, by inserting Eq. (51) in Eq. (55), we get $2ab - c_+^2 - c_-^2 \geq 2$, which is equivalent to

$$2a^2b^2 - ab(c_+^2 + c_-^2) \geq 2ab, \quad (57)$$

since $ab > 0$. In terms of local symplectic invariants, last inequality is equal to

$$2 \det \mathbf{A} \det \mathbf{B} - I_4 \geq 2\sqrt{\det \mathbf{A} \det \mathbf{B}}, \quad (58)$$

where

$$I_4 := \text{Tr}(\mathbf{A}\boldsymbol{\omega}\mathbf{C}\boldsymbol{\omega}\mathbf{B}\boldsymbol{\omega}\mathbf{C}^T\boldsymbol{\omega}) = ab(c_+^2 + c_-^2) \quad (59)$$

is another local symplectic invariant. In fact, the quantity I_4 is connected to the other local symplectic invariants by

$$\det \mathbf{V} = \det \mathbf{A} \det \mathbf{B} + (\det \mathbf{C})^2 - I_4, \quad (60)$$

which holds for every $\mathbf{V} \in \mathcal{S}(4, \mathbb{R})$. Using Eq. (60) in Eq. (58), we then get Eq. (47). ■

V. SEPARABILITY OF A TWO-MODE CORRELATION MATRIX

Few years ago, Ref. [6] showed how to extend the partial transposition and the Peres entanglement criterion [26] to bipartite bosonic systems. In fact, partial transposition $\text{PT}: \rho_{AB} \rightarrow \tilde{\rho}_{AB}$ corresponds in phase space to a ‘‘local time reversal’’ which inverts the momentum of only one of two subsystems. This means that we have the following transformation for the Wigner function:

$$\text{PT}: W(\mathbf{x}) \rightarrow \tilde{W}(\mathbf{x}) := W(\mathbf{\Lambda}\mathbf{x}), \quad (61)$$

where

$$\mathbf{\Lambda} := \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}. \quad (62)$$

For the corresponding CM $\mathbf{V} \in q\mathcal{CM}(2)$, the PT transformation is given by

$$\text{PT}: \mathbf{V} \rightarrow \tilde{\mathbf{V}} := \mathbf{\Lambda}\mathbf{V}\mathbf{\Lambda}, \quad (63)$$

where the partially transposed matrix $\tilde{\mathbf{V}}$ belongs to $\mathcal{P}(4, \mathbb{R})$ [25] but not necessarily to $q\mathcal{CM}(2)$. By writing \mathbf{V} in the block form of Eq. (25), one easily checks that the action of the PT transformation $\mathbf{\Lambda}$ reduces to the following sign flip

$$\det \mathbf{C} \rightarrow -\det \mathbf{C}, \quad (64)$$

at the level of the local symplectic invariants. As a consequence, the positive-definite matrix $\tilde{\mathbf{V}}$ has

$$\Delta(\tilde{\mathbf{V}}) = \det \mathbf{A} + \det \mathbf{B} - 2 \det \mathbf{C} := \tilde{\Delta}(\mathbf{V}), \quad (65)$$

and symplectic eigenvalues

$$\tilde{\nu}_{\pm} = \sqrt{\frac{\tilde{\Delta}(\mathbf{V}) \pm \sqrt{\tilde{\Delta}(\mathbf{V})^2 - 4 \det \mathbf{V}}}{2}}. \quad (66)$$

Once the PT transformation has been extended, also the Peres criterion can be consequently extended via the logical implication

$$\rho_{AB} \text{ separable} \Rightarrow \tilde{\rho}_{AB} \in \mathcal{D}(\mathcal{H}) \Rightarrow \tilde{\mathbf{V}} \in q\mathcal{CM}(2), \quad (67)$$

which becomes an equivalence for Gaussian states under $1 \times n$ mode bipartitions [6,8].

Theorem 6 (Separability). Let us consider a Gaussian state ρ_{AB} with CM $\mathbf{V} \in q\mathcal{CM}(2)$. Then, ρ_{AB} is separable if and only if

$$\tilde{\mathbf{V}} \in q\mathcal{CM}(2), \quad (68)$$

or, equivalently,

$$\tilde{\nu}_- \geq 1, \quad (69)$$

or, equivalently

$$\tilde{\Delta}(\mathbf{V}) \leq 1 + \det \mathbf{V}. \quad (70)$$

Proof. The proof of Eq. (68) follows exactly the same steps of the one in Ref. [6], where the P representation is exploited (see Ref. [27] for recent connections between P representation and separability.) In order to prove Eqs. (69) and (70), let us apply theorem 4 to the positive-definite matrix $\tilde{\mathbf{V}} \in \mathcal{P}(4, \mathbb{R})$. Then, we get

$$\tilde{\mathbf{V}} \in q\mathcal{CM}(2) \Leftrightarrow \tilde{\nu}_- \geq 1 \Leftrightarrow \begin{cases} \det \tilde{\mathbf{V}} \geq 1 \\ \tilde{\Delta}(\mathbf{V}) \leq 1 + \det \tilde{\mathbf{V}} \end{cases} \quad (71)$$

where Eq. (69) is trivially proven. Now, since $\det \tilde{\mathbf{V}} = \det(\mathbf{\Lambda}\mathbf{V}\mathbf{\Lambda}) = \det \mathbf{V} \geq 1$, the first condition in Eq. (71) is al-

ways satisfied and, therefore, the separability condition is reduced to Eq. (70). ■

Let us now derive the algebraic conditions that a generic symmetric matrix must satisfy to represent the CM of a separable or entangled Gaussian state. The following corollary gives an easy recipe to check if a symmetric matrix is a good or bad candidate for this aim.

Corollary 7. An arbitrary

$$\mathbf{V} = \begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^T & \mathbf{B} \end{pmatrix} \in \mathcal{S}(4, \mathbb{R}) \quad (72)$$

represents the CM of a separable Gaussian state if and only if it satisfies

$$\mathbf{V} > 0, \quad \nu_- \geq 1, \quad \tilde{\nu}_- \geq 1, \quad (73)$$

or, equivalently,

$$\mathbf{V} > 0, \quad \det \mathbf{V} \geq 1, \quad \Gamma(\mathbf{V}) \leq 1 + \det \mathbf{V}, \quad (74)$$

or, equivalently,

$$\mathbf{A}, \mathbf{B} > 0, \quad \Gamma(\mathbf{V}) \leq 1 + \det \mathbf{V}, \quad (75)$$

$$2\sqrt{\det \mathbf{A} \det \mathbf{B} + (\det \mathbf{C})^2} \leq \det \mathbf{V} + \det \mathbf{A} \det \mathbf{B}, \quad (76)$$

where $\Gamma(\mathbf{V}) := \det \mathbf{A} + \det \mathbf{B} + 2|\det \mathbf{C}|$. Instead, it represents the CM of an entangled Gaussian state if and only if it satisfies

$$\mathbf{V} > 0, \quad \nu_- \geq 1, \quad \tilde{\nu}_- < 1, \quad (77)$$

or, equivalently,

$$\mathbf{V} > 0, \quad \det \mathbf{V} \geq 1, \quad \Delta(\mathbf{V}) \leq 1 + \det \mathbf{V} < \tilde{\Delta}(\mathbf{V}), \quad (78)$$

or, equivalently,

$$\mathbf{A}, \mathbf{B} > 0, \quad \Delta(\mathbf{V}) \leq 1 + \det \mathbf{V} < \tilde{\Delta}(\mathbf{V}), \quad (79)$$

$$2\sqrt{\det \mathbf{A} \det \mathbf{B} + (\det \mathbf{C})^2} \leq \det \mathbf{V} + \det \mathbf{A} \det \mathbf{B}. \quad (80)$$

Proof. In order to represent the CM of a separable Gaussian state, the symmetric matrix $\mathbf{V} \in \mathcal{S}(4, \mathbb{R})$ must simultaneously satisfy

$$\mathbf{V} \in q\mathcal{CM}(2), \quad \tilde{\mathbf{V}} \in q\mathcal{CM}(2). \quad (81)$$

The bona fide condition $\mathbf{V} \in q\mathcal{CM}(2)$ is equivalently expressed by the conditions of Eqs. (40) and (41) in theorem 4. Then, for every $\mathbf{V} \in q\mathcal{CM}(2)$, the separability condition $\tilde{\mathbf{V}} \in q\mathcal{CM}(2)$ is equivalent to Eqs. (69) and (70) in theorem 6. By combining Eq. (40) with Eq. (69) and Eq. (41) with Eq. (70), one easily gets Eqs. (73) and (74), where $\max\{\Delta(\mathbf{V}), \tilde{\Delta}(\mathbf{V})\} \leq 1 + \det \mathbf{V} \Leftrightarrow \Gamma(\mathbf{V}) \leq 1 + \det \mathbf{V}$. According to theorem 6, for every $\mathbf{V} \in q\mathcal{CM}(2)$ the entanglement condition is expressed by $\tilde{\nu}_- < 1$ or, equivalently, by $\tilde{\Delta}(\mathbf{V}) > 1 + \det \mathbf{V}$. Then, it is trivial to derive the corresponding Eqs. (77) and (78). The proof of Eqs. (75) and (76) and Eqs. (79) and (80) is the same as before except that now we have to combine the Eqs. (45)–(47) of theorem 5 with Eq. (70) for

the separability and with $\tilde{\Delta}(\mathbf{V}) > 1 + \det \mathbf{V}$ for the entanglement. ■

VI. RELATION WITH THE PREVIOUS WORK BY SIMON

In order to make a direct comparison with the previous work by Simon [6], we have to specify some of our results, given for arbitrary symmetric matrices $\mathbf{V} \in \mathcal{S}(4, \mathbb{R})$, to the case of positive-definite matrices, i.e., $\mathbf{V} \in \mathcal{P}(4, \mathbb{R})$. As an immediate consequence of theorem 4, we have the following result.

Corollary 8. An arbitrary $\mathbf{V} \in \mathcal{P}(4, \mathbb{R})$ is a two-mode quantum CM $\mathbf{V} \in q\mathcal{CM}(2)$, i.e., $\mathbf{V} + i\mathbf{\Omega} \geq 0$, if and only if

$$\nu_- \geq 1, \quad (82)$$

or, equivalently,

$$\det \mathbf{V} \geq 1, \quad \Delta(\mathbf{V}) \leq 1 + \det \mathbf{V}, \quad (83)$$

or, equivalently,

$$\det \mathbf{V} \geq 1, \quad (84)$$

$$\det \mathbf{A} \det \mathbf{B} + (1 - \det \mathbf{C})^2 - I_4 \geq \det \mathbf{A} + \det \mathbf{B}, \quad (85)$$

where $I_4 := \text{Tr}(\mathbf{A}\omega\mathbf{C}\omega\mathbf{B}\omega\mathbf{C}^T\omega)$.

Proof. By applying theorem 4 under the assumption $\mathbf{V} > 0$, one trivially derives the equivalent conditions in Eqs. (82) and (83). In order to prove Eqs. (84) and (85), let us reduce the positive-definite matrix \mathbf{V} to its standard form of Eq. (30). Under local symplectic transformations, we then have the equivalence

$$\Delta(\mathbf{V}) \leq 1 + \det \mathbf{V} \Leftrightarrow a^2 + b^2 + 2c_+c_- \leq 1 + (ab - c_+^2)(ab - c_-^2). \quad (86)$$

Note that Eq. (86) can be equivalently written as

$$a^2b^2 + (1 - c_+c_-)^2 - ab(c_+^2 + c_-^2) \geq a^2 + b^2. \quad (87)$$

In terms of local symplectic invariants, last relation can be written as in Eq. (85), where Eq. (59) has been also used. ■

Notice that Eq. (85) corresponds to the Eq. (17) of Ref. [6], up to notation factors [28]. In Ref. [6], this condition is incorrectly claimed to be equivalent to the Heisenberg principle $\mathbf{V} + i\mathbf{\Omega} \geq 0$ (this equivalence is claimed under the positivity constraint $\mathbf{V} > 0$, which is a sufficient condition for the reduction to standard form used in the corresponding proof of Ref. [6]). In order to have a full equivalence with the Heisenberg principle $\mathbf{V} + i\mathbf{\Omega} \geq 0$, the supplementary condition of Eq. (84) is mandatory. It is indeed rather simple to construct a positive-definite matrix $\mathbf{V} \in \mathcal{P}(4, \mathbb{R})$ which satisfies Eq. (85) but violates $\mathbf{V} + i\mathbf{\Omega} \geq 0$. As an example, let us consider the following real and symmetric matrix

$$\mathbf{V}(x) = \frac{1}{2} \begin{pmatrix} 1+4x & 0 & -1+4x & 0 \\ 0 & 1+4x & 0 & -4x \\ -1+4x & 0 & 1+4x & 0 \\ 0 & -4x & 0 & 1+4x \end{pmatrix}, \quad (88)$$

which is positive-definite for every $x > 0$. It is easy to verify that the Hermitian matrix $\mathbf{V} + i\mathbf{\Omega}$ has the following real eigenvalues:

$$\lambda_{\pm} = \frac{1}{4}(1 + 8x \pm \sqrt{17 - 16x + 64x^2}), \quad (89)$$

$$\theta_{\pm} = \frac{1}{4}(3 + 8x \pm \sqrt{17 - 16x + 64x^2}). \quad (90)$$

Since λ_{-} is the minimal eigenvalue, the Heisenberg principle $\mathbf{V} + i\mathbf{\Omega} \geq 0$ is equivalent to $\lambda_{-} \geq 0$, which gives

$$x \geq 1/2. \quad (91)$$

Now, let us explicitly compute Eqs. (84) and (85). It is easy to show that Eq. (84) is equivalent to

$$x(8x + 1) \geq 1 \Leftrightarrow x \geq (\sqrt{33} - 1)/16 \approx 0.3, \quad (92)$$

while Eq. (85) (Simon's genuineness condition) is equivalent to

$$\frac{1}{2} + x(8x - 5) \geq 0 \Leftrightarrow 0 < x \leq \frac{1}{8} \text{ OR } x \geq \frac{1}{2}. \quad (93)$$

From Eq. (93), one can see that Simon's condition alone does not exclude the matrices $\mathbf{V}(x)$ for $0 < x \leq 1/8$, which are clearly unphysical since they violate the Heisenberg condition of Eq. (91). A complete equivalence with Eq. (91) is retrieved by coupling Eq. (93) with Eq. (92), where the latter equation excludes the nonphysical region $0 < x \leq 1/8$.

This imprecision in Simon's work leads to a common misunderstanding of the subsequent separability condition [Eq. (19) of Ref. [6]], which in our notation corresponds to

$$\det \mathbf{A} \det \mathbf{B} + (1 - |\det \mathbf{C}|)^2 - I_4 \geq \det \mathbf{A} + \det \mathbf{B}. \quad (94)$$

In this condition, the Heisenberg principle is erroneously claimed to be included (in fact, it is only partially included). Hence, Simon's separability condition of Eq. (94) is actually valid *only if* $\mathbf{V} \in q\mathcal{CM}(2)$, i.e., the positive-definite matrix $\mathbf{V} \in \mathcal{P}(4, \mathbb{R})$ is already known to be a bona fide quantum CM. In other words, the separability criterion of Eq. (94) must be tested on positive-definite matrices which are already known to describe the second statistical moments of a physical quantum state. However, under this assumption of physicality, Simon's separability criterion of Eq. (94) displays a *redundant* modulus and must be simplified to

$$\det \mathbf{A} \det \mathbf{B} + (1 + \det \mathbf{C})^2 - I_4 \geq \det \mathbf{A} + \det \mathbf{B}. \quad (95)$$

Criterion 9 (Separability). Let us consider a two-mode quantum state ρ_{AB} having quantum CM

$$\mathbf{V} = \begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^T & \mathbf{B} \end{pmatrix} \in q\mathcal{CM}(2). \quad (96)$$

The separability of ρ_{AB} implies

$$\tilde{\Delta}(\mathbf{V}) \leq 1 + \det \mathbf{V}, \quad (97)$$

or, equivalently,

$$\det \mathbf{A} \det \mathbf{B} + (1 + \det \mathbf{C})^2 - I_4 \geq \det \mathbf{A} + \det \mathbf{B}. \quad (98)$$

In particular, if ρ_{AB} is a Gaussian state, then it is separable if and only if Eq. (97) [or Eq. (98)] holds.

Proof. The proof is straightforward. Since \mathbf{V} is a quantum CM, it is positive-definite and satisfies the condition $\det \mathbf{V} \geq 1$. Now, suppose that the corresponding two-mode state ρ_{AB} is separable. Then we have

$$\begin{aligned} \rho_{AB} \text{ separable} &\Rightarrow \tilde{\rho}_{AB} \in \mathcal{D}(\mathcal{H}) \Rightarrow \tilde{\mathbf{V}} \in q\mathcal{CM}(2) \\ &\Leftrightarrow \tilde{\mathbf{V}} + i\mathbf{\Omega} \geq 0. \end{aligned} \quad (99)$$

By applying corollary 8 to the positive-definite matrix $\tilde{\mathbf{V}} = \mathbf{A}\mathbf{V}\mathbf{A}$, we have

$$\tilde{\mathbf{V}} + i\mathbf{\Omega} \geq 0 \Leftrightarrow \det \tilde{\mathbf{V}} \geq 1, \quad \Delta(\tilde{\mathbf{V}}) \leq 1 + \det \tilde{\mathbf{V}}. \quad (100)$$

Since $\det \tilde{\mathbf{V}} = \det \mathbf{V}$, we have that $\det \tilde{\mathbf{V}} \geq 1$ is automatically satisfied in the previous Eq. (100). Then, we get

$$\rho_{AB} \text{ separable} \Rightarrow \tilde{\Delta}(\mathbf{V}) \leq 1 + \det \mathbf{V}, \quad (101)$$

where $\tilde{\Delta}(\mathbf{V}) := \Delta(\tilde{\mathbf{V}})$ is defined in Eq. (65). Using Eq. (60), one easily proves the equivalence between Eqs. (97) and (98). Finally, the full equivalence which holds for Gaussian ρ_{AB} is a direct application of theorem 6. ■

This criterion represents a simplification of Simon's separability criterion. Now, it is important to notice that the separability criterion becomes a bit more involved when arbitrary positive-definite matrices $\mathbf{V} \in \mathcal{P}(4, \mathbb{R})$ are considered, without any other *a priori* assumption. For a generic $\mathbf{V} \in \mathcal{P}(4, \mathbb{R})$, both the Heisenberg principle ($\mathbf{V} + i\mathbf{\Omega} \geq 0$) and the separability property ($\tilde{\mathbf{V}} + i\mathbf{\Omega} \geq 0$) must be explicitly considered and combined together, in order to get a complete set of algebraic conditions. Thanks to these conditions, one easily checks when a positive-definite matrix $\mathbf{V} \in \mathcal{P}(4, \mathbb{R})$ can represent the quantum CM of a Gaussian state ρ_{AB} which is separable or entangled. By applying corollary 7, we get the following criterion for positive-definite matrices.

Criterion 10. An arbitrary $\mathbf{V} \in \mathcal{P}(4, \mathbb{R})$ represents the CM of a separable Gaussian state if and only if

$$\det \mathbf{V} \geq 1, \quad (102)$$

$$\Gamma(\mathbf{V}) \leq 1 + \det \mathbf{V}, \quad (103)$$

or, equivalently,

$$\det \mathbf{V} \geq 1, \quad (104)$$

$$\det \mathbf{A} \det \mathbf{B} + (1 - |\det \mathbf{C}|)^2 - I_4 \geq \det \mathbf{A} + \det \mathbf{B}. \quad (105)$$

Instead, it represents the CM of an entangled Gaussian state if and only if

$$\det \mathbf{V} \geq 1, \quad (106)$$

$$\Delta(\mathbf{V}) \leq 1 + \det \mathbf{V} < \tilde{\Delta}(\mathbf{V}), \quad (107)$$

or, equivalently,

$$\det \mathbf{V} \geq 1, \quad (108)$$

$$\begin{aligned} (1 + \det \mathbf{C})^2 &< \det \mathbf{A} + \det \mathbf{B} - \det \mathbf{A} \det \mathbf{B} + I_4 \\ &\leq (1 - \det \mathbf{C})^2. \end{aligned} \quad (109)$$

The proof is a trivial application of corollary 7, together with Eq. (60), used to state the equivalences between Eqs. (103) and (105), and between Eqs. (107) and (109). According to Eq. (109), positive-definite matrices with $\det \mathbf{C} \geq 0$ can only be associated to separable Gaussian states [6]. Notice that the original Simon's separability criterion, i.e., Eq. (105), must be coupled with the mandatory condition of Eq. (104) in order to investigate correctly the separability properties of a generic positive-definite matrix.

VII. SUMMARY

In theorem 4, we have rederived and explicitly stated all the precise algebraic conditions a symmetric matrix must satisfy to represent the CM of a two-mode bosonic (or canonical) quantum system, including the (critical and often neglected) definite positivity condition. Such conditions are expressed in terms of global symplectic invariants. In theorem 5, we have derived a new and alternative set of conditions, which are expressed in terms of local symplectic invariants. In these local conditions the positivity check is restricted to the submatrices \mathbf{A} and \mathbf{B} . Finally, in theorem 6, the necessary and sufficient condition for the separability of two-mode Gaussian states has been reviewed and cast in a compact form. We should stress that such a condition is valid only under the assumption that physicality is also met [$\mathbf{V} \in q\mathcal{CM}(2)$]. In corollary 7, both the physicality and separability have been explicitly taken into account. Then, we have derived a complete set of (global or local) conditions that a generic symmetric matrix must satisfy in order to represent the CM of a separable (or entangled) Gaussian state of two bosonic modes. In Sec. VI, some of our results have been specified for positive-definite matrices and a comparison with the previous results by Simon has been thoroughly presented. The rigorous agreement with *all* the conditions here considered should constitute a constant reference in both the theoretical practice and the analysis of experimental data involving quantum systems of two canonical degrees of freedom.

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APPENDIX A: SIMPLE PROOF OF WILLIAMSON'S THEOREM

Let us construct the diagonalizing symplectic according to the decomposition

$$\mathbf{S} = \mathbf{W}^{1/2} \mathbf{R} \mathbf{V}^{-1/2}, \quad (A1)$$

with a *suitable* $\mathbf{R} \in \text{SO}(2n)$. In fact

$$\begin{aligned} \mathbf{S} \mathbf{V} \mathbf{S}^T &= (\mathbf{W}^{1/2} \mathbf{R} \mathbf{V}^{-1/2}) \mathbf{V} (\mathbf{V}^{-1/2} \mathbf{R}^T \mathbf{W}^{1/2}) = \mathbf{W}^{1/2} \mathbf{R} \mathbf{R}^T \mathbf{W}^{1/2} \\ &= \mathbf{W}^{1/2} \mathbf{I} \mathbf{W}^{1/2} = \mathbf{W}. \end{aligned} \quad (A2)$$

Notice that Eq. (A1) is well defined since \mathbf{V} and \mathbf{W} are positive-definite (therefore, nonsingular). However, the rotation \mathbf{R} in Eq. (A1) is not arbitrary but must be chosen in order to make \mathbf{S} symplectic.

Let us apply Eq. (A1) to the symplectic condition $\mathbf{S} \mathbf{O} \mathbf{S}^T = \mathbf{O}$. Then, we have

$$\begin{aligned} (\mathbf{W}^{1/2} \mathbf{R} \mathbf{V}^{-1/2}) \mathbf{O} (\mathbf{V}^{-1/2} \mathbf{R}^T \mathbf{W}^{1/2}) &= \mathbf{O} \\ \Leftrightarrow \mathbf{R} (\mathbf{V}^{-1/2} \mathbf{O} \mathbf{V}^{-1/2}) \mathbf{R}^T &= \mathbf{W}^{-1/2} \mathbf{O} \mathbf{W}^{-1/2} \\ \Leftrightarrow \mathbf{R} \mathbf{X} \mathbf{R}^T &= \mathbf{Y}, \end{aligned} \quad (A3)$$

where

$$\mathbf{X} := \mathbf{V}^{-1/2} \mathbf{O} \mathbf{V}^{-1/2}, \quad \mathbf{Y} := \mathbf{W}^{-1/2} \mathbf{O} \mathbf{W}^{-1/2} \quad (A4)$$

are *antisymmetric* (because \mathbf{V} and \mathbf{W} are symmetric, while \mathbf{O} is antisymmetric). In particular, we have

$$\mathbf{Y} = \bigoplus_{k=1}^n \begin{pmatrix} 0 & \nu_k^{-1} \\ -\nu_k^{-1} & 0 \end{pmatrix}. \quad (A5)$$

Now the existence of \mathbf{R} in Eq. (A3) is assured by the following theorem on the block-diagonalization of real antisymmetric matrices (specialized to even dimensions) [29]

Theorem 11. For every $\mathbf{A} = -\mathbf{A}^T \in \mathcal{M}(2n, \mathbb{R})$, there exists a (unique) $\mathbf{O} \in \text{SO}(2n)$ such that

$$\mathbf{O} \mathbf{A} \mathbf{O}^T = \bigoplus_{k=1}^n a_k \boldsymbol{\omega} := \tilde{\mathbf{A}}, \quad (A6)$$

where the (unique) block diagonal form $\tilde{\mathbf{A}}$ has $a_k > 0$.

APPENDIX B: FINDING THE DIAGONALIZING SYMPLECTIC MATRIX

Let us show a possible procedure for deriving the proper rotation \mathbf{O} that block-diagonalizes a generic antisymmetric matrix \mathbf{A} as in theorem 11. We can easily prove the following connection between the block-diagonalization of \mathbf{A} and its unitary diagonalization

Theorem 12. The proper rotation \mathbf{O} performing the block-diagonalization of Eq. (A6) is given by

$$\mathbf{O} = \mathbf{\Gamma} \mathbf{U}^\dagger, \quad (B1)$$

where

$$\mathbf{\Gamma} = \frac{1}{\sqrt{2}} \bigoplus_{k=1}^n \boldsymbol{\gamma}, \quad \boldsymbol{\gamma} := \frac{1}{\sqrt{2}} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}, \quad (B2)$$

and \mathbf{U} is an arbitrary unitary performing the diagonalization of \mathbf{A} , i.e.,

$$\mathbf{U}^\dagger \mathbf{A} \mathbf{U} = \bigoplus_{k=1}^n ia_k \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} := \mathbf{A}_D. \quad (B3)$$

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- [23] For every $\mathbf{M} \in \mathcal{M}(n, \mathbb{R})$ there always exists a pair of proper rotations $\mathbf{R}_1, \mathbf{R}_2 \in \text{SO}(n)$ such that $\mathbf{M} = \mathbf{R}_1 \mathbf{D} \mathbf{R}_2^T$ with \mathbf{D} diagonal and real.
- [24] For the possibility of symplectically diagonalizing quadratic forms under different positivity conditions, see Appendix 6 of Ref. [30].
- [25] According to the Sylvester’s law of inertia, congruence transformations preserve the signs of the eigenvalues.
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- [28] Notice that Ref. [6] adopts the commutation relations $[\hat{x}_i, \hat{x}_m] = i\Omega_{im}$, so that the variance of the vacuum noise is equal to $1/2$. In this notation, our Eq. (85) becomes $\det \mathbf{A} \det \mathbf{B} + [(1/4) - \det \mathbf{C}]^2 - I_4 \geq (\det \mathbf{A} + \det \mathbf{B})/4$, which is exactly the Eq. (17) of Ref. [6].
- [29] This theorem is a specialization of the orthogonal block diagonalization which is valid for all the normal matrices [see, e.g., R. A. Horn and C. R. Johnson, *Matrix Analysis* (Cambridge University Press, New York, 2006), Chap. 2.5]. Clearly, the uniqueness of $\tilde{\mathbf{A}}$ and \mathbf{O} holds up to permutation in the spectrum.
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