

# Uncertainty relation of mixed states by means of Wigner-Yanase-Dyson information

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The variance of an observable in a quantum state is usually used to describe Heisenberg uncertainty relation. For mixed states, the variance includes quantum and classical uncertainties. By means of the skew information and the decomposition of the variance, a stronger uncertainty relation was presented by Luo [Phys. Rev. A **72**, 042110 (2005)]. In this paper, by using Wigner-Yanase-Dyson information which is a generalization of the skew information, we propose a general uncertainty relation of mixed states.

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## I. INTRODUCTION

In quantum measurement theory, the Heisenberg uncertainty principle provides a fundamental limit for the measurements of incompatible observables. On the other hand, as dictated by Cramer-Rao's lower bound, there is also an ultimate limit for the resolution of any unbiased parameter (see, for instance, [1]) and this lower bound is given by a quantity called Fisher information. A long time ago, Wigner [2,3] demonstrated that it is more difficult to measure observables that do not commute with some additive conserved quantities. Thus, observables not commuting with some conserved quantities cannot be measured exactly and only approximate measurement is possible [4,5]. This tradeoff in measurement forms the basis of the well-known Wigner-Araki-Yanase theorem. In their study of quantum measurement theory, Wigner and Yanase introduced a quantity called the skew information. As shown in [6], the skew information is essentially a form of Fisher information.

The skew information for a mixed state  $\rho$  relative to a self-adjoint "observable"  $A$  is defined as  $I(\rho, A) = -\frac{1}{2}\text{Tr}[\rho^{1/2}, A]^2$ . This definition was subsequently generalized by Dyson as  $I_\alpha(\rho, A) = -\frac{1}{2}\text{Tr}([\rho^\alpha, A][\rho^{1-\alpha}, A])$ , where  $0 < \alpha < 1$  [7]. When  $\alpha = 1/2$ ,  $I_\alpha(\rho, X)$  is reduced to the skew information. The convexity of  $I_\alpha(\rho, A)$  was finally resolved by Lieb and Ruskai [8,9].

The von Neumann entropy of  $\rho$ , defined as  $S(\rho) = -\text{Tr} \rho \ln \rho$ , has been widely used as a measure of the uncertainty of a mixed state. This quantity, profoundly rooted in quantum-statistical mechanics, possesses several remarkable and satisfactory properties. Like all measures, the von Neumann entropy, together with its classical analog called the Shannon entropy, is not always the best measure under certain contexts. In [6,10–12], the skew information was proposed as means to unify the study of Heisenberg uncertainty relation for mixed states.

It is well known in the standard textbooks that the Heisenberg uncertainty relation for any two self-adjoint operators  $X$  and  $Y$  is given by

$$V(\rho, X)V(\rho, Y) \geq \frac{1}{4} \|\text{Tr}(\rho[X, Y])\|^2. \quad (1)$$

Note that  $[,]$  is the usual commutator, i.e.,  $[A, B] = AB - BA$  and the variance of the observable  $X$  with respect to  $\rho$  is

$$V(\rho, X) = \text{Tr}(\rho X^2) - [\text{Tr}(\rho X)]^2. \quad (2)$$

A similar definition applies to  $V(\rho, Y)$ .

When  $\rho$  is a mixed state, Luo showed that the variance comprises of two terms: a quantum uncertainty term and a classical uncertainty term [10,11]. He separated the variance into its quantum and classical parts using the skew information. He interpreted  $I(\rho, X)$  as the quantum uncertainty of  $X$  in  $\rho$  through the Bohr complementary principle and  $V(\rho, X) - I(\rho, X)$  as the classical uncertainty of the mixed state. He then considered  $U(\rho, X) = \sqrt{V^2(\rho, X) - [I(\rho, X)]^2}$  as a measure of quantum uncertainty. Thus, he obtained the following two inequalities for the uncertainty relation [10]:

$$I(\rho, X)J(\rho, Y) \geq \frac{1}{4} \|\text{Tr}(\rho[X, Y])\|^2, \quad (3)$$

$$U(\rho, X)U(\rho, Y) \geq \frac{1}{4} \|\text{Tr}(\rho[X, Y])\|^2, \quad (4)$$

where  $J(\rho, Y) = \frac{1}{2}\text{Tr}\{\rho^{1/2}, Y_0\}^2$  and  $Y_0 = Y - \text{Tr}(\rho Y)$ . The notation  $\{ \}$  here is the anticommutator, i.e.,  $\{A, B\} = AB + BA$ .

This paper is organized as follows. In Sec. II, we discuss various properties of the Wigner-Yanase-Dyson information. We show using a counterexample that it needs not to satisfy the uncertainty relation obtained from the skew information. In Sec. III, we formulate an uncertainty relation for Wigner-Yanase-Dyson information. Finally, in Sec. IV, we reiterate our main results. We have also provided two appendixes concerning the proof of the present uncertainty principle and additivity of the Wigner-Yanase-Dyson information.

**II. WIGNER-YANASE-DYSON INFORMATION VIOLATES HEISENBERG UNCERTAINTY RELATION**

In this paper, we extend the above discussion to Wigner-Yanase-Dyson information. The skew information proposed by Dyson can also be written as

$$I_\alpha(\rho, X) = \text{Tr}(\rho X^2) - \text{Tr}(\rho^\alpha X \rho^{1-\alpha} X) \\ = \text{Tr}(\rho X_0^2) - \text{Tr}(\rho^\alpha X_0 \rho^{1-\alpha} X_0), \quad (5)$$

where  $X_0 = X - \text{Tr}(\rho X)$ .  $I_\alpha(\rho, X)$  is positive from Eq. (A5). Similarly, we define  $J_\alpha(\rho, Y) = \frac{1}{2} \text{Tr}(\{\rho^\alpha, Y_0\} \{\rho^{1-\alpha}, Y_0\})$ . When  $\alpha = 1/2$ ,  $J_\alpha(\rho, Y)$  is reduced to  $J(\rho, Y)$ . As well, we can define  $J_\alpha(\rho, X)$ ,  $J_\alpha(\rho, A)$ , and  $J_\alpha(\rho, B)$ . By calculating,

$$J_\alpha(\rho, Y) = \text{Tr}(\rho Y_0^2) + \text{Tr}(\rho^\alpha Y_0 \rho^{1-\alpha} Y_0) \\ = \text{Tr}(\rho Y^2) + \text{Tr}(\rho^\alpha Y \rho^{1-\alpha} Y) - 2[\text{Tr}(\rho Y)]^2. \quad (6)$$

$J_\alpha(\rho, Y)$  is also positive from Eq. (A9) in this paper.

Adopting Luo's interpretations and based on the following properties of Wigner-Yanase-Dyson information, we interpret  $I_\alpha(\rho, X)$  as quantum uncertainty of  $X$  in  $\rho$ ,  $V(\rho, X) - I_\alpha(\rho, X)$  as the classical mixing uncertainty, and  $U_\alpha(\rho, X) = \sqrt{V^2(\rho, X) - [V(\rho, X) - I_\alpha(\rho, X)]^2}$  as a measure of quantum uncertainty. Lieb studied the properties of Wigner-Yanase-Dyson information in [8]. The Wigner-Yanase-Dyson information satisfies the following requirements:

(1) Wigner-Yanase-Dyson conjecture about the convexity of  $I_\alpha(\rho, X)$  with respect to  $\rho$  was proved by Lieb [8].

(2) Wigner-Yanase-Dyson information  $I_\alpha(\rho, X)$  is additive in the following sense (see [6,8]). Let  $\rho_1$  and  $\rho_2$  be two density operators of two subsystems, and  $A_1$  ( $A_2$ ) be a self-adjoint operator on  $H^1$  ( $H^2$ ).  $I_\alpha(\rho, X)$  is additive if  $I_\alpha(\rho_1 \otimes \rho_2, A_1 \otimes I_2 + I_1 \otimes A_2) = I_\alpha(\rho_1, A_1) + I_\alpha(\rho_2, A_2)$ , where  $I_1$  and  $I_2$  are the identity operators for the first and second systems, respectively. For the proof see Appendix B.

(3)  $J_\alpha(\rho, Y)$  is also additive. For the proof see Appendix B.

(4) However, Hansen showed that Wigner-Yanase-Dyson information is not subadditive [13]. For the definition of subadditivity, see [8,13].

(5)  $J_\alpha(\rho, Y)$  is concave with respect to  $\rho$ . This is because  $\text{Tr}(\rho Y_0^2)$  is a linear operator with respect to  $\rho$  and  $\text{Tr}(\rho^\alpha Y_0 \rho^{1-\alpha} Y_0)$  is concave with respect to  $\rho$ .

(6) When  $\rho$  is pure,  $V(\rho, X) = I_\alpha(\rho, X)$ . Thus, Wigner-Yanase-Dyson information reduces to the variance. Thus, we may say that the variance  $V(\rho, X)$  does not include classical mixing uncertainty. In other words, the variance only includes quantum uncertainty of  $X$  in  $\rho$ . The case in which  $\alpha = 1/2$  was discussed in [11].

The above fact can be argued as follows. When  $\rho$  is pure,  $\text{Tr}(\rho^\alpha X_0 \rho^{1-\alpha} X_0) = [\text{Tr}(\rho X_0)]^2 = 0$ . Thus,  $I_\alpha(\rho, X) = \text{Tr}(\rho X_0^2) = V(\rho, X)$ .

(7) When  $\rho$  is a mixed state,  $V(\rho, X) \geq I_\alpha(\rho, X)$ . This is because  $\text{Tr}(\rho^\alpha X \rho^{1-\alpha} X) = \text{Tr}[(\rho^{\alpha/2} X \rho^{(1-\alpha)/2})(\rho^{\alpha/2} X \rho^{(1-\alpha)/2})^\dagger] \geq 0$ . Also, see Eq. (A3) in this paper. The case in which  $\alpha = 1/2$  was discussed in [11].

(8) When  $\rho$  and  $A$  commute, according to the discussion for the skew information in [10,12], the quantum uncertainty should vanish and, thus, the variance only includes the clas-

sical uncertainty. We can argue that the above conclusion is also true for Wigner-Yanase-Dyson information. When  $\rho$  and  $A$  commute, it is well known that  $\rho$  and  $A$  have the same orthonormal eigenvector basis [14]. Hence,  $\rho^\alpha$  and  $A$  also commute. By the definition in Eq. (5), Wigner-Yanase-Dyson information  $I_\alpha(\rho, X)$  vanishes.

However,  $I_\alpha(\rho, X)$  and  $J_\alpha(\rho, Y)$  do not satisfy Eq. (3). We give the following counterexample for Eq. (3).

Let  $n=2$ ,  $\alpha=1/4$ , and  $\rho$  has the eigenvalues  $\lambda_1=1/4$  and  $\lambda_2=3/4$ . Since  $A$  and  $B$  are self-adjoint, we write

$$A = \begin{pmatrix} x & u + iv \\ u - iv & y \end{pmatrix}, \quad B = \begin{pmatrix} a & c + di \\ c - di & b \end{pmatrix}.$$

In this example,  $u=4$ ,  $v=2$ ,  $a=b=0$ ,  $c=1$ , and  $d=-5$ . By calculating  $I_\alpha(\rho, A)$  in Eq. (A5) and  $J_\alpha(\rho, B)$  in Eq. (A8),  $I_\alpha(\rho, A)J_\alpha(\rho, B) = [1 - (\lambda_1^\alpha \lambda_2^{1-\alpha} + \lambda_2^\alpha \lambda_1^{1-\alpha})^2](u^2 + v^2)(c^2 + d^2) = 99.83$ . By calculating  $\text{Tr}(\rho[A, B])$  in Eq. (A11),  $\frac{1}{4} \|\text{Tr}(\rho[A, B])\|^2 = (\lambda_1 - \lambda_2)^2 (cv - du)^2 = 121$ . Hence, it violates Eq. (3). It implies that the bound on the right side of the inequality in Eq. (3) is too large in this example. We need to get the appropriate lower bound for Wigner-Yanase-Dyson information, i.e., we need to modify the term on the right-hand side (RHS) of the inequality.

As a digression, we also note that in [15] Rivas gave a counterexample for Theorem 2 in [6]. It means that the inequality  $\|\text{Tr}(\rho[k_1, k_2])\|^2 \leq 4I(\rho, k_1)I(\rho, k_2)$  in [6] is not true.

**III. GENERAL UNCERTAINTY RELATION**

We replace  $\text{Tr}(\rho[X, Y])$  with  $I_\alpha(\rho, X, Y)$  which is defined as follows:

$$I_\alpha(\rho, X, Y) = \text{Tr}(\rho[X, Y]) - \text{Tr}(\rho^{2\alpha-1}[X, Y]). \quad (7)$$

When  $\alpha=1/2$ ,  $I_\alpha(\rho, X, Y)$  reduces to  $\text{Tr}(\rho[X, Y])$ . In [10], Luo defined  $k = i[\rho^{1/2}, X_0]t + \{\rho^{1/2}, Y_0\}$ , where  $t \in \mathbb{R}$  and  $i$  is the imaginary unit. From  $\text{Tr}(kk^\dagger) \geq 0$ , by expanding  $\text{Tr}(kk^\dagger)$ , he derived  $\text{Tr}(kk^\dagger) = 2\{I[\rho, X]t^2 + i[\text{Tr}(\rho[X, Y])]t + J[\rho, Y]\} \geq 0$ . Since the above inequality is true for any real  $t$ , Luo obtained the inequality in Eq. (3). However, unlike his previous case, the form of  $I_\alpha(\rho, X)$  does not allow us to employ the trick  $k = i[\rho^\alpha, X_0]t + \{\rho^\alpha, Y_0\}$  nor  $k = i[\rho^{1-\alpha}, X_0]t + \{\rho^{1-\alpha}, Y_0\}$  to derive the uncertainty relation from  $\text{Tr}(kk^\dagger) \geq 0$ . The proof becomes more involved and one needs to modify the RHS of the previous uncertainty relation.

In Appendix A, we see that if  $A$  and  $B$  are self-adjoint observables, then

$$I_\alpha(\rho, A)J_\alpha(\rho, B) \geq \frac{1}{4} \|I_\alpha(\rho, A, B)\|^2, \quad (8)$$

and

$$I_\alpha(\rho, B)J_\alpha(\rho, A) \geq \frac{1}{4} \|I_\alpha(\rho, A, B)\|^2. \quad (9)$$

Let  $U_\alpha(\rho, A) = \sqrt{V^2(\rho, A) - [V(\rho, A) - I_\alpha(\rho, A)]^2}$  and  $U_\alpha(\rho, B) = \sqrt{V^2(\rho, B) - [V(\rho, B) - I_\alpha(\rho, B)]^2}$ . By Eqs. (2), (5), and (6),  $U_\alpha(\rho, A) = \sqrt{I_\alpha(\rho, A)J_\alpha(\rho, A)}$  and  $U_\alpha(\rho, B) = \sqrt{I_\alpha(\rho, B)J_\alpha(\rho, B)}$ . Thus, we obtain our main result from Eqs. (8) and (9),

$$U_\alpha(\rho, A)U_\alpha(\rho, B) \geq \frac{1}{4} \|l_\alpha(\rho, A, B)\|^2. \quad (10)$$

For the counterexample in Sec. II, a direct calculation of Eq. (A13) yields  $\frac{1}{4} \|l_\alpha(\rho, A, B)\|^2 = 8.6874$ . Therefore, the inequality in Eq. (8) holds in this case. When  $\alpha = 1/2$ , Eq. (10) reduces to Luo's result in Eq. (4).

#### IV. SUMMARY

In [10], Luo presented a refined Heisenberg uncertainty relation. In this paper, we demonstrate some properties of Wigner-Yanase-Dyson information and provide a counterexample to show that Wigner-Yanase-Dyson information does not in general satisfy Heisenberg uncertainty relation. We have also proposed a general uncertainty relation of mixed states based on Wigner-Yanase-Dyson information. Bell-type inequalities based on the skew information have been proposed as nonlinear entanglement witnesses [16]. We note here that similar Bell-type inequalities with the advantage of an additional  $\alpha$  parameter for fine adjustments could also be constructed from the uncertainty principle derived from the Wigner-Yanase-Dyson information.

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#### APPENDIX A: PROOF OF UNCERTAINTY RELATION

By spectral decomposition, there exists an orthonormal basis  $\{|x_1, \dots, x_n\rangle$  consisting of eigenvectors of  $\rho$ . Let  $\lambda_1, \dots, \lambda_n$  be the corresponding eigenvalues, where  $\lambda_1 + \dots + \lambda_n = 1$  and  $\lambda_i \geq 0$ . Thus,  $\rho$  has a spectral representation,

$$\rho = \lambda_1 |x_1\rangle\langle x_1| + \dots + \lambda_n |x_n\rangle\langle x_n|. \quad (A1)$$

##### 1. Calculating $I_\alpha(\rho, A)$

By Eq. (A1),  $\rho A^2 = \lambda_1 |x_1\rangle\langle x_1| A^2 + \dots + \lambda_n |x_n\rangle\langle x_n| A^2$  and

$$\begin{aligned} \text{Tr}(\rho A^2) &= \lambda_1 \langle x_1 | A^2 | x_1 \rangle + \dots + \lambda_n \langle x_n | A^2 | x_n \rangle \\ &= \lambda_1 \|A|x_1\rangle\|^2 + \dots + \lambda_n \|A|x_n\rangle\|^2. \end{aligned} \quad (A2)$$

Moreover, since  $\rho^\alpha A = \lambda_1^\alpha |x_1\rangle\langle x_1| A + \dots + \lambda_n^\alpha |x_n\rangle\langle x_n| A$  and  $\rho^{1-\alpha} A = \lambda_1^{1-\alpha} |x_1\rangle\langle x_1| A + \dots + \lambda_n^{1-\alpha} |x_n\rangle\langle x_n| A$ , we have  $\rho^\alpha A \rho^{1-\alpha} A = \sum_{i,j=1}^n \lambda_i^\alpha \lambda_j^{1-\alpha} |x_i\rangle\langle x_i| A |x_j\rangle\langle x_j| A$ . Thus

$$\begin{aligned} \text{Tr}(\rho^\alpha A \rho^{1-\alpha} A) &= \sum_{i,j=1}^n \lambda_i^\alpha \lambda_j^{1-\alpha} \langle x_i | A | x_j \rangle \langle x_j | A | x_i \rangle \\ &= \sum_{i,j=1}^n \lambda_i^\alpha \lambda_j^{1-\alpha} \|\langle x_i | A | x_j \rangle\|^2. \end{aligned} \quad (A3)$$

From Eqs. (5), (A2), and (A3),

$$I_\alpha(\rho, A) = \sum_{i=1}^n \lambda_i \|A|x_i\rangle\|^2 - \sum_{i,j=1}^n \lambda_i^\alpha \lambda_j^{1-\alpha} \|\langle x_i | A | x_j \rangle\|^2. \quad (A4)$$

Let  $A = \{A_{ij}\}$  ( $B = \{B_{ij}\}$ ) be the matrix representation of the operator  $A$  ( $B$ ) corresponding to the orthonormal basis  $\{|x_1, \dots, x_n\rangle$ . Then  $\langle x_i | A | x_j \rangle = A_{ij}$  and

$$\begin{aligned} I_\alpha(\rho, A) &= \sum_{i \neq j} (\lambda_i - \lambda_i^\alpha \lambda_j^{1-\alpha}) \|A_{ij}\|^2 \\ &= \sum_{i < j} (\lambda_i + \lambda_j - \lambda_i^\alpha \lambda_j^{1-\alpha} - \lambda_i^{1-\alpha} \lambda_j^\alpha) \|A_{ij}\|^2. \end{aligned} \quad (A5)$$

##### 2. Calculating $J_\alpha(\rho, B)$

Similarly, from Eqs. (6) and (A1), we can obtain

$$\begin{aligned} J_\alpha(\rho, B) &= \sum_{i=1}^n \lambda_i \|B|x_i\rangle\|^2 + \sum_{i,j=1}^n \lambda_i^\alpha \lambda_j^{1-\alpha} \|\langle x_i | B | x_j \rangle\|^2 \\ &\quad - 2 \left( \sum \lambda_i \langle x_i | B | x_i \rangle \right)^2. \end{aligned} \quad (A6)$$

Let  $\langle x_i | B | x_j \rangle = B_{ij}$ . Then, from Eq. (A6),

$$\begin{aligned} J_\alpha(\rho, B) &= 2 \sum_{i=1}^n \lambda_i |B_{ii}|^2 - 2 \left( \sum_{i=1}^n \lambda_i B_{ii} \right)^2 \\ &\quad + \sum_{i \neq j} (\lambda_i + \lambda_j + \lambda_i^\alpha \lambda_j^{1-\alpha}) \|B_{ij}\|^2. \end{aligned} \quad (A7)$$

By simplifying,

$$\begin{aligned} J_\alpha(\rho, B) &= 2 \sum_{i=1}^n \lambda_i |B_{ii}|^2 - 2 \left( \sum_{i=1}^n \lambda_i B_{ii} \right)^2 \\ &\quad + \sum_{i < j} (\lambda_i + \lambda_j + \lambda_i^\alpha \lambda_j^{1-\alpha} + \lambda_i^{1-\alpha} \lambda_j^\alpha) \|B_{ij}\|^2. \end{aligned} \quad (A8)$$

Since  $x^2$  is convex,  $(\sum_{i=1}^n \lambda_i B_{ii})^2 \leq \sum_{i=1}^n \lambda_i |B_{ii}|^2$ . So from Eq. (A8),

$$J_\alpha(\rho, B) \geq \sum_{i < j} (\lambda_i + \lambda_j + \lambda_i^\alpha \lambda_j^{1-\alpha} + \lambda_i^{1-\alpha} \lambda_j^\alpha) \|B_{ij}\|^2. \quad (A9)$$

##### 3. Calculating $I_\alpha(\rho, A, B)$

First we calculate  $\text{Tr}(\rho[A, B])$ . From Eq. (A1),  $\rho[A, B] = \lambda_1 |x_1\rangle\langle x_1| [A, B] + \dots + \lambda_n |x_n\rangle\langle x_n| [A, B]$  and  $\text{Tr}(\rho[A, B]) = \lambda_1 \langle x_1 | [A, B] | x_1 \rangle + \dots + \lambda_n \langle x_n | [A, B] | x_n \rangle$ . It is well known that  $\text{Re}\langle x_i | [A, B] | x_i \rangle = 0$  and  $\langle x_i | [A, B] | x_i \rangle = i(2 \text{Im}\langle x_i | AB | x_i \rangle)$ , where  $i$  is the imaginary unit. Consequently,  $\text{Tr}(\rho[A, B]) = 2i(\lambda_1 \text{Im}\langle x_1 | AB | x_1 \rangle + \dots + \lambda_n \text{Im}\langle x_n | AB | x_n \rangle)$ . Therefore we obtain

$$\begin{aligned} \text{Tr}(\rho[A, B]) &= 2i \text{Im}(\lambda_1 \langle x_1 | AB | x_1 \rangle + \dots + \lambda_n \langle x_n | AB | x_n \rangle) \\ &= 2i \text{Im} \sum_{j \neq i} \lambda_i A_{ij} B_{ji}. \end{aligned} \quad (A10)$$

Note that in Eq. (A10)  $A_{ii}$  and  $B_{ii}$  are real because  $A$  and  $B$  are self-adjoint. Since  $A_{ij} B_{ji} = (A_{ji} B_{ij})^*$ ,  $\sum_{j \neq i} \lambda_i A_{ij} B_{ji} = \text{Im} \sum_{i < j} (\lambda_i - \lambda_j) A_{ij} B_{ji}$ . Thus, by simplifying,

$$\text{Tr}(\rho[A, B]) = 2i \text{Im} \sum_{i < j} (\lambda_i - \lambda_j) A_{ij} B_{ji}. \quad (A11)$$

Moreover,

$$\text{Tr}(\rho^{2\alpha-1}[A, B]) = 2i \text{Im} \sum_{i < j} (\lambda_i^{2\alpha-1} - \lambda_j^{2\alpha-1}) A_{ij} B_{ji}. \tag{A12}$$

Hence, from Eqs. (7), (A11), and (A12),

$$I_\alpha(\rho, A, B) = 2i \sum_{i < j} [\lambda_i - \lambda_j - (\lambda_i^{2\alpha-1} - \lambda_j^{2\alpha-1})] \text{Im}(A_{ij} B_{ji}). \tag{A13}$$

**4. Proof of the uncertainty relation**

From Eqs. (A5), (A9), and (A13), for Eq. (8) we need to show

$$\left[ \sum_{i < j} (\lambda_i + \lambda_j - \lambda_i^\alpha \lambda_j^{1-\alpha} - \lambda_i^{1-\alpha} \lambda_j^\alpha) \|A_{ij}\|^2 \right] \times \left[ \sum_{i < j} (\lambda_i + \lambda_j + \lambda_i^\alpha \lambda_j^{1-\alpha} + \lambda_i^{1-\alpha} \lambda_j^\alpha) \|B_{ij}\|^2 \right] \geq \left\{ \sum_{i < j} [\lambda_i - \lambda_j - (\lambda_i^{2\alpha-1} - \lambda_j^{2\alpha-1})] \text{Im}(A_{ij} B_{ji}) \right\}^2. \tag{A14}$$

It is easy to know that  $[\text{Im}(A_{ij} B_{ji})]^2 \leq \|A_{ij}\|^2 \|B_{ij}\|^2$ . Note that  $\lambda_i + \lambda_j - \lambda_i^\alpha \lambda_j^{1-\alpha} - \lambda_i^{1-\alpha} \lambda_j^\alpha = (\lambda_i - \lambda_j) (\lambda_i^{1-\alpha} - \lambda_j^{1-\alpha}) \geq 0$ . By the Cauchy-Schwarz inequality, the left-hand side (LHS) of the inequality in Eq. (A14) is  $\geq \{\sum_{i < j} [(\lambda_i + \lambda_j)^2 - (\lambda_i^\alpha \lambda_j^{1-\alpha} + \lambda_i^{1-\alpha} \lambda_j^\alpha)^2]^{1/2} \text{Im}(A_{ij} B_{ji})\}^2$ . Finally, what needs to be shown is

$$(\lambda_i + \lambda_j)^2 - (\lambda_i^\alpha \lambda_j^{1-\alpha} + \lambda_i^{1-\alpha} \lambda_j^\alpha)^2 \geq |(\lambda_i - \lambda_j) - (\lambda_i^{2\alpha-1} - \lambda_j^{2\alpha-1})|^2. \tag{A15}$$

It is easy to see that

$$(\lambda_i + \lambda_j)^2 - (\lambda_i^\alpha \lambda_j^{1-\alpha} + \lambda_i^{1-\alpha} \lambda_j^\alpha)^2 = (\lambda_i - \lambda_j)^2 - (\lambda_i^\alpha \lambda_j^{1-\alpha} - \lambda_i^{1-\alpha} \lambda_j^\alpha)^2.$$

When  $\alpha \geq 1/2$ ,

$$\begin{aligned} & (\lambda_i - \lambda_j)^2 - (\lambda_i^\alpha \lambda_j^{1-\alpha} - \lambda_i^{1-\alpha} \lambda_j^\alpha)^2 \\ &= (\lambda_i - \lambda_j)^2 - \lambda_i^{2(1-\alpha)} \lambda_j^{2(1-\alpha)} (\lambda_i^{2\alpha-1} - \lambda_j^{2\alpha-1})^2 \\ &\geq (\lambda_i - \lambda_j)^2 - (\lambda_i^{2\alpha-1} - \lambda_j^{2\alpha-1})^2 \\ &= |(\lambda_i - \lambda_j) - (\lambda_i^{2\alpha-1} - \lambda_j^{2\alpha-1})| \\ &\quad \times |(\lambda_i - \lambda_j) + (\lambda_i^{2\alpha-1} - \lambda_j^{2\alpha-1})| \\ &\geq |(\lambda_i - \lambda_j) - (\lambda_i^{2\alpha-1} - \lambda_j^{2\alpha-1})|^2. \end{aligned}$$

Note that the last inequality holds because  $(\lambda_i - \lambda_j)$  and  $(\lambda_i^{2\alpha-1} - \lambda_j^{2\alpha-1})$  have the same sign. Also, when  $0 < \alpha \leq 1/2$ , we can prove the inequality in Eq. (A15) as follows: let  $\beta = 1 - \alpha$  with  $1/2 \leq \beta < 1$ . Replacing  $\alpha$  in Eq. (A15) with  $1 - \beta$ , we obtain  $(\lambda_i + \lambda_j)^2 - (\lambda_i^{1-\beta} \lambda_j^\beta + \lambda_i^\beta \lambda_j^{1-\beta})^2 \geq |(\lambda_i - \lambda_j) - (\lambda_i^{2\beta-1} - \lambda_j^{2\beta-1})|^2$ . This ends the proof.

**APPENDIX B: ADDITIVITY**

The quantity  $J_\alpha(\rho, B)$  is additive in the following sense:  $J_\alpha(\rho_1 \otimes \rho_2, B_1 \otimes I_2 + I_1 \otimes B_2) = J_\alpha(\rho_1, B_1) + J_\alpha(\rho_2, B_2)$ . Using the notation in [8], the proof proceeds by letting  $\rho_{12} = \rho_1 \otimes \rho_2$  and  $L = B_1 \otimes I_2 + I_1 \otimes B_2$ . Setting  $\rho_{12}^\alpha = \rho_1^\alpha \otimes \rho_2^\alpha$ , we have

$$\begin{aligned} \rho_{12}^\alpha L \rho_{12}^{1-\alpha} L &= \rho_1^\alpha B_1 \rho_1^{1-\alpha} B_1 \otimes \rho_2 + \rho_1^\alpha B_1 \rho_1^{1-\alpha} \otimes \rho_2 B_2 \\ &\quad + \rho_1 B_1 \otimes \rho_2^\alpha B_2 \rho_2^{1-\alpha} + \rho \otimes \rho_2^\alpha B_2 \rho_2^{1-\alpha} B_2 \end{aligned}$$

and

$$\begin{aligned} \text{Tr}(\rho_{12}^\alpha L \rho_{12}^{1-\alpha} L) &= \text{Tr}(\rho_1^\alpha B_1 \rho_1^{1-\alpha} B_1) + 2 \text{Tr}(\rho_1 B_1) \text{Tr}(\rho_2 B_2) \\ &\quad + \text{Tr}(\rho_2^\alpha B_2 \rho_2^{1-\alpha} B_2). \end{aligned} \tag{B1}$$

Similarly,

$$\text{Tr}(\rho_{12}^\alpha L^2) = \text{Tr}(\rho_1 B_1^2) + 2 \text{Tr}(\rho_1 B_1) \text{Tr}(\rho_2 B_2) + \text{Tr}(\rho_2 B_2^2). \tag{B2}$$

From the above equations (B1) and (B2), we can derive  $I_\alpha(\rho, B)$  as additive.

Similarly,

$$\text{Tr}(\rho_{12}^\alpha L) = \text{Tr}(\rho_1 B_1) + \text{Tr}(\rho_2 B_2). \tag{B3}$$

By Eqs. (B1)–(B3), and the definition of  $J_\alpha(\rho, B)$  in Eq. (6), we can conclude that  $J_\alpha(\rho, B)$  is additive.

[1] Fisher information has been discussed extensively in literature on statistical estimation theory. A particularly insightful survey of the connection between Fisher information and the Heisenberg uncertainty principle can be found in e-print arXiv:quant-ph/0309184.  
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