

## Mutually unbiased bases and generalized Bell states

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We employ a straightforward relation between mutually unbiased and Bell bases to extend the latter in terms of a direct construction for the former. We analyze in detail the properties of these generalized Bell states, showing that they constitute an appropriate tool for testing entanglement in bipartite multidigit systems.

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### I. INTRODUCTION

Entanglement is probably the most intriguing feature of the quantum world, the hallmark of correlations that delimits the boundary between classical and quantum behavior. Although some amazing aspects of this phenomenon were already noticed by Schrödinger in the early stages of quantum theory [1], it was not until quite recently that it attracted a considerable attention as a crucial resource for quantum information processing [2].

The simplest instance of entanglement is most clearly illustrated by the maximally entangled states between a pair of qubits (known as the Bell states), whose properties can be found in many textbooks [3]. Despite their simplicity, they are of utmost importance for the analysis of many experiments [4].

In consequence, as any sound concept, the Bell states deserve an appropriate generalization. However, this is a touchy business since thoughtful notions for a pair of qubits may become fuzzy for more complex systems. There are two sensible ways to proceed: the first, is to investigate multipartite entanglement of qubits. While the standard Bell basis defines (for pure states) a natural unit of entanglement, it has recently become clear that for qubits shared by more parties there is a rich phenomenology of entangled states [5–11].

The second possibility involves examining bipartite entanglement between two multidimensional systems [12–16]. Again there is no unique way of looking at the problem, and different definitions focus on different aspects and capture different features of this quantum phenomenon.

We wish to approach this subject from a new perspective: our starting point is the notion of mutually unbiased bases (MUBs), which emerged in the seminal work of Schwinger [17], and it has turned into a cornerstone of quantum information, mainly due to the elegant work of Wootters and co-workers [18–21]. Since MUBs carry complete information about single-particle attributes and Bell's bases about bipartite entanglement, one is led to look for a relation between them.

In this paper we confirm such a relation for qudits [22] and take advantage of the well-established MUB machinery (in prime power dimensions) to propose a straightforward generalization of the Bell states for any dimension. The resulting bases are analyzed in detail, paying special attention to their symmetry properties. In view of our results, we con-

clude that these states constitute an ideal instrument to analyze bipartite multidigit systems.

### II. BIPARTITE QUDIT SYSTEMS

#### A. Mutually unbiased bases for qudits

We start by considering a qudit, which lives in a Hilbert space  $\mathcal{H}_d$ , whose dimension  $d$  is assumed for now to be a prime number. The different outcomes of a maximal test constitute an orthogonal basis of  $\mathcal{H}_d$ . One can also look for other orthogonal bases that, in addition, are “as different as possible.”

To formalize this idea, we suppose we have a number of orthonormal bases described by vectors  $|\psi_\ell^n\rangle$ , where  $\ell$  ( $\ell=0, 1, \dots, d-1$ ) labels the vectors in the  $n$ th basis. These are MUBs if each state of one basis gives rise to the same probabilities when measured in other basis:

$$|\langle \psi_{\ell'}^{n'} | \psi_\ell^n \rangle|^2 = \frac{1}{d}, \quad n \neq n'. \quad (2.1)$$

Equivalently, this can be concisely reformulated as

$$|\langle \psi_{\ell'}^{n'} | \psi_\ell^n \rangle|^2 = \delta_{\ell\ell'} \delta_{nn'} + \frac{1}{d}(1 - \delta_{nn'}). \quad (2.2)$$

Note in passing that the Hermitian product of two MUBs is a generalized Hadamard matrix, i.e., a unitary matrix whose entries all have the same absolute value [23].

If one wants to determine the state of a system, given only a limited supply of copies, the optimal strategy is to perform measurements with respect to MUBs [24]. They have also been used in cryptographic protocols [25] due to the complete uncertainty about the outcome of a measurement in some basis after the preparation of the system in another if the bases are mutually unbiased. MUBs are also important for quantum error correction codes [26,27] and in quantum game theory [28–31].

The maximum number of MUBs can be at most  $d+1$  [32]. Actually, it is known that if  $d$  is prime or power of prime (which is precisely our case), the maximal number of MUBs can be achieved.

Unbiasedness also applies to measurements: two nondegenerate tests are mutually unbiased if the bases formed by their eigenstates are MUBs. For example, the measurements of the components of a qubit along  $x$ ,  $y$ , and  $z$  axes are all

unbiased. It is also clear that for finite quantum systems unbiasedness is tantamount of complementarity [33,34].

The construction of MUBs is closely related to the possibility of finding  $d+1$  disjoint classes, each one having  $d-1$  commuting operators (excluding the identity) so that the corresponding eigenstates form sets of MUBs [35]. Different explicit methods in prime dimensions have been suggested in a number of recent papers [36–41], but we follow here the one introduced in Ref. [42], since it is especially germane for our purposes.

First, in the computational basis  $|\ell\rangle$  of  $\mathcal{H}_d$  we introduce the basic operators

$$X|\ell\rangle = |\ell+1\rangle, \quad Z|\ell\rangle = \omega(\ell)|\ell\rangle, \quad (2.3)$$

where addition and multiplication must be understood modulo  $d$  and, for simplicity, we employ the notation

$$\omega(\ell) = \omega^\ell = \exp(i2\pi\ell/d), \quad (2.4)$$

$\omega = \exp(i2\pi/d)$  being a  $d$ th root of the unity. These operators  $X$  and  $Z$ , which are generalizations of the Pauli matrices, were studied long ago by Weil [43]. They generate a group under multiplication known as the generalized Pauli group and obey  $ZX = \omega XZ$ , which is the finite-dimensional version of the Weyl form of the commutation relations [44].

We consider the following sets of operators:

$$\tilde{\Lambda}(m) = X^m, \quad \Lambda(m, n) = Z^m X^{nm}, \quad (2.5)$$

with  $m = 1, \dots, d-1$  and  $n = 0, \dots, d-1$ . They fulfill the pairwise orthogonality relations

$$\text{Tr}[\tilde{\Lambda}(m)\tilde{\Lambda}^\dagger(m')] = d\delta_{mm'}.$$

$$\text{Tr}[\Lambda(m, n)\Lambda^\dagger(m', n')] = d\delta_{mm'}\delta_{nn'}, \quad (2.6)$$

which indicate that, for every value of  $n$ , we generate a maximal set of  $d-1$  commuting operators and that all these classes are disjoint. In addition, the common eigenstates of each class  $n$  form different sets of MUBs.

If one recalls that the finite Fourier transform  $F$  is [45]

$$F = \frac{1}{\sqrt{d}} \sum_{\ell, \ell'=0}^{d-1} \omega(\ell\ell')|\ell\rangle\langle\ell'|, \quad (2.7)$$

then one easily verifies that

$$Z = FXF^\dagger, \quad (2.8)$$

much in the spirit of the standard way of looking at complementary variables in the infinite-dimensional Hilbert space [46]: the position and momentum eigenstates are the Fourier transform one of the other.

The operators  $\Lambda(m, n)$  can be written as

$$\Lambda(m, n) = e^{i\phi(m, n)} V^n Z^m V^{\dagger n}, \quad (2.9)$$

where  $V$  turns out to be ( $d > 2$ )

$$V = \sum_{\ell=0}^{d-1} \omega(-2^{-1}\ell^2)|\tilde{\ell}\rangle\langle\tilde{\ell}|, \quad (2.10)$$

and the phase  $\phi(m, n)$  is [47,48]

$$\phi(m, n) = \omega(2^{-1}nm^2). \quad (2.11)$$

Here  $2^{-1}$  denotes the multiplicative inverse of 2 mod  $d$  [that is,  $2^{-1} = (d+1)/2$ ] and  $|\tilde{\ell}\rangle$  is the conjugate basis, which is defined by the action of the Fourier transform on the computational basis, namely,  $|\tilde{\ell}\rangle = F|\ell\rangle$ .

The case of qubits ( $d=2$ ) requires minor modifications:  $V$  is now

$$V = \frac{1}{2} \begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}, \quad (2.12)$$

while its action reads as  $VZV^\dagger = -iZX$ .

The operator  $V$  has quite an important property: its powers generate MUBs when acting on the computational basis. Indeed, if

$$|\psi_\ell^n\rangle = V^n|\ell\rangle, \quad (2.13)$$

one can check by a direct calculation that the states  $|\psi_\ell^n\rangle$  fulfill Eq. (2.2), which confirms the unbiasedness. If we denote  $\Lambda_{\ell\ell'}(m, n) = \langle\ell|\Lambda(m, n)|\ell'\rangle$ , according to Eq. (2.9), we have

$$\Lambda_{\ell\ell'}(m, n) = e^{i\phi(m, n)} \langle\psi_\ell^n|Z^m|\psi_{\ell'}^n\rangle. \quad (2.14)$$

Therefore, up to an unessential phase factor,  $\Lambda_{\ell\ell'}(m, n)$  are the matrix elements of the powers of the diagonal operator  $Z$  in the corresponding MUB. This provides an elegant interpretation of these objects, which will play an essential role in what follows.

## B. Qudit Bell states

For the case of two qudits, a sensible generalization of Bell states was devised in Ref. [49], namely,

$$|\Psi_{mn}\rangle = \frac{1}{\sqrt{d}} \sum_{\ell=0}^{d-1} \omega(m\ell)|\ell\rangle_A |\ell+n\rangle_B, \quad (2.15)$$

where, to simplify as much as possible the notation, we drop the subscript  $AB$  from  $|\Psi_{mn}\rangle$ , since here we deal only with bipartite states. For further use, we also define

$$|\tilde{\Psi}_m\rangle = \frac{1}{\sqrt{d}} \sum_{\ell=0}^{d-1} |\ell\rangle_A |\ell+m\rangle_B. \quad (2.16)$$

In the same vein, some generalized gates have been proposed to create these  $d^2$  states [50,51]. Relations between maximally entangled vectors and orthonormal unitary bases have also been highlighted [52,53].

This set of states is orthonormal

$$\langle\Psi_{mn}|\Psi_{m'n'}\rangle = \delta_{mm'}\delta_{nn'}, \quad \langle\tilde{\Psi}_m|\tilde{\Psi}_{m'}\rangle = \delta_{mm'},$$

$$\langle\Psi_{mn}|\tilde{\Psi}_{m'}\rangle = \delta_{m0}\delta_{m'0}, \quad (2.17)$$

and allows for a resolution of the identity

$$\sum_{m=1}^{d-1} \sum_{n=0}^{d-1} |\Psi_{mn}\rangle\langle\Psi_{mn}| + \sum_{m=1}^{d-1} |\tilde{\Psi}_m\rangle\langle\tilde{\Psi}_m| = 1, \quad (2.18)$$

so they constitute a *bona fide* basis for any bipartite qudit system. As anticipated in the introduction, there must be then a connection with MUBs. And this is indeed the case: it suffices to observe that states (2.15) and (2.16) can be recast as

$$\begin{aligned} |\Psi_{mn}\rangle &= \frac{1}{\sqrt{d}} \sum_{\ell, \ell'=0}^{d-1} \Lambda_{\ell\ell'}(m, n) |\ell\rangle_A |\ell'\rangle_B, \\ |\tilde{\Psi}_m\rangle &= \frac{1}{\sqrt{d}} \sum_{\ell, \ell'=0}^{d-1} \tilde{\Lambda}_{\ell\ell'}(m) |\ell\rangle_A |\ell'\rangle_B, \end{aligned} \quad (2.19)$$

which can be checked by a direct calculation and  $\Lambda_{\ell\ell'}(m, n)$  and  $\tilde{\Lambda}_{\ell\ell'}(m)$  are the matrix elements of operator (2.5).

The matrices  $\Lambda$  possess quite an interesting symmetry property

$$\Lambda_{\ell\ell'}(m, n) = \omega(m^2 n) \Lambda_{\ell'\ell}(m, n), \quad \tilde{\Lambda}_{\ell\ell'}(m) = \tilde{\Lambda}_{\ell'\ell}(m). \quad (2.20)$$

In consequence,  $\tilde{\Lambda}(m)$  are always totally symmetric under the permutation of subsystems  $A$  and  $B$  and so are the corresponding Bell states. Whenever  $\omega(m^2 n) = \pm 1$ ,  $\Lambda(m, n)$  are either symmetric or antisymmetric. This happens for  $mn=0 \pmod{d}$ , and this is only possible for qubits: the symmetric matrices are  $\tilde{\Lambda}(0)$ ,  $\tilde{\Lambda}(1)$ , and  $\Lambda(1, 0)$ , while the antisymmetric is  $\Lambda(1, 1)$ . The symmetric states are  $|\tilde{\Psi}_0\rangle = |\Phi_+\rangle$ ,  $|\tilde{\Psi}_1\rangle = |\Psi_+\rangle$ , and  $|\Psi_{1,0}\rangle = |\Phi_-\rangle$ , and  $|\Psi_{1,1}\rangle = |\Psi_-\rangle$  is the antisymmetric one.

Finally, we can sum up the projectors of the bipartite states [Eq. (2.15)] over  $m$ , obtaining the following interesting property:

$$\begin{aligned} \sum_{m=0}^{d-1} |\Psi_{mn}\rangle \langle \Psi_{mn}| &= \frac{1}{d} \sum_{\ell=0}^{d-1} (X^{n\ell} Z^{-\ell})_A \otimes (X^{n\ell} Z^{\ell})_B, \\ \sum_{m=0}^{d-1} |\tilde{\Psi}_m\rangle \langle \tilde{\Psi}_m| &= \frac{1}{d} \sum_{\ell=0}^{d-1} (X^{\ell})_A \otimes (X^{\ell})_B. \end{aligned} \quad (2.21)$$

In words, this means that the sum of projectors over the index  $m$  is the sum of direct products of commuting operators for each particle. The proof of this statement involves a tedious yet direct calculation.

For the case of two qubits, this means that

$$\begin{aligned} \sum_{m=0,1} |\Psi_{m1}\rangle \langle \Psi_{m1}| &= \frac{1}{2} [1 + (XZ)_A \otimes (XZ)_B], \\ \sum_{m=0,1} |\tilde{\Psi}_m\rangle \langle \tilde{\Psi}_m| &= \frac{1}{2} [1 + (X)_A \otimes (X)_B]. \end{aligned} \quad (2.22)$$

### III. BIPARTITE MULTIQUIDIT SYSTEMS

#### A. Mutually unbiased bases for $n$ qudits

The previous ideas can be extended for  $n$ -qudit systems. Instead of natural numbers, it is then convenient to use ele-

ments of the finite field  $\mathbb{F}_{d^n}$  to label states, since then we can almost directly translate all the properties studied before for a single qudit. In the Appendix we briefly summarize the basic notions of finite fields needed to proceed.

We denote as  $|\lambda\rangle$  (from here on, Greek letters will represent elements in the field  $\mathbb{F}_{d^n}$ ) an orthonormal basis in the Hilbert space of the quantum system. Operationally, the elements of the basis can be labeled by powers of a primitive element, which can be found as a root of a minimal irreducible polynomial of degree  $n$  over  $\mathbb{Z}_d$ .

The generators of the generalized Pauli group are now

$$X_{\mu} |\lambda\rangle = |\lambda + \mu\rangle, \quad Z_{\mu} |\lambda\rangle = \chi(\lambda\mu) |\lambda\rangle, \quad (3.1)$$

where  $\chi(\lambda)$  is an additive character (defined in the Appendix). The Weyl form of the commutation relations reads as  $Z_{\mu} X_{\nu} = \chi(\mu\nu) X_{\nu} Z_{\mu}$ .

In agreement with Eq. (2.5), we introduce the set of monomials

$$\tilde{\Lambda}(\mu) = X_{\mu}, \quad \Lambda(\mu, \nu) = Z_{\mu} X_{\nu\mu}, \quad (3.2)$$

and their corresponding eigenstates also form a complete set of  $d^n + 1$  MUBs.

The finite Fourier transform in our case reduces to [54]

$$F = \frac{1}{\sqrt{d^n}} \sum_{\lambda, \lambda' \in \mathbb{F}_{d^n}} \chi(\lambda\lambda') |\lambda\rangle \langle \lambda'|, \quad (3.3)$$

and thus

$$Z_{\mu} = F X_{\mu} F^{\dagger}. \quad (3.4)$$

The rotation operator  $V_{\nu}$  transforms the diagonal  $Z_{\mu}$  into an arbitrary monomial according to

$$\Lambda(\mu, \nu) = e^{i\varphi(\mu, \nu)} V_{\nu} Z_{\mu} V_{\nu}^{\dagger}, \quad (3.5)$$

and is diagonal in the conjugate basis (defined, as before, via the Fourier transform  $|\tilde{\lambda}\rangle = F|\lambda\rangle$ )

$$V_{\nu} = \sum_{\lambda} c_{\lambda\nu} |\tilde{\lambda}\rangle \langle \tilde{\lambda}|, \quad (3.6)$$

where the coefficients  $c_{\lambda\nu}$  satisfy the following relations:

$$c_{0\nu} = 1, \quad c_{\lambda+\lambda'\nu} c_{\lambda\nu}^* = c_{\lambda'\nu} \chi(-\nu\lambda'\lambda), \quad (3.7)$$

When  $d \neq 2$ , a particular solution of Eq. (3.7) is

$$c_{\lambda\nu} = \chi(-2^{-1}\lambda^2\nu). \quad (3.8)$$

Again, if we define the states

$$|\psi_{\lambda}^{\mu}\rangle = V_{\mu} |\lambda\rangle, \quad (3.9)$$

they are unbiased and  $\Lambda_{\lambda\lambda'}(\mu, \nu)$  are the matrix elements of the diagonal operator  $Z_{\mu}$  in the corresponding MUB

$$\Lambda_{\lambda\lambda'}(\mu, \nu) = e^{i\varphi(\mu, \nu)} \langle \psi_{\lambda}^{\nu} | Z_{\mu} | \psi_{\lambda'}^{\mu} \rangle. \quad (3.10)$$

#### B. Multiqudit Bell states

For a bipartite system of  $n$  qudits, it seems natural to extend the previous construction in Eq. (2.19) by introducing the  $d^{2n}$  states

$$|\Psi_{\mu\nu}\rangle = \frac{1}{\sqrt{d^n}} \sum_{\lambda, \lambda'} \Lambda_{\lambda\lambda'}(\mu, \nu) |\lambda\rangle_A |\lambda'\rangle_B,$$

$$|\tilde{\Psi}_{\mu}\rangle = \frac{1}{\sqrt{d^n}} \sum_{\lambda, \lambda'} \tilde{\Lambda}_{\lambda\lambda'}(\mu) |\lambda\rangle_A |\lambda'\rangle_B. \quad (3.11)$$

This means that, apart from an unessential global phase, we have

$$|\Psi_{\mu\nu}\rangle = \frac{1}{\sqrt{d^n}} \sum_{\lambda} \chi(\mu\lambda) |\lambda\rangle_A |\lambda + \nu\rangle_B,$$

$$|\tilde{\Psi}_{\mu}\rangle = \frac{1}{\sqrt{d^n}} \sum_{\lambda} |\lambda\rangle_A |\lambda + \nu\rangle_B, \quad (3.12)$$

which look as quite a reasonable generalization. One can prove the orthogonality

$$\langle \Psi_{\mu\nu} | \Psi_{\mu'\nu'} \rangle = \delta_{\mu\mu'} \delta_{\nu\nu'}, \quad \langle \tilde{\Psi}_{\mu} | \tilde{\Psi}_{\mu'} \rangle = \delta_{\mu\mu'},$$

$$\langle \Psi_{\mu\nu} | \tilde{\Psi}_{\mu'} \rangle = \delta_{\mu 0} \delta_{\mu' 0}, \quad (3.13)$$

and the completeness relation

$$\sum_{\mu \neq 0, \nu} |\Psi_{\mu\nu}\rangle \langle \Psi_{\mu\nu}| + \sum_{\mu} |\tilde{\Psi}_{\mu}\rangle \langle \tilde{\Psi}_{\mu}| = \mathbb{1}, \quad (3.14)$$

which confirms that they constitute a basis. Moreover, the reduced density matrices for both subsystems are completely random,

$$\text{Tr}_A(|\Psi_{\mu\nu}\rangle \langle \Psi_{\mu\nu}|) = \frac{1}{d^n} \sum_{\lambda} |\lambda\rangle_{BB} \langle \lambda| \quad (3.15)$$

(and other analogous equation with  $A$  and  $B$  interchanged), showing that they are maximally entangled states.

The concept of symmetric and antisymmetric states can be worked out for systems of  $n$  qubits, which constitutes a nontrivial generalization of our previous discussion [55]. The symmetric states [i.e.,  $\Lambda_{\lambda\lambda'}(\mu, \nu) = \Lambda_{\lambda'\lambda}(\mu, \nu)$ ], correspond to those pairs  $(\mu, \nu)$  such that

$$\text{tr}(\nu\mu^2) = 0, \quad (3.16)$$

where  $\text{tr}$ , in small case, denotes the trace map in the field. Clearly,  $|\Psi_{\mu 0}\rangle$  and  $|\tilde{\Psi}_{\mu}\rangle$  are symmetric. The antisymmetric ones [i.e.,  $\Lambda_{\lambda\lambda'}(\mu, \nu) = -\Lambda_{\lambda'\lambda}(\mu, \nu)$ ] are defined by the pairs  $(\mu, \nu)$  such that

$$\text{tr}(\nu\mu^2) = 1. \quad (3.17)$$

Finally, a property similar to Eq. (2.21) is fulfilled: summing up the projectors over  $\mu$  one obtains

$$\sum_{\mu} |\Psi_{\mu\nu}\rangle \langle \Psi_{\mu\nu}| = \sum_{\lambda} (X_{\lambda\nu} Z_{-\lambda})_A \otimes (X_{\lambda\nu} Z_{\lambda})_B,$$

$$\sum_{\mu} |\tilde{\Psi}_{\mu}\rangle \langle \tilde{\Psi}_{\mu}| = \sum_{\lambda} (X_{\lambda})_A \otimes (X_{\lambda})_B, \quad (3.18)$$

whose interpretation is otherwise the same as for qudits.

### C. Examples

Since we are dealing with  $n$ -qudit systems, we can map the abstract Hilbert space  $\mathcal{H}_{d^n}$  into  $n$  single-qudit Hilbert spaces. This is achieved by expanding the field elements in a convenient basis  $\{\theta_j\}$  (with  $j=1, \dots, n$ ) so that

$$\lambda = \sum_j \ell_j \theta_j, \quad (3.19)$$

where  $\ell_j \in \mathbb{Z}_d$ . We can thus represent the states as  $|\lambda\rangle = |\ell_1, \dots, \ell_n\rangle$  and the coefficients  $\ell_j$  play the role of quantum numbers for each qudit.

For example, for two qubits, the abstract state  $(|0\rangle + |\sigma^3\rangle)/\sqrt{2}$ , where  $\sigma$  is a primitive element, can be mapped onto the physical state  $(|00\rangle + |10\rangle)/\sqrt{2}$  in the polynomial basis  $\{1, \sigma\}$ , whereas in the self-dual basis  $\{\sigma, \sigma^2\}$  it is associated with  $(|00\rangle + |11\rangle)/\sqrt{2}$ . Observe that, while the first state is factorizable, the other is entangled.

The use of the self-dual basis (or the almost self-dual if the latter does not exist) is especially advantageous since only then the Fourier transform and the basic operators factorize in terms of single-qudit analogs:

$$X_{\lambda} = X^{\ell_1} \otimes \dots \otimes X^{\ell_n}, \quad Z_{\lambda} = Z^{\ell_1} \otimes \dots \otimes Z^{\ell_n}. \quad (3.20)$$

For a bipartite  $4 \times 4$  system the states are represented as  $|\lambda\rangle = |\ell_1, \ell_2\rangle$ , with  $\ell_j \in \mathbb{Z}_2$ . The Bell basis can be expressed as

$$|m_1, n_1; m_2, n_2\rangle$$

$$= \frac{(-1)^{m_1 n_2 + m_2 n_1}}{2} \sum_{\ell_1, \ell_2} (-1)^{m_1 \ell_1 + m_2 \ell_2}$$

$$\times |\ell_1 + m_1 n_2 + m_2 n_1, \ell_2 + m_1 n_1 + m_2 n_2\rangle_A |\ell_1, \ell_2\rangle_B,$$

$$\widetilde{|m_1, m_2\rangle} = \frac{1}{2} \sum_{\ell_1, \ell_2} |\ell_1 + n_1, \ell_2 + m_2\rangle_A |\ell_1; \ell_2\rangle_B. \quad (3.21)$$

The conditions

$$m_1 n_2 + m_2 n_1 = \begin{cases} 0 \\ 1 \end{cases} \quad (3.22)$$

determine the symmetric and antisymmetric states, respectively. The solutions of this equation show that there are ten symmetric states and six antisymmetric ones, whose explicit form can be directly computed from previous formulas.

So far we have been dealing with systems made of  $n$  qudits. However, sometimes they can be treated instead as a single ‘‘particle’’ with  $d^n$  levels. For example, a four-dimensional system can be taken as two qubits or as a ququart. If, for some physical reason, we choose for the ququart, we can still use Eq. (2.15), as in Ref. [49] even if now the dimension is not a prime number. However, if we proceed in this way the resulting basis contains six symmetric and two antisymmetric states, while the other eight do not have any explicit symmetry, contrary to our results.

To conclude, we observe that, for the prime-dimensional case, the labeling of the Bell states [Eq. (2.19)] by indices of the corresponding MUB operators is unique since only straight lines in the discrete phase space define MUBs. For multi-qudit systems things are different: although in

Eq. (3.11) we have used only the standard set of MUBs operators to label such states, another construction of MUBs exists. It has been shown in Ref. [56] that different and non-isomorphic sets of MUBs can be associated with some specific curves in phase space. This opens the possibility to label generalized Bell states also with points of such curves, producing quite different form of  $\Lambda$  matrices:  $\Lambda(\mu, \nu) = Z_{\mu(\kappa)} X_{\nu(\kappa)}$ , where  $[\mu(\kappa), \nu(\kappa)]$  are points of curve such that  $[Z_{\mu(\kappa)} X_{\nu(\kappa)}, Z_{\mu(\kappa')} X_{\nu(\kappa')}] = 0$  and  $\kappa$  is a parameter along the curve. Unfortunately, in this case no direct generalization of sum rules [Eq. (3.18)] can be obtained.

**IV. CONCLUDING REMARKS**

In summary, we have provided a complete MUB-based construction of Bell states that fulfills all the requirements needed for a good description of maximally entangled states of bipartite multidigit systems.

Mutually unbiasedness is a very sound concept arising from the exact formulation of complementarity. The deep connection shown in this paper with the Bell bases is more than a mere academic curiosity for it is immediately applicable to a variety of experiments involving qudit systems.

**APPENDIX: FINITE FIELDS**

In this appendix we briefly recall the minimum background needed in this paper. The reader interested in more mathematical details is referred, e.g., to the excellent monograph by Lidl and Niederreiter [57].

A commutative ring is a nonempty set  $R$  furnished with two binary operations, called addition and multiplication, such that it is an Abelian group with respect to addition, and the multiplication is associative. Perhaps, the motivating example is the ring of integers  $\mathbb{Z}$ , with the standard sum and multiplication. On the other hand, the simplest example of a finite ring is the set  $\mathbb{Z}_n$  of integers mod  $n$ , which has exactly  $n$  elements.

A field  $F$  is a commutative ring with division, that is, such that 0 does not equal 1 and all elements of  $F$  except 0 have a multiplicative inverse (note that 0 and 1 here stand for the identity elements for the addition and multiplication, respectively, which may differ from the familiar real numbers 0 and 1). Elements of a field form Abelian groups with respect to addition and multiplication (in this latter case, the zero element is excluded).

The characteristic of a finite field is the smallest integer  $d$  such that

$$d1 = \underbrace{1 + 1 + \dots + 1}_d = 0, \tag{A1}$$

and it is always a prime number. Any finite field contains a prime subfield  $\mathbb{Z}_d$  and has  $d^n$  elements, where  $n$  is a natural number. Moreover, the finite field containing  $d^n$  elements is unique and is called the Galois field  $\mathbb{F}_{d^n}$ .

Let us denote as  $\mathbb{Z}_d[x]$  the ring of polynomials with coefficients in  $\mathbb{Z}_d$ . Let  $P(x)$  be an irreducible polynomial of degree  $n$  (i.e., one that cannot be factorized over  $\mathbb{Z}_d$ ). Then, the quotient space  $\mathbb{Z}_d[x]/P(x)$  provides an adequate representation of  $\mathbb{F}_{d^n}$ . Its elements can be written as polynomials that

are defined modulo the irreducible polynomial  $P(x)$ . The multiplicative group of  $\mathbb{F}_{d^n}$  is cyclic and its generator is called a primitive element of the field.

As a simple example of a nonprime field, we consider the polynomial  $x^2+x+1=0$ , which is irreducible in  $\mathbb{Z}_2$ . If  $\sigma$  is a root of this polynomial, the elements  $\{0, 1, \sigma, \sigma^2=\sigma+1=\sigma^{-1}\}$  form the finite field  $\mathbb{F}_{2^2}$  and  $\sigma$  is a primitive element.

A basic map is the trace

$$\text{tr}(\lambda) = \lambda + \lambda^2 + \dots + \lambda^{d^n-1}. \tag{A2}$$

It is always in the prime field  $\mathbb{Z}_d$  and satisfies

$$\text{tr}(\lambda + \lambda') = \text{tr}(\lambda) + \text{tr}(\lambda'). \tag{A3}$$

In terms of it we define the additive characters as

$$\chi(\lambda) = \exp\left[\frac{2\pi i}{p} \text{tr}(\lambda)\right], \tag{A4}$$

which posses two important properties:

$$\chi(\lambda + \lambda') = \chi(\lambda)\chi(\lambda'), \quad \sum_{\lambda' \in \mathbb{F}_{d^n}} \chi(\lambda\lambda') = d^n \delta_{0,\lambda}. \tag{A5}$$

Any finite field  $\mathbb{F}_{d^n}$  can be also considered as an  $n$ -dimensional linear vector space. Given a basis  $\{\theta_j\}$ , ( $j=1, \dots, n$ ) in this vector space, any field element can be represented as

$$\lambda = \sum_{j=1}^n \ell_j \theta_j, \tag{A6}$$

with  $\ell_j \in \mathbb{Z}_d$ . In this way, we map each element of  $\mathbb{F}_{d^n}$  onto an ordered set of natural numbers  $\lambda \Leftrightarrow (\ell_1, \dots, \ell_n)$ .

Two bases  $\{\theta_1, \dots, \theta_n\}$  and  $\{\theta'_1, \dots, \theta'_n\}$  are dual when

$$\text{tr}(\theta_k \theta'_l) = \delta_{kl}. \tag{A7}$$

A basis that is dual to itself is called self-dual.

There are several natural bases in  $\mathbb{F}_{d^n}$ . One is the polynomial basis defined as

$$\{1, \sigma, \sigma^2, \dots, \sigma^{n-1}\}, \tag{A8}$$

where  $\sigma$  is a primitive element. An alternative is the normal basis constituted of

$$\{\sigma, \sigma^d, \dots, \sigma^{d^{n-1}}\}. \tag{A9}$$

The choice of the appropriate basis depends on the specific problem at hand. For example, in  $\mathbb{F}_{2^2}$  the elements  $\{\sigma, \sigma^2\}$  are both roots of the irreducible polynomial. The polynomial basis is  $\{1, \sigma\}$  and its dual is  $\{\sigma^2, 1\}$ , while the normal basis  $\{\sigma, \sigma^2\}$  is self-dual.

The self-dual basis exists if and only if either  $d$  is even or both  $n$  and  $d$  are odd. However for every prime power  $d^n$ , there exists an almost self-dual basis of  $\mathbb{F}_{d^n}$ , which satisfies the properties:  $\text{tr}(\theta_i \theta_j) = 0$  when  $i \neq j$  and  $\text{tr}(\theta_i^2) = 1$ , with one possible exception. For instance, in the case of two qutrits  $\mathbb{F}_{3^2}$ , a self-dual basis does not exist and two elements  $\{\sigma^2, \sigma^4\}$ ,  $\sigma$  being a root of the irreducible polynomial  $x^2+x+2=0$ , form a self-dual basis

$$\text{tr}(\sigma^2 \sigma^2) = 1, \quad \text{tr}(\sigma^4 \sigma^4) = 2, \quad \text{tr}(\sigma^2 \sigma^4) = 0. \tag{A10}$$

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