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Local entanglability and multipartite entanglement

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We introduce a physically motivated classification of pure quantum states describing n qubits. We characterize all multipartite states which can be maximally entangled to local auxiliary systems using controlled operations. A state has this property if and only if one can construct out of it an orthonormal basis by applying independent local unitary operations. This implies that those states can be used to encode locally the maximum amount of n independent bits. Examples of these states are the so-called stabilizer states, which are used for quantum error correction and one-way quantum computing. We give a simple characterization of these states and construct a complete set of commuting unitary observables which characterize the state uniquely. Furthermore we show how these states can be prepared and discuss their applications.

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I. INTRODUCTION

Many of the applications in the context of quantum information are due to the subtle properties of multipartite entangled states [1]. Thus, the investigation of these properties is at the heart of quantum information theory. Several entanglement measures which quantify the entanglement of arbitrary multipartite states have been introduced [2]. Moreover, different classes of entangled states have been identified [3] and a normal form of multipartite states has been presented [4]. Those investigations are not only relevant in the context of quantum information theory, but are also of interest in other fields of physics which deal with many-body systems [5].

The ultimate goal in this context to cope with the properties of arbitrary multipartite states is far from being reached. Therefore, several special classes of multipartite pure states have been introduced and identified to be useful for certain tasks. For instance, stabilizer states can be employed for quantum informational tasks, like quantum error correction, one-way quantum computing and quantum secret sharing [6,7]. Another class of states are the topologically ordered states which are relevant in condensed-matter physics [8]. Projected entangled pair states have been shown to be very useful to determine the ground state properties of many-body Hamiltonians and to simulate many-body systems [9]. Once, the proper class of states has been identified, the relevant properties of those states are analyzed. For instance in order to determine the relationship between phases of matter and the entanglement properties of the corresponding state. Apart from that, the characterization of the properties of the states within a certain class led to applications of multipartite states. Thus, in order to gain a better understanding of the properties and applications of multipartite states, classifications and characterizations of them are required.

Here, we propose an approach to achieve this task. In contrast to other classifications we do not only consider the system qubits, but classify multipartite states according to their ability to get entangled to local auxiliary systems. Here and in the following, we use the term "local" whenever we refer to a single site. Before we discuss the operational meaning of this approach, let us precisely state the situation we investigate here. We consider an n qubit pure quantum state $|\Psi\rangle$. Each party uses a local auxiliary qubit to entangle it to its system qubit in such a way that the global state is a maximally entangled state between the system and the auxiliary qubits [22]. The operations which are used by the parties are so-called controlled operations, which we denote by C_l, with $C_l=\Sigma_{i=0}^{1}U_l^{(i)}\otimes|i\rangle_{l_a}\langle i|$, where $U_l^{(i)}$ are unitary operations acting on system l and $|i\rangle_{l_a}\langle i|$ is acting on the auxiliary system attached to l. If there exist local control gates C_l such that the state $C_1\otimes C_2\otimes\ldots\otimes C_n|\Psi\rangle|+\rangle^{\otimes n}$, with $|+\rangle=1/\sqrt{2}(|0\rangle+|1\rangle)$ is a maximally entangled state between the system and the auxiliary systems, we call the state $|\Psi\rangle$ locally maximally entanglable (LME). Important examples of these states are all stabilizer states.

The motivation for this investigation is not only that it leads to a, physically motivated, classification of multipartite pure states, but also that a state is LME if and only if (iff) it can be used for a certain task, namely, the optimal local encoding of independent classical bits.

We will show here that LME states (LMESs) have the following properties: (i) According to their definition, the global quantum information of LMESs can be washed out by local operations. This is due to the fact that the system qubits are in a maximally mixed state, after successfully attaching the local auxiliary qubits. This seems to be a crucial property shared by those states which are useful for one-way quantum computing, quantum error correction and quantum secret sharing. Note that the local information can always be washed out [23]. However, there exist states, e.g. the W state [3] for which it is not possible to wash out the global information in this way. Therefore, these states are fundamentally different from LMESs. (ii) A state is LME iff it can be used to encode by local unitary operations the maximum amount of n independent bits. Each party encodes a bit value by applying certain local unitary operations to the qubit at his disposal. We will show that the 2^n states obtained in this way are all orthogonal and therefore globally perfectly distinguishable. However, no party can gain locally any information about the bits owned by the other parties. (iii) Any LMES is local unitary equivalent (LU-equivalent) to a state of the form $\sqrt{\frac{1}{2^n}}\Sigma_{i_1,\ldots,i_n=0}^{1}e^{i\alpha_{i_1,\ldots,i_n}}|i_1,\ldots,i_n\rangle$, where $|i_k\rangle$ denotes the computational basis and all $\alpha(\mathbf{i})\equiv\alpha_{i_1,\ldots,i_n}$ are real. The entanglement contained in this state is completely determined by the classical phases $\alpha(\mathbf{i})$ and their correlations. (iv) Any LMES can be prepared by applying generalized phase gates to a product state. The number of qubits on which the phase gates are acting depends on the correlations of the phases $\alpha(\mathbf{i})$. Thus, LMESs can be entangled in many different, but hierarchical ways. Product states but also stabilizer states are all LME. Stabilizer states or more generally, weighted Graph states [6,10], are for instance those LMESs which require only two-qubit phase gates for their preparation. (v) For any LMES $|\Psi\rangle$, one can construct a complete set of commuting unitary observables such that $|\Psi\rangle$ is the unique eigenstate with eigenvalue one for all these observables (the so-called generalized stabilizer). This cannot only be used to construct frustration free Hamiltonians for which $|\Psi\rangle$ is the unique ground state [9], but also to design dissipative processes for which $|\Psi\rangle$ is the unique stationary state [11].

The sequel of the paper is organized as follows. In Sec. II, we introduce a standard form of multipartite states. Any multipartite state can be transformed into its unique standard form by local unitary operations. In Sec. III, we show that a state is LME iff it can be extended to an ON–basis by independent local unitary operations. Using these results we derive a simple characterization of all LMESs and demonstrate their properties. Next, we show that the three-qubit W state is not LME and that, in fact, two of the three parties can protect some information in the state.

II. NOTATION AND TRACE DECOMPOSITION

Let us start by introducing our notation. By X, Y, and Z we denote the Pauli operators. The subscript of an operator will always denote the system it is acting on, or the system it is describing. For instance ρ_i is the single qubit reduced state of system i of a state $|\Psi\rangle$, i.e. $\rho_i=\mathrm{tr}_{\text{all but } i}(|\Psi\rangle\langle\Psi|)$ and $\langle W_i\rangle=\mathrm{tr}(|\Psi\rangle\langle\Psi|W_i)$ denotes the expectation value of the operator W_i acting on system i. W^i denotes the ith power of the operator W with $W^0\equiv\mathbb{1}$ for any operator W. We denote by $\mathbf{i}$ the classical bit–string $(i_1,\ldots,i_n)$ with $i_k\in\{0,1\}$ $\forall\ k\in\{1,\ldots,n\}$, e.g. $|\mathbf{0}\rangle=|0,\ldots,0\rangle$. We say that a state, $|\Psi\rangle$ is LU equivalent to $|\Phi\rangle$ ($|\Psi\rangle\simeq_{LU}|\Phi\rangle$) if there exist local unitary operators, $U_1,\ldots,U_n$, such that $|\Psi\rangle=U_1\otimes\cdots\otimes U_n|\Phi\rangle$.

In order to investigate LMESs we introduce the trace decomposition of multipartite states. Let $|\Psi\rangle$ be an n qubit state with single qubit reduced states $\{\rho_i\}$. We write each single qubit reduced state ρ_i in its spectral decomposition, $\rho_i=U_i^\dagger D_i U_i$, with $D_i=\mathrm{diag}(\lambda_1^i,\lambda_2^i)$, where $\sqrt{\lambda_k^i}$ are the Schmidt coefficients of the bipartite splitting qubit i and the rest [12]. We call any such decomposition, $U_1\otimes\cdots\otimes U_n|\Psi\rangle$, trace decomposition of $|\Psi\rangle$. The trace decomposition has the property that the reduced states are all diagonal in the computational basis. In this paper we will only make use of the trace decomposition. However, it should be noted that this decomposition can be used to define a unique standard form of multipartite states [13]. For $D_i\not\propto\mathbb{1}\ \forall\ i$ the trace decomposition can be easily made unique, by requiring that $\lambda_1^i\geq\lambda_2^i$, and imposing certain conditions on the phases of the coefficients of the states in the computational basis. If $\rho_i=\frac{1}{2}\mathbb{1}$, for some system i, the standard form can be defined as $\lim_{\epsilon\to 0}|\Psi(\epsilon)\rangle$, where $|\Psi(\epsilon)\rangle$ denotes the unique standard form of $\sqrt{1-\epsilon}|\Psi\rangle+\sqrt{\epsilon}|\mathbf{0}\rangle$ [24]. Any state can be transformed by local unitary operations into its standard form [13]. Thus, it is easy to verify that if the standard forms of two states are equivalent, then the states are LU equivalent. Note that the standard form coincides with the Schmidt decomposition [12] for two qubits and can be generalized to d-level systems.

III. LME STATES

A. Basic properties

Let us now characterize the LMESs. First of all, we show that a state is LME iff it is extendable by independent local unitary operations to an ON basis.

Lemma 1. An n-qubit state $|\Psi\rangle$ is LME iff there exists for each party l a unitary operation U_l such that the set $\{U_1^{i_1}\otimes\ldots,\otimes U_n^{i_n}|\Psi\rangle\}_{i_l=0,1}$, forms an ON–basis.

Proof. Only if: If $|\Psi\rangle$ is LME then there exist control operations $C_l=\Sigma_i V_l^{(i)}\otimes|i\rangle_{l_a}\langle i|$ such that $|\Phi\rangle=C_1\otimes C_2\otimes\ldots\otimes C_n|\Psi\rangle|+\rangle^{\otimes n}$ is maximally entangled in the splitting system versus auxiliary systems. Applying $(V_l^{(0)})^\dagger$ to each system l does not change the entanglement properties and therefore $\rho_{1,\ldots,n}=\frac{1}{2^n}\mathcal{E}_1\circ\ldots\circ\mathcal{E}_n(|\Phi\rangle\langle\Phi|)=\frac{1}{2^n}\mathbb{1}$, where $\mathcal{E}_l(\sigma)=\sigma+U_l\sigma U_l^\dagger$, with $U_l=(V_l^{(0)})^\dagger V_l^{(1)}$. Since $\mathcal{E}_1\circ\ldots\circ\mathcal{E}_n(|\Phi\rangle\langle\Phi|)=\Sigma_{\mathbf{i}}|\Psi_{\mathbf{i}}\rangle\langle\Psi_{\mathbf{i}}|$, with $|\Psi_{\mathbf{i}}\rangle=U_1^{i_1}\otimes\ldots\otimes U_n^{i_n}|\Psi\rangle$ is a sum of 2^n projectors, this can only be fulfilled if $\{|\Psi_{\mathbf{i}}\rangle\}$ is an ON–basis. To see the inverse, one only has to define $C_l=\mathbb{1}\otimes|0\rangle\langle 0|+U_l\otimes|1\rangle\langle 1|$. ∎

Note that the proof implies that after successfully attaching the auxiliary qubits, each of the system qubits is maximally entangled with the rest since its reduced state is then $\rho_i+U_i\rho_iU_i^\dagger=\mathbb{1}$. We are going to show now that these unitary operations are of a special form. Note that $\{U_1^{i_1}\otimes\ldots\otimes U_n^{i_n}|\Psi\rangle\}_{i_l=0,1}$ is an ON-basis iff $\{(V_1U_1V_1^\dagger)^{i_1}\otimes\ldots\otimes(V_nU_nV_n^\dagger)^{i_n}(V_1\otimes\cdots\otimes V_n|\Psi\rangle)\}_{i_l=0,1}$ is an ON–basis, implying that a state is LME iff any LU-equivalent state is LME. Therefore, we can restrict ourselves to some trace decompositions of the state $|\Psi\rangle$. For $\rho_i\not\propto\mathbb{1}$ and ρ_i diagonal the necessary condition, $\rho_i+U_i\rho_iU_i^\dagger=\mathbb{1}$, can only be fulfilled by $U_i=R_{z_i}(\alpha)X_iR_{z_i}(-\alpha)$, where $R_{z_i}(\alpha_i)\equiv e^{i\alpha_i/2Z_i}$. For

$\rho_i\propto\mathbb{1}$, we also find that $U_i=V_iX_iV_i^\dagger$ for some unitary V_i (up to a global phase). This is due to the fact that $U_i|\Psi\rangle$ must be orthogonal to $|\Psi\rangle$ (lemma 1) and therefore $\mathrm{tr}(U_i)=0$. Thus, we only have to consider X operations which implies that a state $|\Psi\rangle$ is LME iff $|\Psi\rangle\simeq_{LU}|\Phi\rangle$, where $\{X^{i_1}\otimes\ldots\otimes X^{i_n}|\Phi\rangle\}$, is an ON-basis, i.e., $\langle\Phi|X^{i_1}\otimes\ldots\otimes X^{i_n}|\Phi\rangle=0\,\forall\,\mathbf{i}\neq\mathbf{0}$. Using all that it is now easy to show the following theorem:

Theorem 2. A state $|\Phi\rangle$ is LME iff $|\Phi\rangle$ is LU equivalent to a state $|\Psi\rangle$ with

$$|\Psi\rangle=\sqrt{\frac{1}{2^n}}\sum_{\mathbf{i}}e^{i\alpha_{\mathbf{i}}}|\mathbf{i}\rangle\equiv U_{ph}^{\Psi}|+\rangle^{\otimes n},\qquad(1)$$

where $\alpha_{\mathbf{i}}\in\boldsymbol{R}$ and U_{ph}^{Ψ} denotes the diagonal unitary operator with the entries $e^{i\alpha_{\mathbf{i}}}$ [25].

Proof. As we have seen before, $|\Phi\rangle$ is LME iff $|\Phi\rangle \simeq_{LU} |\Psi\rangle$ with $\langle\Psi|X^{i_1}\otimes\ldots\otimes X^{i_n}|\Psi\rangle=0\,\forall\ \mathbf{i}\neq\mathbf{0}$ or, equivalently, $|\Phi\rangle\simeq_{LU}|\Psi\rangle$ with $\langle\Psi|Z^{i_1}\otimes\ldots\otimes Z^{i_n}|\Psi\rangle=0\,\forall\ \mathbf{i}\neq\mathbf{0}$. We write $|\Psi\rangle$ in the computational basis, $|\Psi\rangle=\Sigma_{\mathbf{i}}\lambda_{\mathbf{i}}|\mathbf{i}\rangle$ and use that $|i_k\rangle\langle i_k|=1/2(\mathbb{1}+(-1)^{i_k}Z)_k$. Then we have $|\lambda_{\mathbf{i}}|^2=\langle|\mathbf{i}\rangle\langle\mathbf{i}|\rangle=2^{-n}\langle(\mathbb{1}+(-1)^{i_1}Z)_1\otimes\cdots\otimes(\mathbb{1}+(-1)^{i_n}Z)_n\rangle$. Since all expectation values of the operators where at least one Z operator occurs vanish we have $|\lambda_{\mathbf{i}}|^2=2^{-n}$. ■

Thus, a state is LME iff there exists a product basis such that all the coefficients of the state in this basis are phases. The control gates used to create the maximally entangled state between the system [described by $|\Psi\rangle$ in Eq. (2)] and auxiliary qubits are the two-qubit π-phase gates, $\tilde{C}=|0\rangle\langle 0|\otimes\mathbb{1}+|1\rangle\langle 1|\otimes Z$. Note that, given an n qubit LMES [Eq. (2)] one can construct an $n+1$ qubit LMES by entangling an additional qubit via $\tilde{C}$ to some system j. The phases would change to $\alpha_{i_1,\ldots,i_{n+1}}=\alpha_{i_1,\ldots,i_n}+\pi i_j i_{n+1}$. In this way one can attach arbitrarily many qubits.

Since there are 2^n real parameters many multipartite states have the property of being LME. For instance any two-qubit state is LME. This can be easily verified using the Schmidt decomposition (standard form) of the state, $|\Psi\rangle=\alpha|00\rangle+\sqrt{1-\alpha^2}|11\rangle$, with $\alpha\in\mathbb{R}$, $\alpha\geq 0$ and choosing $U_1=X$ and $U_2=Y$. Prominent examples of LMESs are all stabilizer states (which are LU equivalent to the graph states) and the weighted graph states [6,10]. There the phases $\alpha_{\mathbf{i}}$ are quadratic functions of the index $\mathbf{i}=(i_1,\ldots,i_n)$, i.e. $\alpha_{\mathbf{i}}=\pi\mathbf{i}^T\Gamma\mathbf{i}$, where the $n\times n$ matrix Γ is the so-called adjacency matrix [10]. Note that any product state is LME, however, it is very simple to distinguish product states from entangled states using this notion. If $|\Psi\rangle$ is a product state then the state $C_1\otimes\ldots\otimes C_n|\Psi\rangle|+\rangle^n$ is maximally entangled between the system and the auxiliary systems iff each party creates a maximally entangled state (locally). Thus, considering the difference between the local entanglement (each qubit with its auxiliary system) and the global entanglement allows us to distinguish product states from entangled states. Similar arguments can be used to distinguish biseparable states from truly multipartite entangled states [13]. In the following we consider the general LMES $|\Psi\rangle$ given in Eq. (1) and denote by $|\Psi_{\mathbf{i}}\rangle\equiv|\Psi_{i_1,\ldots,i_n}\rangle=Z^{i_1}\otimes\cdots\otimes Z^{i_n}|\Psi\rangle$ the elements of the ON-basis ($|\Psi\rangle\equiv|\Psi_0\rangle$)

B. Applications

Let us now discuss some applications of LMESs. An LMES can be used to encode classical information locally. If n parties share the LMES $|\Psi\rangle$ [Eq. (1)], each party can encode a single bit value by applying either $\mathbb{1}$ (corresponding to the bit value 0), or Z (corresponding to the bit value 1), to the qubit at his possession. The 2^n states obtained in this way are globally perfectly distinguishable (since they are all orthogonal due to Lemma 1), but locally, no information can be gained. Note that for instance for the W state, which is not LME, as we shall see below, it is possible to find local unitary operations $V_i\otimes W_i\otimes U_i$ such that $\{V_i\otimes W_i\otimes U_i|W\rangle\}$ is an ON–basis [14]. However, in this case the unitary operators which generate the ON–basis depend on each other which prevents us from using the state to encode locally n independent classical bits. Apart from that, LMESs can also be used to implement certain non–local unitary operations. In order to see that, we use the Jamiołkowski isomorphism which is a one-to-one mapping between quantum states and quantum operations [15]. For an LMES $|\Psi\rangle$, the operation which corresponds to the state $\tilde{C}_1\otimes\ldots\otimes\tilde{C}_n|\Psi\rangle|+\rangle^n$ is $U_\Psi=\Sigma_{\mathbf{i}}|\Psi_{\mathbf{i}}\rangle\langle\mathbf{i}|=U_{ph}^{\Psi}H^{\otimes n}$, where H is the Hadamard gate. Thus, having $|\Psi\rangle$ at ones disposal, one can implement (up to local Pauli operators) the unitary operation U_Ψ on an arbitrary state using only local operations [15]. Note that $|\Psi\rangle$ can also be employed to implement certain transformations on a state describing less than n qubits. For instance, for the one-way quantum computer [7], it was possible to show that the special properties of the used LMESs, the 2D-cluster states [16] allow for the implementation of an arbitrary unitary operator.

C. Generation

Let us now briefly discuss how LMESs can be generated. We write any LMES $|\Psi\rangle$ as

$$|\Psi\rangle = U_{1,\ldots,n}\prod U_{i_{k_1},\ldots,i_{k_{n-1}}}\cdots\prod U_i|+\rangle^n, \qquad (2)$$

where $U_{i_{k_1},\ldots,i_{k_l}}$ is a phase gate acting on l qubits. For instance, U_{123} maps $|111\rangle_{123}$ to $e^{i\phi_{123}}|111\rangle_{123}$, with $\phi_{123}\in\boldsymbol{R}$ and leaves the rest unchanged. It is straightforward to see that in this hierarchical way the 2^n phases $\alpha_{\mathbf{i}}$ can be generated. Thus, any LMES can be prepared using generalized phase gates, which could result from a generalized Ising interaction. If $\alpha(\mathbf{i})$ is a polynomial of degree k (as a function of $\mathbf{i}=(i_1,\ldots,i_n)$) then the corresponding state can be prepared using only k-body interactions. E.g. graph states or weighted graph states, where the phases $\alpha_{\mathbf{i}}$ are polynomials of degree 2 can be created using only two–qubit phase gates. This shows that the correlations in the coefficients are directly related to a preparation scheme and therefore to the entanglement contained in the state. In order to discuss different methods for the preparation of any LMES $|\Psi\rangle$, we construct its generalized stabilizer [6]. We define $W_k=U_\Psi Z_k U_\Psi^\dagger=U_{ph}^{\Psi}X_k(U_{ph}^{\Psi})^\dagger$. Then $W_k|\Phi\rangle=|\Phi\rangle\,\forall\ k$ iff $|\Phi\rangle=|\Psi\rangle$. Note that all these unitary observables have as a common eigenbasis the basis $\{|\Psi_{\mathbf{i}}\rangle\}$ and that $W_k^2=\mathbb{1}$. Similarly to the stabilizer states, we have $\Sigma_{W\in\mathcal{W}}W=|\Psi\rangle\langle\Psi|$, where $\mathcal{W}$ denotes the group generated by $\{W_1,\ldots,W_n\}$. Depending only on the phases $\alpha_{\mathbf{i}}$, which define the LME, $|\Psi\rangle$, the generators of the generalized stabilizer can be quasi–local, i.e. act non trivially on a small set of (neighboring) qubits [13]. In this case, the methods developed in [11] can be employed to derive a quasi–local dissipative process for which the unique stationary state is $|\Psi\rangle$. Apart form that, one can also easily construct frustration free Hamiltonians for which the unique ground–state is $|\Psi\rangle$, e.g. $H=\mathbb{1}-\Sigma_{W\in\mathcal{W}}W$.

IV. NON-LME STATES

An example of a state which is not LME is the three qubit W state, $|W\rangle=\frac{1}{\sqrt{3}}(|001\rangle+|010\rangle+|100\rangle)$. Due to the fact that $|W\rangle$ is already in its standard decomposition, the unitary operations we have to consider are of the form

$U_i = R_{z_i}(2\alpha_i) X_i R_{z_i}(-2\alpha_i)$. Since $\langle U_i \otimes U_j \rangle \propto \cos(\alpha_i - \alpha_j)$ it is impossible that all these expectation values vanish for any pair of unitary operations. Consequently $|W\rangle$ is not LME and it is only possible to choose U_1, U_2 such that the set $\{|W\rangle, U_1 \otimes \mathbb{1}|W\rangle, \mathbb{1} \otimes U_2|W\rangle, U_1 \otimes U_2|W\rangle\}$ is orthogonal, for instance with $U_1 = X$, $U_2 = Y$. One can also show that if two parties maximally entangled their system qubit with a local auxiliary qubit then the third party cannot adequately entangle his auxiliary qubit to his system qubit, even if he would apply a general two-qubit gate [26]. Thus, two of the three parties can protect some information in the state by entangling their system to auxiliary systems.

V. CONCLUSION

A deep understanding of the manipulation and description of LMESs might lead, similarly to its simplest subclass, the stabilizer states, to applications of multipartite states. We plan to investigate the entanglement properties of LMESs with the aim to find both, operational entanglement measures and applications of multipartite states. Moreover, we will generalize the known quantum informational tasks, which use stabilizer states, like quantum computing, and quantum communication tasks [17] employing more general LMESs. It should be noted here that LMESs can have [13], in contrast to stabilizer states, an exponentially large quantum Kolmogorov complexity [18]. Those states are necessarily highly entangled [19]. It might also be feasible to define the minimal set of reversible entangled states for LMESs [20]. Furthermore, this notion can also be used to study the separability problem [13]. Apart from that, considering a restricted set of LMESs, where for instance only certain three qubit phase gates are required to generate the states, might allow us to generalize the well–known Gottesman-Knill Theorem [12]. Identifying a large enough subset of these states might also be relevant for the simulation of quantum systems [21]. Furthermore, the states which are not LME might be used for protecting information and avoiding certain errors.

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[22] We allow only one qubit per site, because, if we would consider a four-level system per site, each party could implement a completely depolarizing map leading to a maximally entangled state between the auxiliary systems and the system qubits.
[23] That is, for any single qubit state ρ there exists a unitary operator U, such that $\rho + U\rho U^\dagger = \mathbb{1}$.
[24] There always exists an $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$ none of the reduced states is proportional to the identity. Since $|\Psi(\epsilon)\rangle$ is a continuous function of ϵ in this region, the limit exists. The phase gates are chosen such that the first n non–vanishing coefficients in the computational basis of the resulting state are real and positive.
[25] Note that these states can be easily transformed to a trace decomposition by applying the local unitary operations HU_i with $U_i = \mathrm{diag}(e^{ix_i}, 1)$, where $\cot(x_i) = \frac{\langle X_i \rangle}{\langle Y_i \rangle}$.
[26] Here, one needs to consider instead of the local unitary operators local POVMs [13].